EXERCISES 7.7

Common Values of Inverse Trignonometric Functions

Use reference triangles like those in Examples 1–3 to find the angles in Exercises 1–12.

Exercises

1. **a.** $\tan^{-1} 1$ **b.** $\tan^{-1}(-\sqrt{3})$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2. **a.** $\tan^{-1}(-1)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

3. **a.** $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

4. **a.** $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

5. **a.** $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

6. **a.** $\cos^{-1}\left(\frac{-1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

7. a. $\sec^{-1}(-\sqrt{2})$ b. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ c. $\sec^{-1}(-2)$

8. a. $\sec^{-1}\sqrt{2}$ b. $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c. $\sec^{-1} 2$

9. a. $\csc^{-1}\sqrt{2}$ b. $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c. $\csc^{-1} 2$

10. a. $\csc^{-1}(-\sqrt{2})$ b. $\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$ c. $\csc^{-1}(-2)$

11. a. $\cot^{-1}(-1)$ b. $\cot^{-1}(\sqrt{3})$ c. $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

12. a. $\cot^{-1}(1)$ b. $\cot^{-1}(-\sqrt{3})$ c. $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Trigonometric Function Values

13. Given that $\alpha = \sin^{-1}(5/13)$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

14. Given that $\alpha = \tan^{-1}(4/3)$, find $\sin\alpha$, $\cos\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

15. Given that $\alpha = \sec^{-1}(-\sqrt{5})$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

16. Given that $\alpha = \sec^{-1}(-\sqrt{13}/2)$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

Evaluating Trigonometric and Inverse Trigonometric Terms

Find the values in Exercises 17–28.

17. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

18. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

19. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

20. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

21. $\csc(\sec^{-1} 2) + \cos(\tan^{-1}(-\sqrt{3}))$

22. $\tan(\sec^{-1} 1) + \sin(\csc^{-1}(-2))$

23. $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)\right)$

24. $\cot\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sec^{-1} 2\right)$

25. $\sec(\tan^{-1} 1 + \csc^{-1} 1)$

26. $\sec(\cot^{-1}\sqrt{3} + \csc^{-1}(-1))$

27. $\sec^{-1}\left(\sec\left(-\frac{\pi}{6}\right)\right)$ (The answer is *not* $-\pi/6$.)

28. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$ (The answer is *not* $-\pi/4$.)

Finding Trigonometric Expressions

Evaluate the expressions in Exercises 29–40.

29. $\sec\left(\tan^{-1}\frac{x}{2}\right)$

30. $\sec(\tan^{-1} 2x)$

31. $\tan(\sec^{-1} 3y)$

32. $\tan\left(\sec^{-1}\frac{y}{5}\right)$

33. $\cos(\sin^{-1} x)$

34. $\tan(\cos^{-1} x)$

35. $\sin(\tan^{-1}\sqrt{x^2 - 2x})$, $x \geq 2$

36. $\sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right)$

37. $\cos\left(\sin^{-1}\frac{2y}{3}\right)$

38. $\cos\left(\sin^{-1}\frac{y}{5}\right)$

39. $\sin\left(\sec^{-1}\frac{x}{4}\right)$

40. $\sin\sec^{-1}\left(\frac{\sqrt{x^2+4}}{x}\right)$

Limits

Find the limits in Exercises 41–48. (If in doubt, look at the function's graph.)

41. $\lim_{x\to 1^-}\sin^{-1} x$

42. $\lim_{x\to -1^+}\cos^{-1} x$

43. $\lim_{x\to\infty}\tan^{-1} x$

44. $\lim_{x\to -\infty}\tan^{-1} x$

45. $\lim_{x\to\infty}\sec^{-1} x$

46. $\lim_{x\to -\infty}\sec^{-1} x$

47. $\lim_{x\to\infty}\csc^{-1} x$

48. $\lim_{x\to -\infty}\csc^{-1} x$

Finding Derivatives

In Exercises 49–70, find the derivative of y with respect to the appropriate variable.

49. $y = \cos^{-1}(x^2)$

50. $y = \cos^{-1}(1/x)$

51. $y = \sin^{-1}\sqrt{2}\,t$

52. $y = \sin^{-1}(1 - t)$

53. $y = \sec^{-1}(2s + 1)$

54. $y = \sec^{-1} 5s$

55. $y = \csc^{-1}(x^2 + 1)$, $x > 0$

56. $y = \csc^{-1}\frac{x}{2}$

57. $y = \sec^{-1}\frac{1}{t}$, $0 < t < 1$

58. $y = \sin^{-1}\frac{3}{t^2}$

59. $y = \cot^{-1}\sqrt{t}$

60. $y = \cot^{-1}\sqrt{t - 1}$

61. $y = \ln(\tan^{-1} x)$

62. $y = \tan^{-1}(\ln x)$

63. $y = \csc^{-1}(e^t)$

64. $y = \cos^{-1}(e^{-t})$

65. $y = s\sqrt{1 - s^2} + \cos^{-1} s$

66. $y = \sqrt{s^2 - 1} - \sec^{-1} s$

67. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x$, $x > 1$

68. $y = \cot^{-1}\frac{1}{x} - \tan^{-1} x$

69. $y = x\sin^{-1} x + \sqrt{1 - x^2}$

70. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\frac{x}{2}\right)$

Evaluating Integrals

Evaluate the integrals in Exercises 71–94.

71. $\int \frac{dx}{\sqrt{9 - x^2}}$

72. $\int \frac{dx}{\sqrt{1 - 4x^2}}$

73. $\int \frac{dx}{17 + x^2}$

74. $\int \frac{dx}{9 + 3x^2}$

75. $\int \frac{dx}{x\sqrt{25x^2 - 2}}$

76. $\int \frac{dx}{x\sqrt{5x^2 - 4}}$

77. $\int_0^1 \frac{4\,ds}{\sqrt{4 - s^2}}$

78. $\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

79. $\int_0^2 \frac{dt}{8 + 2t^2}$

80. $\int_{-2}^2 \frac{dt}{4 + 3t^2}$

81. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$

82. $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$

83. $\int \frac{3\,dr}{\sqrt{1 - 4(r - 1)^2}}$

84. $\int \frac{6\,dr}{\sqrt{4 - (r + 1)^2}}$

85. $\int \frac{dx}{2 + (x - 1)^2}$

86. $\int \frac{dx}{1 + (3x + 1)^2}$

87. $\int \frac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$

88. $\int \frac{dx}{(x + 3)\sqrt{(x + 3)^2 - 25}}$

89. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1 + (\sin\theta)^2}$

90. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1 + (\cot x)^2}$

91. $\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1 + e^{2x}}$

92. $\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1 + \ln^2 t)}$

93. $\int \frac{y\,dy}{\sqrt{1 - y^4}}$

94. $\int \frac{\sec^2 y\,dy}{\sqrt{1 - \tan^2 y}}$

Evaluate the integrals in Exercises 95–104.

95. $\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

96. $\int \frac{dx}{\sqrt{2x - x^2}}$

97. $\int_{-1}^0 \frac{6\,dt}{\sqrt{3 - 2t - t^2}}$

98. $\int_{1/2}^1 \frac{6\,dt}{\sqrt{3 + 4t - 4t^2}}$

99. $\int \frac{dy}{y^2 - 2y + 5}$

100. $\int \frac{dy}{y^2 + 6y + 10}$

101. $\int_1^2 \frac{8\,dx}{x^2 - 2x + 2}$

102. $\int_2^4 \frac{2\,dx}{x^2 - 6x + 10}$

103. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$

104. $\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

Evaluate the integrals in Exercises 105–112.

105. $\int \frac{e^{\sin^{-1} x}\,dx}{\sqrt{1 - x^2}}$

106. $\int \frac{e^{\cos^{-1} x}\,dx}{\sqrt{1 - x^2}}$

107. $\int \frac{(\sin^{-1} x)^2\,dx}{\sqrt{1 - x^2}}$

108. $\int \frac{\sqrt{\tan^{-1} x}\,dx}{1 + x^2}$

109. $\int \frac{dy}{(\tan^{-1} y)(1 + y^2)}$

110. $\int \frac{dy}{(\sin^{-1} y)\sqrt{1 - y^2}}$

111. $\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$

112. $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$

Limits

Find the limits in Exercises 113–116.

113. $\lim_{x \to 0} \frac{\sin^{-1} 5x}{x}$

114. $\lim_{x \to 1^+} \frac{\sqrt{x^2 - 1}}{\sec^{-1} x}$

115. $\lim_{x \to \infty} x \tan^{-1} \frac{2}{x}$

116. $\lim_{x \to 0} \frac{2\tan^{-1} 3x^2}{7x^2}$

Integration Formulas

Verify the integration formulas in Exercises 117–120.

117. $\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C$

118. $\int x^3 \cos^{-1} 5x\,dx = \frac{x^4}{4}\cos^{-1} 5x + \frac{5}{4}\int \frac{x^4\,dx}{\sqrt{1 - 25x^2}}$

119. $\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2}\sin^{-1} x + C$

120. $\int \ln(a^2 + x^2)\,dx = x\ln(a^2 + x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

Initial Value Problems

Solve the initial value problems in Exercises 121–124.

121. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, \quad y(0) = 0$

122. $\frac{dy}{dx} = \frac{1}{x^2 + 1} - 1, \quad y(0) = 1$

123. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}, \quad x > 1; \quad y(2) = \pi$

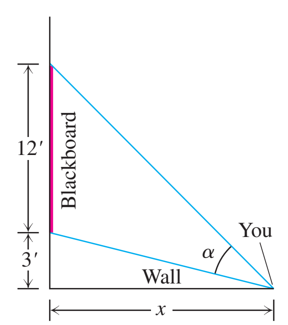
124. $\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{2}{\sqrt{1 - x^2}}, \quad y(0) = 2$

Applications and Theory

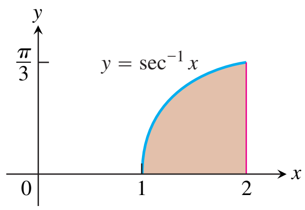
125. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

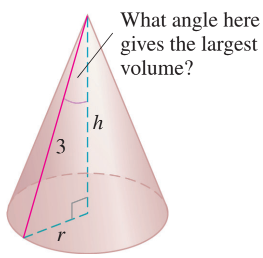
if you are x ft from the front wall.



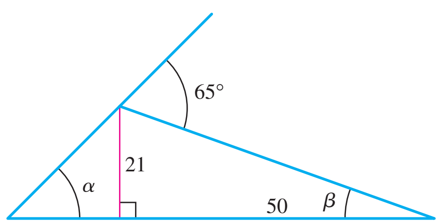
126. The region between the curve $y = \sec^{-1} x$ and the x-axis from $x = 1$ to $x = 2$ (shown here) is revolved about the y-axis to generate a solid. Find the volume of the solid.



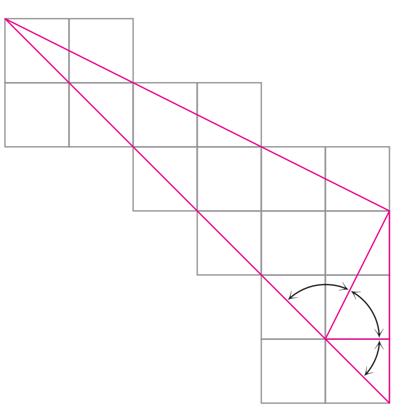
127. The slant height of the cone shown here is 3 m. How large should the indicated angle be to maximize the cone's volume?



128. Find the angle α.

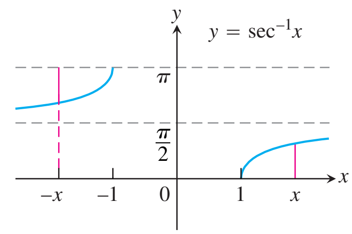


129. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



130. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$

a. (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



b. (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (3)}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (5)}$$

131. The identity $\sin^{-1} x + \cos^{-1} x = \pi/2$ Figure 7.21 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1$, 0, and -1. Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (1) and (3) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

132. Show that the sum $\tan^{-1} x + \tan^{-1}(1/x)$ is constant.

Which of the expressions in Exercises 133–136 are defined, and which are not? Give reasons for your answers.

133. **a.** $\tan^{-1} 2$ **b.** $\cos^{-1} 2$

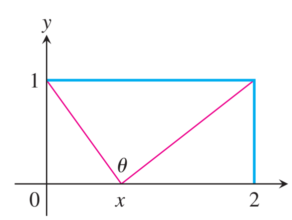
134. **a.** $\csc^{-1}(1/2)$ **b.** $\csc^{-1} 2$

135. **a.** $\sec^{-1} 0$ **b.** $\sin^{-1}\sqrt{2}$

136. **a.** $\cot^{-1}(-1/2)$ **b.** $\cos^{-1}(-5)$

137. (*Continuation of Exercise 125.*) You want to position your chair along the wall to maximize your viewing angle α. How far from the front of the room should you sit?

138. What value of x maximizes the angle θ shown here? How large is θ at that point? Begin by showing that $\theta = \pi - \cot^{-1} x - \cot^{-1}(2 - x)$.



139. Can the integrations in (a) and (b) both be correct? Explain.

a. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$

b. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$

140. Can the integrations in (a) and (b) both be correct? Explain.

a. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$

b. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-du}{\sqrt{1-(-u)^2}}$ $\quad x = -u, \; dx = -du$

$$= \int \frac{-du}{\sqrt{1-u^2}}$$

$$= \cos^{-1} u + C$$

$$= \cos^{-1}(-x) + C \qquad u = -x$$

141. Use the identity

$$\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u$$

to derive the formula for the derivative of $\csc^{-1} u$ in Table 7.3 from the formula for the derivative of $\sec^{-1} u$.

142. Derive the formula

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

for the derivative of $y = \tan^{-1} x$ by differentiating both sides of the equivalent equation $\tan y = x$.

143. Use the Derivative Rule in Section 7.1, Theorem 1, to derive

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

144. Use the identity

$$\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u$$

to derive the formula for the derivative of $\cot^{-1} u$ in Table 7.3 from the formula for the derivative of $\tan^{-1} u$.

145. What is special about the functions

$$f(x) = \sin^{-1}\frac{x-1}{x+1}, \quad x \ge 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$

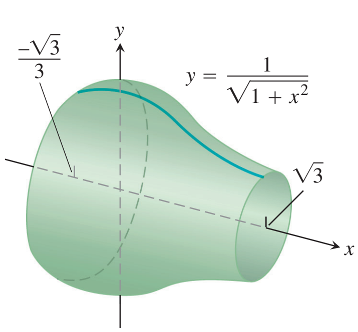
Explain.

146. What is special about the functions

$$f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1}\frac{1}{x}?$$

Explain.

147. Find the volume of the solid of revolution shown here.



148. Arc length Find the length of the curve $y = \sqrt{1-x^2}$, $-1/2 \le x \le 1/2$.

Volumes by Slicing

Find the volumes of the solids in Exercises 149 and 150.

149. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are

a. circles whose diameters stretch from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

b. vertical squares whose base edges run from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

150. The solid lies between planes perpendicular to the x-axis at $x = -\sqrt{2}/2$ and $x = \sqrt{2}/2$. The cross-sections perpendicular to the x-axis are

a. circles whose diameters stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

b. squares whose diagonals stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

T Calculator and Grapher Explorations

151. Find the values of

a. $\sec^{-1} 1.5$ **b.** $\csc^{-1}(-1.5)$ **c.** $\cot^{-1} 2$

152. Find the values of

a. $\sec^{-1}(-3)$ **b.** $\csc^{-1} 1.7$ **c.** $\cot^{-1}(-2)$

In Exercises 153–155, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

153. a. $y = \tan^{-1}(\tan x)$ **b.** $y = \tan(\tan^{-1} x)$

154. a. $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1} x)$

155. a. $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1} x)$

156. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

157. Newton's serpentine Graph Newton's serpentine, $y = 4x/(x^2+1)$. Then graph $y = 2\sin(2\tan^{-1} x)$ in the same graphing window. What do you see? Explain.

158. Graph the rational function $y = (2-x^2)/x^2$. Then graph $y = \cos(2\sec^{-1} x)$ in the same graphing window. What do you see? Explain.

159. Graph $f(x) = \sin^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

160. Graph $f(x) = \tan^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.