7.2 Natural Logarithms

For any positive number a, the function value $f(x) = a^x$ is easy to define when x is an integer or rational number. When x is irrational, the meaning of a^x is not so clear. Similarly, the definition of the logarithm $\log_a x$, the inverse function of $f(x) = a^x$, is not completely obvious. In this section we use integral calculus to define the *natural logarithm* function, for which the number a is a particularly important value. This function allows us to define and analyze general exponential and logarithmic functions, $y = a^x$ and $y = \log_a x$.

Logarithms originally played important roles in arithmetic computations. Historically, considerable labor went into producing long tables of logarithms, correct to five, eight, or even more, decimal places of accuracy. Prior to the modern age of electronic calculators and computers, every engineer owned slide rules marked with logarithmic scales. Calculations with logarithms made possible the great seventeenth-century advances in offshore navigation and celestial mechanics. Today we know such calculations are done using calculators or computers, but the properties and numerous applications of logarithms are as important as ever.

Definition of the Natural Logarithm Function

One solid approach to defining and understanding logarithms begins with a study of the natural logarithm function defined as an integral through the Fundamental Theorem of Calculus. While this approach may seem indirect, it enables us to derive quickly the familiar properties of logarithmic and exponential functions. The functions we have studied so far were analyzed using the techniques of calculus, but here we do something more fundamental. We use calculus for the very definition of the logarithmic and exponential functions.

The natural logarithm of a positive number x, written as $\ln x$, is the value of an integral.

DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 7.9). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1. The function is not defined for $x \le 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

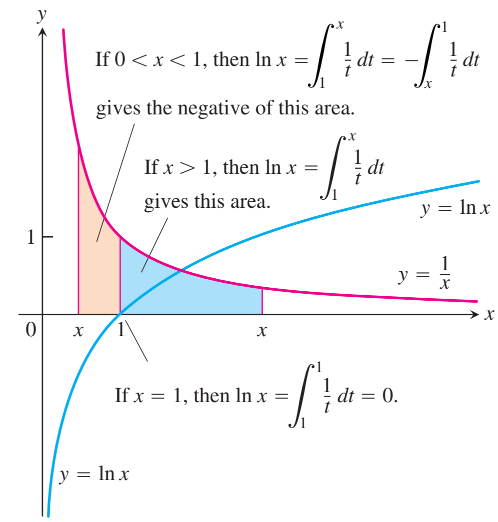

FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

Notice that we show the graph of $y = 1/x$ in Figure 7.9 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x}\,dx,$$

with x meaning two different things. So we change the variable of integration to t.

By using rectangles to obtain finite approximations of the area under the graph of $y = 1/t$ and over the interval between $t = 1$ and $t = x$, as in Section 5.1, we can approximate the values of the function $\ln x$. Several values are given in Table 7.1. There is an important number whose natural logarithm equals 1.

DEFINITION The Number *e*

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

Geometrically, the number e corresponds to the point on the x-axis for which the area under the graph of $y = 1/t$ and above the interval $[1, e]$ is the exact area of the unit square. The area of the region shaded blue in Figure 7.9 is 1 sq unit when $x = e$.

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (Section 5.4),

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$. Notice that the derivative is always positive so the natural logarithm is an increasing function, hence it is one-to-one and invertible. Its inverse is studied in Section 7.3.

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)$$

EXAMPLE 1 Derivatives of Natural Logarithms

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$

(b) Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2 + 3) = \frac{1}{x^2 + 3}\cdot\frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3}\cdot 2x = \frac{2x}{x^2 + 3}.$$

■

Notice the remarkable occurrence in Example 1a. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln ax$ for any positive number a:

$$\frac{d}{dx}\ln ax = \frac{1}{ax}\cdot\frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x}. \tag{2}$$

Since they have the same derivative, the functions $y = \ln ax$ and $y = \ln x$ differ by a constant.

Historical Biography

John Napier
(1550–1617)

Properties of Logarithms

Logarithms were invented by John Napier and were the single most important improvement in arithmetic calculation before the modern electronic computer. What made them so useful is that the properties of logarithms enable multiplication of positive numbers by addition of their logarithms, division of positive numbers by subtraction of their logarithms, and exponentiation of a number by multiplying its logarithm by the exponent. We summarize these properties as a series of rules in Theorem 2. For the moment, we restrict the exponent r in Rule 4 to be a rational number; you will see why when we prove the rule.

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ — r rational

We illustrate how these rules apply.

EXAMPLE 2 Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$ — Product

(b) $\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$ — Quotient

(c) $\ln\frac{1}{8} = -\ln 8$ — Reciprocal

$= -\ln 2^3 = -3\ln 2$ — Power ■

EXAMPLE 3 Applying the Properties to Function Formulas

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ — Product

(b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ — Quotient

(c) $\ln \sec x = \ln \frac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x + 1} = \ln (x + 1)^{1/3} = \frac{1}{3} \ln (x + 1)$ Power ■

We now give the proof of Theorem 2. The steps in the proof are similar to those used in solving problems involving logarithms.

Proof that $\ln ax = \ln a + \ln x$ The argument is unusual—and elegant. It starts by observing that $\ln ax$ and $\ln x$ have the same derivative (Equation 2). According to Corollary 2 of the Mean Value Theorem, then, the functions must differ by a constant, which means that

$$\ln ax = \ln x + C$$

for some C.

Since this last equation holds for all positive values of x, it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln (a \cdot 1) &= \ln 1 + C \\ \ln a &= 0 + C \qquad \text{ln } 1 = 0 \\ C &= \ln a. \end{aligned}$$

By substituting we conclude,

$$\ln ax = \ln a + \ln x.$$

Proof that $\ln x^r = r \ln x$ (assuming r rational) We use the same-derivative argument again. For all positive values of x,

$$\begin{aligned} \frac{d}{dx} \ln x^r &= \frac{1}{x^r} \frac{d}{dx}(x^r) \qquad \text{Eq. (1) with } u = x^r \\ &= \frac{1}{x^r} r x^{r-1} \\ &= r \cdot \frac{1}{x} = \frac{d}{dx}(r \ln x). \end{aligned}$$

Here is where we need r to be rational, at least for now. We have proved the Power Rule only for rational exponents.

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r \ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero, and we're done.

You are asked to prove Rule 2 in Exercise 84. Rule 3 is a special case of Rule 2, obtained by setting $a = 1$ and noting that $\ln 1 = 0$. So we have established all cases of Theorem 2. ■

We have not yet proved Rule 4 for r irrational; we will return to this case in Section 7.3. The rule does hold for all r, rational or irrational.

The Graph and Range of $\ln x$

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down.

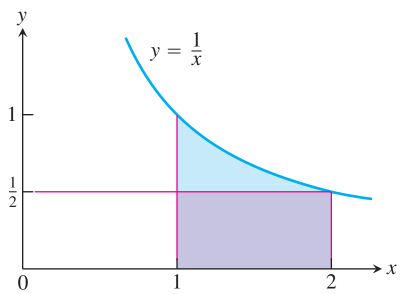

FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

We can estimate the value of $\ln 2$ by considering the area under the graph of $y = 1/x$ and above the interval [1, 2]. In Figure 7.10 a rectangle of height 1/2 over the interval [1, 2] fits under the graph. Therefore the area under the graph, which is $\ln 2$, is greater than the area, 1/2, of the rectangle. So $\ln 2 > 1/2$. Knowing this we have,

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and

$$\ln 2^{-n} = -n \ln 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}.$$

It follows that

$$\lim_{x\to\infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \ln x = -\infty .$$

We defined $\ln x$ for $x > 0$, so the domain of $\ln x$ is the set of positive real numbers. The above discussion and the Intermediate Value Theorem show that its range is the entire real line giving the graph of $y = \ln x$ shown in Figure 7.9.

The Integral $\int (1/u)\, du$

Equation (1) leads to the integral formula

$$\int \frac{1}{u}\, du = \ln u + C \tag{3}$$

when u is a positive differentiable function, but what if u is negative? If u is negative, then $-u$ is positive and

$$\int \frac{1}{u}\, du = \int \frac{1}{(-u)}\, d(-u) \qquad \text{Eq. (3) with } u \text{ replaced by } -u$$

$$= \ln(-u) + C. \tag{4}$$

We can combine Equations (3) and (4) into a single formula by noticing that in each case the expression on the right is $\ln |u| + C$. In Equation (3), $\ln u = \ln |u|$ because $u > 0$; in Equation (4), $\ln(-u) = \ln |u|$ because $u < 0$. Whether u is positive or negative, the integral of $(1/u)\, du$ is $\ln |u| + C$.

> If u is a differentiable function that is never zero,
>
> $$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

Equation (5) applies anywhere on the domain of $1/u$, the points where $u \neq 0$.

We know that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1 \text{ and rational}$$

Equation (5) explains what to do when n equals -1. Equation (5) says integrals of a certain *form* lead to logarithms. If $u = f(x)$, then $du = f'(x)\,dx$ and

$$\int \frac{1}{u}\,du = \int \frac{f'(x)}{f(x)}\,dx.$$

So Equation (5) gives

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

EXAMPLE 4 Applying Equation (5)

(a)
$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$
$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

$u = x^2 - 5,\quad du = 2x\,dx,$
$u(0) = -5,\quad u(2) = -1$

(b)
$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$
$$= 2\ln|u|\Big]_1^5$$
$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

$u = 3 + 2\sin\theta,\quad du = 2\cos\theta\,d\theta,$
$u(-\pi/2) = 1,\quad u(\pi/2) = 5$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Integrals of tan *x* and cot *x*

Equation (5) tells us at last how to integrate the tangent and cotangent functions. For the tangent function,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$
$$= -\int \frac{du}{u} = -\ln|u| + C$$
$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C$$
$$= \ln|\sec x| + C.$$

$u = \cos x > 0$ on $(-\pi/2, \pi/2)$,
$du = -\sin x\,dx$

Reciprocal Rule

For the cotangent,

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$$
$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$u = \sin x,$
$du = \cos x\,dx$

$$\int \tan u\, du = -\ln |\cos u| + C = \ln |\sec u| + C$$

$$\int \cot u\, du = \ln |\sin u| + C = -\ln |\csc x| + C$$

EXAMPLE 5

$$\int_0^{\pi/6} \tan 2x\, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\, du$$

$$= \frac{1}{2} \ln |\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$ ■

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 6 Using Logarithmic Differentiation

Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\ln y = \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}$$

$$= \ln ((x^2 + 1)(x + 3)^{1/2}) - \ln (x - 1) \qquad \text{Rule 2}$$

$$= \ln (x^2 + 1) + \ln (x + 3)^{1/2} - \ln (x - 1) \qquad \text{Rule 1}$$

$$= \ln (x^2 + 1) + \frac{1}{2} \ln (x + 3) - \ln (x - 1). \qquad \text{Rule 3}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

A direct computation in Example 6, using the Quotient and Product Rules, would be much longer.