7.3 The Exponential Function

Having developed the theory of the function $\ln x$, we introduce the exponential function $\exp x = e^x$ as the inverse of $\ln x$. We study its properties and compute its derivative and integral. Knowing its derivative, we prove the power rule to differentiate x^n when n is *any* real number, rational or irrational.

The Inverse of ln *x* and the Number *e*

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.11,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \ln^{-1} x = 0.$$

The function $\ln^{-1} x$ is also denoted by $\exp x$.

In Section 7.2 we defined the number e by the equation $\ln(e) = 1$, so $e = \ln^{-1}(1) = \exp(1)$. Although e is not a rational number, later in this section we see one way to express it as a limit. In Chapter 11, we will calculate its value with a computer to as many places of accuracy as we want with a different formula (Section 11.9, Example 6). To 15 places,

$$e = 2.718281828459045.$$

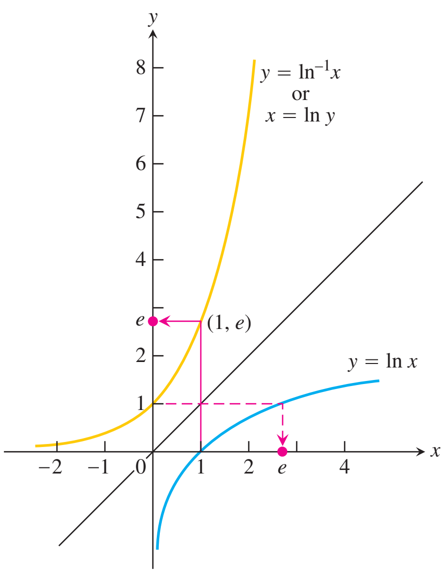

FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

The Function $y = e^x$

We can raise the number e to a rational power r in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take the logarithm, we find that

$$\ln e^r = r \ln e = r \cdot 1 = r.$$

Since $\ln x$ is one-to-one and $\ln(\ln^{-1} r) = r$, this equation tells us that

$$e^r = \ln^{-1} r = \exp r \qquad \text{for } r \text{ rational.} \tag{1}$$

We have not yet found a way to give an obvious meaning to e^x for x irrational. But $\ln^{-1} x$ has meaning for any x, rational or irrational. So Equation (1) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all x, so we use it to assign a value to e^x at every point where e^x had no previous definition.

DEFINITION The Natural Exponential Function
For every real number x, $e^x = \ln^{-1} x = \exp x$.

For the first time we have a precise meaning for an irrational exponent. Usually the exponential function is denoted by e^x rather than $\exp x$. Since $\ln x$ and e^x are inverses of one another, we have

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \qquad (2)$$

$$\ln(e^x) = x \qquad (\text{all } x) \qquad (3)$$

Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers that are not algebraic are called **transcendental**, like e and π. In 1873, Charles Hermite proved the transcendence of e in the sense that we describe. In 1882, C.L.F. Lindemann proved the transcendence of π.

Today, we call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

The domain of $\ln x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.

EXAMPLE 1 Using the Inverse Equations

(a) $\ln e^2 = 2$

(b) $\ln e^{-1} = -1$

(c) $\ln \sqrt{e} = \dfrac{1}{2}$

(d) $\ln e^{\sin x} = \sin x$

(e) $e^{\ln 2} = 2$

(f) $e^{\ln(x^2+1)} = x^2 + 1$

(g) $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way

(h) $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way ■

EXAMPLE 2 Solving for an Exponent

Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$\begin{aligned} e^{2k} &= 10 \\ \ln e^{2k} &= \ln 10 \\ 2k &= \ln 10 && \text{Eq. (3)} \\ k &= \frac{1}{2}\ln 10. \end{aligned}$$

■

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$.

HISTORICAL BIOGRAPHY

Siméon Denis Poisson
(1781–1840)

Biography

EXAMPLE 3 Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$ ■

We study the calculus of general exponential functions and their inverses in the next section. Here we need the definition in order to discuss the laws of exponents for e^x.

Laws of Exponents

Even though e^x is defined in a seemingly roundabout way as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra. Theorem 3 shows us that these laws are consequences of the definitions of $\ln x$ and e^x.

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Proof of Law 1 Let

$$y_1 = e^{x_1} \quad \text{and} \quad y_2 = e^{x_2}. \tag{4}$$

Then

$$\begin{aligned}
x_1 &= \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 && \text{Take logs of both sides of Eqs. (4).}\\
x_1 + x_2 &= \ln y_1 + \ln y_2 \\
&= \ln y_1 y_2 && \text{Product Rule for logarithms}\\
e^{x_1 + x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.}\\
&= y_1 y_2 && e^{\ln u} = u\\
&= e^{x_1} e^{x_2}.
\end{aligned}$$ ■

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercise 78).

EXAMPLE 4 Applying the Exponent Laws

(a) $e^{x + \ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

(b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

(c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

(d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4 ■

Theorem 3 is also valid for a^x, the exponential function with base a. For example,

$$\begin{aligned} a^{x_1} \cdot a^{x_2} &= e^{x_1 \ln a} \cdot e^{x_2 \ln a} && \text{Definition of } a^x \\ &= e^{x_1 \ln a + x_2 \ln a} && \text{Law 1} \\ &= e^{(x_1 + x_2)\ln a} && \text{Factor } \ln a \\ &= a^{x_1 + x_2}. && \text{Definition of } a^x \end{aligned}$$

The Derivative and Integral of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero (Theorem 1). We calculate its derivative using Theorem 1 and our knowledge of the derivative of $\ln x$. Let

$$f(x) = \ln x \qquad \text{and} \qquad y = e^x = \ln^{-1} x = f^{-1}(x).$$

Then,

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}(e^x) &= \frac{d}{dx} \ln^{-1} x \\ &= \frac{d}{dx} f^{-1}(x) \\ &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\ &= \frac{1}{f'(e^x)} && f^{-1}(x) = e^x \\ &= \frac{1}{\left(\frac{1}{e^x}\right)} && f'(z) = \frac{1}{z} \text{ with } z = e^x \\ &= e^x. \end{aligned}$$

That is, for $y = e^x$, we find that $dy/dx = e^x$ so the natural exponential function e^x is its own derivative. We will see in Section 7.5 that the only functions that behave this way are constant multiples of e^x. In summary,

$$\frac{d}{dx} e^x = e^x \tag{5}$$

EXAMPLE 5 Differentiating an Exponential

$$\begin{aligned} \frac{d}{dx}(5e^x) &= 5 \frac{d}{dx} e^x \\ &= 5e^x \end{aligned}$$

■

The Chain Rule extends Equation (5) in the usual way to a more general form.

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{6}$$

EXAMPLE 6 Applying the Chain Rule with Exponentials

(a) $\frac{d}{dx} e^{-x} = e^{-x} \frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (6) with $u = -x$

(b) $\frac{d}{dx} e^{\sin x} = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (6) with $u = \sin x$ ■

The integral equivalent of Equation (6) is

$$\int e^u \, du = e^u + C.$$

EXAMPLE 7 Integrating Exponentials

(a)
$$\begin{aligned} \int_0^{\ln 2} e^{3x} \, dx &= \int_0^{\ln 8} e^u \cdot \frac{1}{3} \, du \\ &= \frac{1}{3} \int_0^{\ln 8} e^u \, du \\ &= \frac{1}{3} e^u \Big]_0^{\ln 8} \\ &= \frac{1}{3}(8 - 1) = \frac{7}{3} \end{aligned}$$

$u = 3x, \quad \frac{1}{3} du = dx, \quad u(0) = 0,$
$u(\ln 2) = 3 \ln 2 = \ln 2^3 = \ln 8$

(b)
$$\begin{aligned} \int_0^{\pi/2} e^{\sin x} \cos x \, dx &= e^{\sin x} \Big]_0^{\pi/2} \\ &= e^1 - e^0 = e - 1 \end{aligned}$$

Antiderivative from Example 6 ■

EXAMPLE 8 Solving an Initial Value Problem

Solve the initial value problem

$$e^y \frac{dy}{dx} = 2x, \qquad x > \sqrt{3}; \qquad y(2) = 0.$$

Solution We integrate both sides of the differential equation with respect to x to obtain

$$e^y = x^2 + C.$$

We use the initial condition $y(2) = 0$ to determine C:

$$\begin{aligned} C &= e^0 - (2)^2 \\ &= 1 - 4 = -3. \end{aligned}$$

This completes the formula for e^y:

$$e^y = x^2 - 3.$$

To find y, we take logarithms of both sides:

$$\begin{aligned} \ln e^y &= \ln(x^2 - 3) \\ y &= \ln(x^2 - 3). \end{aligned}$$

Notice that the solution is valid for $x > \sqrt{3}$.

Let's check the solution in the original equation.

$$\begin{aligned} e^y \frac{dy}{dx} &= e^y \frac{d}{dx} \ln(x^2 - 3) \\ &= e^y \frac{2x}{x^2 - 3} && \text{Derivative of } \ln(x^2 - 3) \\ &= e^{\ln(x^2-3)} \frac{2x}{x^2 - 3} && y = \ln(x^2 - 3) \\ &= (x^2 - 3)\frac{2x}{x^2 - 3} && e^{\ln y} = y \\ &= 2x. \end{aligned}$$

The solution checks. ■

The Number *e* Expressed as a Limit

We have defined the number e as the number for which $\ln e = 1$, or the value $\exp(1)$. We see that e is an important constant for the logarithmic and exponential functions, but what is its numerical value? The next theorem shows one way to calculate e as a limit.

THEOREM 4 The Number *e* as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) && \ln 1 = 0 \\ &= \lim_{x \to 0} \ln(1+x)^{1/x} = \ln\left[\lim_{x \to 0}(1+x)^{1/x}\right] && \text{ln is continuous.} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x\to 0}(1+x)^{1/x}\right] = 1$$

Therefore,

$$\lim_{x\to 0}(1+x)^{1/x} = e \qquad \text{ln } e = 1 \text{ and ln is one-to-one}$$ ■

By substituting $y = 1/x$, we can also express the limit in Theorem 4 as

$$e = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y. \tag{7}$$

At the beginning of the section we noted that $e = 2.718281828459045$ to 15 decimal places.

The Power Rule (General Form)

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n\ln x}$. Therefore, the n in the equation $\ln x^n = n\ln x$ no longer needs to be rational—it can be any number as long as $x > 0$:

$$\ln x^n = \ln\left(e^{n\ln x}\right) = n\ln x \qquad \ln e^u = u, \text{ any } u$$

Together, the law $a^x/a^y = a^{x-y}$ and the definition $x^n = e^{n\ln x}$ enable us to establish the Power Rule for differentiation in its final form. Differentiating x^n with respect to x gives

$$\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}e^{n\ln x} && \text{Definition of } x^n,\ x > 0 \\
&= e^{n\ln x}\cdot\frac{d}{dx}(n\ln x) && \text{Chain Rule for } e^u \\
&= x^n\cdot\frac{n}{x} && \text{The definition again} \\
&= nx^{n-1}.
\end{aligned}$$

In short, as long as $x > 0$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

The Chain Rule extends this equation to the Power Rule's general form.

Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 9 Using the Power Rule with Irrational Powers

(a) $\frac{d}{dx}x^{\sqrt{2}} = \sqrt{2}x^{\sqrt{2}-1} \qquad (x > 0)$

(b) $$\begin{aligned}\frac{d}{dx}(2 + \sin 3x)^{\pi} &= \pi(2 + \sin 3x)^{\pi-1}(\cos 3x) \cdot 3 \\ &= 3\pi(2 + \sin 3x)^{\pi-1}(\cos 3x).\end{aligned}$$

■