7.5 Exponential Growth and Decay

Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in a wide variety of natural and industrial situations. The variety of models based on these functions partly accounts for their importance.

The Law of Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Examples of such quantities include the amount of a decaying radioactive material, funds earning interest in a bank account, the size of a population, and the temperature difference between a hot cup of coffee and the room in which it sits. Such quantities change according to the *law of exponential change*, which we derive in this section.

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

$$\begin{aligned} &\text{Differential equation:} && \frac{dy}{dt} = ky \\ &\text{Initial condition:} && y = y_0 \quad \text{when} \quad t = 0. \end{aligned} \tag{1}$$

If y is positive and increasing, then k is positive, and we use Equation (1) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Equation (1) to say that the rate of decay is proportional to the amount still left.

We see right away that the constant function $y = 0$ is a solution of Equation (1) if $y_0 = 0$. To find the nonzero solutions, we divide Equation (1) by y:

$$\frac{1}{y} \cdot \frac{dy}{dt} = k$$

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt \qquad \text{Integrate with respect to } t;$$

$$\ln|y| = kt + C \qquad \int (1/u)\,du = \ln|u| + C.$$

$$|y| = e^{kt+C} \qquad \text{Exponentiate.}$$

$$|y| = e^{C} \cdot e^{kt} \qquad e^{a+b} = e^{a} \cdot e^{b}$$

$$y = \pm e^{C} e^{kt} \qquad \text{If } |y| = r, \text{ then } y = \pm r.$$

$$y = Ae^{kt}. \qquad A \text{ is a shorter name for } \pm e^{C}.$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^{C}$, we can include the solution $y = 0$ in the formula.

We find the value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = Ae^{k \cdot 0} = A.$$

The solution of the initial value problem is therefore $y = y_0 e^{kt}$.

Quantities changing in this way are said to undergo **exponential growth** if $k > 0$, and **exponential decay** if $k < 0$.

The Law of Exponential Change

$$y = y_0 e^{kt} \qquad (2)$$

Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

The derivation of Equation (2) shows that the only functions that are their own derivatives are constant multiples of the exponential function.

Unlimited Population Growth

Strictly speaking, the number of individuals in a population (of people, plants, foxes, or bacteria, for example) is a discontinuous function of time because it takes on discrete values. However, when the number of individuals becomes large enough, the population can be approximated by a continuous function. Differentiability of the approximating function is another reasonable hypothesis in many settings, allowing for the use of calculus to model and predict population sizes.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. Let's assume, too, that the death rate of the population is stable and proportional to $y(t)$. If, further, we neglect departures and arrivals, the growth rate

dy/dt is the birth rate minus the death rate, which is the difference of the two proportionalities under our assumptions. In other words, $dy/dt = ky$, so that $y = y_0 e^{kt}$, where y_0 is the size of the population at time $t = 0$. As with all kinds of growth, there may be limitations imposed by the surrounding environment, but we will not go into these here. (This situation is analyzed in Section 9.5.)

In the following example we assume this population model to look at how the number of individuals infected by a disease within a given population decreases as the disease is appropriately treated.

EXAMPLE 1 Reducing the Cases of an Infectious Disease

One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. The number of people cured is proportional to the number that have the disease. Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Solution We use the equation $y = y_0 e^{kt}$. There are three things to find: the value of y_0, the value of k, and the time t when $y = 1000$.

The value of y_0. We are free to count time beginning anywhere we want. If we count from today, then $y = 10{,}000$ when $t = 0$, so $y_0 = 10{,}000$. Our equation is now

$$y = 10{,}000 e^{kt}. \tag{3}$$

The value of k. When $t = 1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

$$\begin{aligned} 8000 &= 10{,}000 e^{k(1)} && \text{Eq. (3) with } t = 1 \text{ and } y = 8000 \\ e^k &= 0.8 \\ \ln(e^k) &= \ln 0.8 && \text{Logs of both sides} \\ k &= \ln 0.8 < 0. \end{aligned}$$

At any given time t,

$$y = 10{,}000 e^{(\ln 0.8)t}. \tag{4}$$

The value of t *that makes* $y = 1000$. We set y equal to 1000 in Equation (4) and solve for t:

$$\begin{aligned} 1000 &= 10{,}000 e^{(\ln 0.8)t} \\ e^{(\ln 0.8)t} &= 0.1 \\ (\ln 0.8)t &= \ln 0.1 && \text{Logs of both sides} \\ t &= \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years}. \end{aligned}$$

It will take a little more than 10 years to reduce the number of cases to 1000. ■

Continuously Compounded Interest

If you invest an amount A_0 of money at a fixed annual interest rate r (expressed as a decimal) and if interest is added to your account k times a year, the formula for the amount of money you will have at the end of t years is

$$A_t = A_0 \left(1 + \frac{r}{k}\right)^{kt}. \tag{5}$$

The interest might be added ("compounded," bankers say) monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently, say by the hour or by the minute. By taking the limit as interest is compounded more and more often, we arrive at the following formula for the amount after t years,

$$\begin{aligned}
\lim_{k\to\infty} A_t &= \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} \\
&= A_0 \lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{\frac{k}{r}\cdot rt} \\
&= A_0\left[\lim_{\frac{r}{k}\to 0}\left(1+\frac{r}{k}\right)^{\frac{k}{r}}\right]^{rt} && \text{As } k\to\infty,\ \frac{r}{k}\to 0 \\
&= A_0\left[\lim_{x\to 0}(1+x)^{1/x}\right]^{rt} && \text{Substitute } x=\frac{r}{k} \\
&= A_0e^{rt} && \text{Theorem 4}
\end{aligned}$$

The resulting formula for the amount of money in your account after t years is

$$A(t) = A_0e^{rt}. \tag{6}$$

Interest paid according to this formula is said to be **compounded continuously**. The number r is called the **continuous interest rate.** The amount of money after t years is calculated with the law of exponential change given in Equation (6).

EXAMPLE 2 A Savings Account

Suppose you deposit $621 in a bank account that pays 6% compounded continuously. How much money will you have 8 years later?

Solution We use Equation (6) with $A_0 = 621$, $r = 0.06$, and $t = 8$:

$$A(8) = 621e^{(0.06)(8)} = 621e^{0.48} = 1003.58 \qquad \text{Nearest cent}$$

Had the bank paid interest quarterly ($k = 4$ in Equation 5), the amount in your account would have been $1000.01. Thus the effect of continuous compounding, as compared with quarterly compounding, has been an addition of $3.57. A bank might decide it would be worth this additional amount to be able to advertise, "We compound interest every second, night and day—better yet, we compound the interest continuously." ■

Radioactivity

Some atoms are unstable and can spontaneously emit mass or radiation. This process is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**. Sometimes when an atom emits some of its mass through this process of radioactivity, the remainder of the atom re-forms to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium, through a number of intermediate radioactive steps, decays into lead.

For radon-222 gas, t is measured in days and $k = 0.18$. For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice), t is measured in years and $k = 4.3 \times 10^{-4}$.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. It is conventional to use

$-k (k > 0)$ here instead of $k (k < 0)$ to emphasize that y is decreasing. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-kt}, \qquad k > 0.$$

EXAMPLE 3 Half-Life of a Radioactive Element

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is an interesting fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$y_0 e^{-kt} = \frac{1}{2} y_0$$

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln \frac{1}{2} = -\ln 2 \qquad \text{Reciprocal Rule for logarithms}$$

$$t = \frac{\ln 2}{k}$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not enter in.

$$\text{Half-life} = \frac{\ln 2}{k} \tag{7}$$

■

EXAMPLE 4 Half-Life of Polonium-210

The effective radioactive lifetime of polonium-210 is so short we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3} t}.$$

Find the element's half-life.

Solution

$$\text{Half-life} = \frac{\ln 2}{k} \qquad \text{Eq. (7)}$$

$$= \frac{\ln 2}{5 \times 10^{-3}} \qquad \text{The } k \text{ from polonium's decay equation}$$

$$\approx 139 \text{ days}$$

■

EXAMPLE 5 Carbon-14 Dating

The decay of radioactive elements can sometimes be used to date events from the Earth's past. In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's surroundings at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 in the organism's remains decreases as the carbon-14 decays.

Scientists who do carbon-14 dating use a figure of 5700 years for its half-life (more about carbon-14 dating in the exercises). Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Solution We use the decay equation $y = y_0 e^{-kt}$. There are two things to find: the value of k and the value of t when y is $0.9y_0$ (90% of the radioactive nuclei are still present). That is, find t when $y_0 e^{-kt} = 0.9y_0$, or $e^{-kt} = 0.9$.

The value of k. We use the half-life Equation (7):

$$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5700} \qquad (\text{about } 1.2 \times 10^{-4})$$

The value of t that makes $e^{-kt} = 0.9$:

$$e^{-kt} = 0.9$$

$$e^{-(\ln 2/5700)t} = 0.9$$

$$-\frac{\ln 2}{5700} t = \ln 0.9 \qquad \text{Logs of both sides}$$

$$t = -\frac{5700 \ln 0.9}{\ln 2} \approx 866 \text{ years}.$$

The sample is about 866 years old. ■

Heat Transfer: Newton's Law of Cooling

Hot soup left in a tin cup cools to the temperature of the surrounding air. A hot silver ingot immersed in a large tub of water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is called *Newton's law of cooling*, although it applies to warming as well, and there is an equation for it.

If H is the temperature of the object at time t and H_S is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_S). \tag{8}$$

If we substitute y for $(H - H_S)$, then

$$\frac{dy}{dt} = \frac{d}{dt}(H - H_S) = \frac{dH}{dt} - \frac{d}{dt}(H_S)$$

$$= \frac{dH}{dt} - 0 \qquad H_S \text{ is a constant.}$$

$$= \frac{dH}{dt}$$

$$= -k(H - H_S) \qquad \text{Eq. (8)}$$

$$= -ky. \qquad H - H_S = y.$$

Now we know that the solution of $dy/dt = -ky$ is $y = y_0 e^{-kt}$, where $y(0) = y_0$. Substituting $(H - H_S)$ for y, this says that

$$H - H_S = (H_0 - H_S)e^{-kt}, \tag{9}$$

where H_0 is the temperature at $t = 0$. This is the equation for Newton's Law of Cooling.

EXAMPLE 6 Cooling a Hard-Boiled Egg

A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 min that have already elapsed. Using Equation (9) with $H_S = 18$ and $H_0 = 98$, the egg's temperature t min after it is put in the sink is

$$H = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k, we use the information that $H = 38$ when $t = 5$:

$$\begin{aligned} 38 &= 18 + 80e^{-5k} \\ e^{-5k} &= \frac{1}{4} \\ -5k &= \ln\frac{1}{4} = -\ln 4 \\ k &= \frac{1}{5}\ln 4 = 0.2\ln 4 \qquad (\text{about } 0.28). \end{aligned}$$

The egg's temperature at time t is $H = 18 + 80e^{-(0.2\ln 4)t}$. Now find the time t when $H = 20$:

$$\begin{aligned} 20 &= 18 + 80e^{-(0.2\ln 4)t} \\ 80e^{-(0.2\ln 4)t} &= 2 \\ e^{-(0.2\ln 4)t} &= \frac{1}{40} \\ -(0.2\ln 4)t &= \ln\frac{1}{40} = -\ln 40 \\ t &= \frac{\ln 40}{0.2\ln 4} \approx 13 \text{ min}. \end{aligned}$$

The egg's temperature will reach 20°C about 13 min after it is put in the water to cool. Since it took 5 min to reach 38°C, it will take about 8 min more to reach 20°C. ■