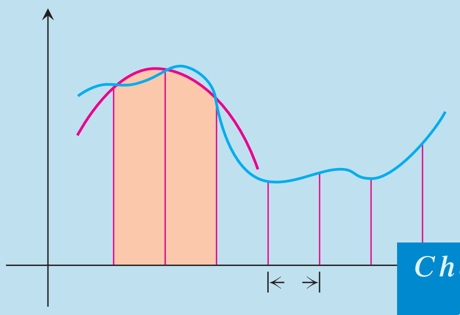

Chapter

8

TECHNIQUES OF INTEGRATION

OVERVIEW The Fundamental Theorem connects antiderivatives and the definite integral. Evaluating the indefinite integral

$$\int f(x)\,dx$$

is equivalent to finding a function F such that $F'(x) = f(x)$, and then adding an arbitrary constant C:

$$\int f(x)\,dx = F(x) + C.$$

In this chapter we study a number of important techniques for finding indefinite integrals of more complicated functions than those seen before. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer. We also extend the idea of the definite integral to *improper integrals* for which the integrand may be unbounded over the interval of integration, or the interval itself may no longer be finite.

8.1 Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

Recall the Substitution Rule from Section 5.5:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I. Success in integration often hinges on the ability to spot what part of the integrand should be called u in order that one will also have du, so that a known formula can be applied. This means that the first requirement for skill in integration is a thorough mastery of the formulas for differentiation.

Table 8.1 shows the basic forms of integrals we have evaluated so far. In this section we present several algebraic or substitution methods to help us use this table. There is a more extensive table at the back of the book; we discuss its use in Section 8.6.

TABLE 8.1 Basic integration formulas

1. $\displaystyle\int du = u + C$
2. $\displaystyle\int k\,du = ku + C \qquad (\text{any number } k)$
3. $\displaystyle\int (du + dv) = \int du + \int dv$
4. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
6. $\displaystyle\int \sin u\,du = -\cos u + C$
7. $\displaystyle\int \cos u\,du = \sin u + C$
8. $\displaystyle\int \sec^2 u\,du = \tan u + C$
9. $\displaystyle\int \csc^2 u\,du = -\cot u + C$
10. $\displaystyle\int \sec u \tan u\,du = \sec u + C$
11. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$
12. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\displaystyle\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\displaystyle\int e^u\,du = e^u + C$
15. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\displaystyle\int \sinh u\,du = \cosh u + C$
17. $\displaystyle\int \cosh u\,du = \sinh u + C$
18. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$
22. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

We often have to rewrite an integral to match it to a standard formula.

Video

You Try It

EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx.$$

Solution

$$\begin{aligned}
\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx &= \int \frac{du}{\sqrt{u}} && u = x^2 - 9x + 1,\ du = (2x-9)\,dx. \\
&= \int u^{-1/2}\,du \\
&= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C && \text{Table 8.1 Formula 4, with } n = -1/2 \\
&= 2u^{1/2} + C \\
&= 2\sqrt{x^2-9x+1} + C
\end{aligned}$$

EXAMPLE 2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x-x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned}
8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\
&= -(x^2 - 8x + 16) + 16 = 16 - (x-4)^2.
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{8x-x^2}} &= \int \frac{dx}{\sqrt{16-(x-4)^2}} \\
&= \int \frac{du}{\sqrt{a^2-u^2}} && a = 4,\ u = (x-4),\ du = dx \\
&= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\
&= \sin^{-1}\left(\frac{x-4}{4}\right) + C.
\end{aligned}$$

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2\,dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}\int (\sec x + \tan x)^2\, dx &= \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\, dx \\ &= 2\int \sec^2 x\, dx + 2\int \sec x \tan x\, dx - \int 1\, dx \\ &= 2\tan x + 2\sec x - x + C.\end{aligned}$$

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EXAMPLE 4 Eliminating a Square Root

You Try It

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution We use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2 \theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\ &= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx && \sqrt{u^2} = |u| \\ &= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx && \text{On } [0, \pi/4],\ \cos 2x \ge 0, \text{ so } |\cos 2x| = \cos 2x. \\ &= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \\ &= \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.\end{aligned}$$

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EXAMPLE 5 Reducing an Improper Fraction

Video

You Try It

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

$$\begin{array}{r} x - 3 \\ 3x + 2\,\overline{\smash{)}\,3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +\ 6 \end{array}$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx = \int \left(x - 3 + \frac{6}{3x + 2}\right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

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Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\, dx, \qquad \text{and} \qquad x\, dx = -\frac{1}{2}\, du.$$

$$\begin{aligned} 3\int \frac{x\, dx}{\sqrt{1 - x^2}} &= 3\int \frac{(-1/2)\, du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\, du \\ &= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1 \end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2\sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = -3\sqrt{1 - x^2} + 2\sin^{-1} x + C. \qquad \blacksquare$$

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

HISTORICAL BIOGRAPHY

George David Birkhoff (1884–1944)

EXAMPLE 7 Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate

$$\int \sec x\, dx.$$

Solution

$$\begin{aligned} \int \sec x\, dx &= \int (\sec x)(1)\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{du}{u} \qquad\qquad u = \tan x + \sec x,\ du = (\sec^2 x + \sec x \tan x)\, dx \\ &= \ln |u| + C = \ln |\sec x + \tan x| + C. \qquad \blacksquare \end{aligned}$$

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

TABLE 8.2 The secant and cosecant integrals

1. $$\int \sec u\, du = \ln|\sec u + \tan u| + C$$

2. $$\int \csc u\, du = -\ln|\csc u + \cot u| + C$$

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\dfrac{2x - 9}{\sqrt{x^2 - 9x + 1}}\, dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x$ $= \sec^2 x + 2\sec x\tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x \tan x - 1$
Eliminating a square root	$\sqrt{1 + \cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2 - 7x}{3x + 2} = x - 3 + \dfrac{6}{3x + 2}$
Separating a fraction	$\dfrac{3x + 2}{\sqrt{1 - x^2}} = \dfrac{3x}{\sqrt{1 - x^2}} + \dfrac{2}{\sqrt{1 - x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$