Chapter 7 Additional and Advanced Exercises

Limits

Find the limits in Exercises 1–6.

1. $\lim_{b\to 1^-}\int_0^b \frac{dx}{\sqrt{1-x^2}}$

2. $\lim_{x\to\infty}\frac{1}{x}\int_0^x \tan^{-1} t\,dt$

3. $\lim_{x\to 0^+}(\cos\sqrt{x})^{1/x}$

4. $\lim_{x\to\infty}(x+e^x)^{2/x}$

5. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$

6. $\lim_{n\to\infty}\frac{1}{n}\left(e^{1/n}+e^{2/n}+\cdots+e^{(n-1)/n}+e^{n/n}\right)$

7. Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y=e^{-x}$, and the vertical line $x=t$, $t>0$. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

   a. $\lim_{t\to\infty} A(t)$ **b.** $\lim_{t\to\infty} V(t)/A(t)$ **c.** $\lim_{t\to 0^+} V(t)/A(t)$

8. **Varying a logarithm's base**

   a. Find $\lim \log_a 2$ as $a\to 0^+, 1^-, 1^+$, and ∞.

   T **b.** Graph $y=\log_a 2$ as a function of a over the interval $0<a\le 4$.

Theory and Examples

9. Find the areas between the curves $y=2(\log_2 x)/x$ and $y=2(\log_4 x)/x$ and the x-axis from $x=1$ to $x=e$. What is the ratio of the larger area to the smaller?

T 10. Graph $f(x)=\tan^{-1}x+\tan^{-1}(1/x)$ for $-5\le x\le 5$. Then use calculus to explain what you see. How would you expect f to behave beyond the interval $[-5,5]$? Give reasons for your answer.

11. For what $x>0$ does $x^{(x^x)}=(x^x)^x$? Give reasons for your answer.

T 12. Graph $f(x)=(\sin x)^{\sin x}$ over $[0,3\pi]$. Explain what you see.

13. Find $f'(2)$ if $f(x)=e^{g(x)}$ and $g(x)=\int_2^x \frac{t}{1+t^4}\,dt$.

14. **a.** Find df/dx if

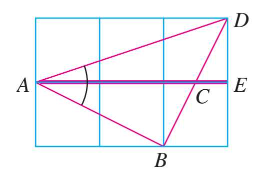
$$f(x)=\int_1^{e^x}\frac{2\ln t}{t}\,dt.$$

   b. Find $f(0)$.

   c. What can you conclude about the graph of f? Give reasons for your answer.

15. The figure here shows an informal proof that

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}.$$

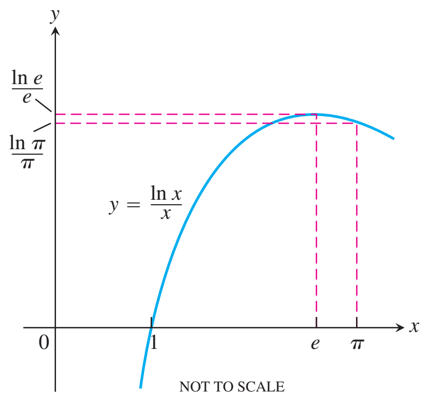


How does the argument go? (*Source*: "Behold! Sums of Arctan," by Edward M. Harris, *College Mathematics Journal*, Vol. 18, No. 2, Mar. 1987, p. 141.)

16. $\boldsymbol{\pi^e < e^\pi}$

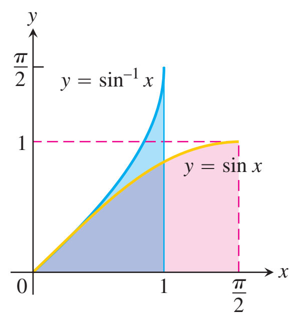
   a. Why does the accompanying figure "prove" that $\pi^e<e^\pi$? (*Source:* "Proof Without Words," by Fouad Nakhil, *Mathematics Magazine*, Vol. 60, No. 3, June 1987, p. 165.)

   b. The accompanying figure assumes that $f(x)=(\ln x)/x$ has an absolute maximum value at $x=e$. How do you know it does?



17. Use the accompanying figure to show that

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x \, dx.$$

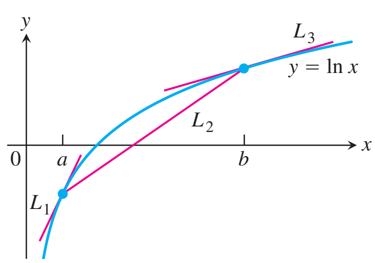


18. **Napier's inequality** Here are two pictorial proofs that

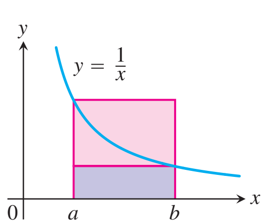
$$b > a > 0 \quad \Rightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

Explain what is going on in each case.

a.



b.



(*Source:* Roger B. Nelson, *College Mathematics Journal*, Vol. 24, No. 2, March 1993, p. 165.)

19. **Even-odd decompositions**

a. Suppose that g is an even function of x and h is an odd function of x. Show that if $g(x) + h(x) = 0$ for all x then $g(x) = 0$ for all x and $h(x) = 0$ for all x.

b. Use the result in part (a) to show that if $f(x) = f_E(x) + f_O(x)$ is the sum of an even function $f_E(x)$ and an odd function $f_O(x)$, then

$$f_E(x) = (f(x) + f(-x))/2 \quad \text{and} \quad f_O(x) = (f(x) - f(-x))/2.$$

c. What is the significance of the result in part (b)?

20. Let g be a function that is differentiable throughout an open interval containing the origin. Suppose g has the following properties:

i. $g(x + y) = \dfrac{g(x) + g(y)}{1 - g(x)g(y)}$ for all real numbers x, y, and $x + y$ in the domain of g.

ii. $\lim_{h \to 0} g(h) = 0$

iii. $\lim_{h \to 0} \dfrac{g(h)}{h} = 1$

a. Show that $g(0) = 0$.

b. Show that $g'(x) = 1 + [g(x)]^2$.

c. Find $g(x)$ by solving the differential equation in part (b).

Applications

21. **Center of mass** Find the center of mass of a thin plate of constant density covering the region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$.

22. **Solid of revolution** The region between the curve $y = 1/(2\sqrt{x})$ and the x-axis from $x = 1/4$ to $x = 4$ is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Find the centroid of the region.

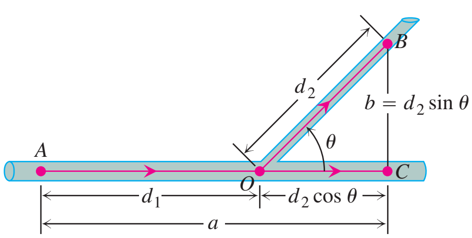
23. **The Rule of 70** If you use the approximation $\ln 2 \approx 0.70$ (in place of 0.69314 . . .), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the Rule of 70 is derived. (A similar "Rule of 72" uses 72 instead of 70, because 72 has more integer factors.)

24. **Free fall in the fourteenth century** In the middle of the fourteenth century, Albert of Saxony (1316–1390) proposed a model of free fall that assumed that the velocity of a falling body was proportional to the distance fallen. It seemed reasonable to think that a body that had fallen 20 ft might be moving twice as fast as a body that had fallen 10 ft. And besides, none of the instruments in use at the time were accurate enough to prove otherwise. Today we can see just how far off Albert of Saxony's model was by

solving the initial value problem implicit in his model. Solve the problem and compare your solution graphically with the equation $s = 16t^2$. You will see that it describes a motion that starts too slowly at first and then becomes too fast too soon to be realistic.

25. The best branching angles for blood vessels and pipes When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction be minimized along the section AOB shown in the accompanying figure. In this diagram, B is a given point to be reached by the smaller pipe, A is a point in the larger pipe upstream from B, and O is the point where the branching occurs. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent flow is proportional to the length of the path and inversely proportional to the fourth power of the radius. Thus, the loss along AO is $(kd_1)/R^4$ and along OB is $(kd_2)/r^4$, where k is a constant, d_1 is the length of AO, d_2 is the length of OB, R is the radius of the larger pipe, and r is the radius of the smaller pipe. The angle θ is to be chosen to minimize the sum of these two losses:

$$L = k\frac{d_1}{R^4} + k\frac{d_2}{r^4}.$$



In our model, we assume that $AC = a$ and $BC = b$ are fixed. Thus we have the relations

$$d_1 + d_2\cos\theta = a \quad d_2\sin\theta = b,$$

so that

$$d_2 = b\csc\theta,$$
$$d_1 = a - d_2\cos\theta = a - b\cot\theta.$$

We can express the total loss L as a function of θ:

$$L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right).$$

a. Show that the critical value of θ for which $dL/d\theta$ equals zero is

$$\theta_c = \cos^{-1}\frac{r^4}{R^4}.$$

b. If the ratio of the pipe radii is $r/R = 5/6$, estimate to the nearest degree the optimal branching angle given in part (a).

The mathematical analysis described here is also used to explain the angles at which arteries branch in an animal's body. (See *Introduction to Mathematics for Life Scientists*, Second Edition, by E. Batschelet [New York: Springer-Verlag, 1976].)

T **26. Group blood testing** During World War II it was necessary to administer blood tests to large numbers of recruits. There are two standard ways to administer a blood test to N people. In method 1, each person is tested separately. In method 2, the blood samples of x people are pooled and tested as one large sample. If the test is negative, this one test is enough for all x people. If the test is positive, then each of the x people is tested separately, requiring a total of $x + 1$ tests. Using the second method and some probability theory it can be shown that, on the average, the total number of tests y will be

$$y = N\left(1 - q^x + \frac{1}{x}\right).$$

With $q = 0.99$ and $N = 1000$, find the integer value of x that minimizes y. Also find the integer value of x that maximizes y. (This second result is not important to the real-life situation.) The group testing method was used in World War II with a savings of 80% over the individual testing method, but not with the given value of q.