EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2 + 1}}$

2. $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1 + 3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$

4. $\displaystyle\int \cot^3 y \csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2 + 2}$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

Exercises

7. $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

8. $\int \frac{dx}{x-\sqrt{x}}$

9. $\int \cot(3-7x)\,dx$

10. $\int \csc(\pi x-1)\,dx$

11. $\int e^{\theta}\csc(e^{\theta}+1)\,d\theta$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\int \sec\frac{t}{3}\,dt$

14. $\int x\sec(x^2-5)\,dx$

15. $\int \csc(s-\pi)\,ds$

16. $\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}\,dx$

18. $\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\,dy$

19. $\int e^{\tan v}\sec^2 v\,dv$

20. $\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\int 3^{x+1}\,dx$

22. $\int \frac{2^{\ln x}}{x}\,dx$

23. $\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\int 10^{2\theta}\,d\theta$

25. $\int \frac{9\,du}{1+9u^2}$

26. $\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$

28. $\int_0^{1} \frac{dt}{\sqrt{4-t^2}}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\int \frac{dx}{e^x+e^{-x}}$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$

36. $\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

Exercises

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2}$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10}$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

Exercises

43. $\int (\sec x+\cot x)^2\,dx$

44. $\int (\csc x-\tan x)^2\,dx$

45. $\int \csc x\sin 3x\,dx$

46. $\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

Exercises

47. $\int \frac{x}{x+1}\,dx$

48. $\int \frac{x^2}{x^2+1}\,dx$

49. $\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx$

50. $\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx$

51. $\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$

52. $\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

Exercises

53. $\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

54. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

55. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$

56. $\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

Exercises

57. $\int \frac{1}{1+\sin x}\,dx$

58. $\int \frac{1}{1+\cos x}\,dx$

59. $\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

60. $\int \frac{1}{\csc\theta+\cot\theta}\,d\theta$

61. $\int \frac{1}{1-\sec x}\,dx$

62. $\int \frac{1}{1-\csc x}\,dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

Exercises

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$

64. $\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

65. $\int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t}\, dt$ **66.** $\int_{-\pi}^{0} \sqrt{1 + \cos t}\, dt$

67. $\int_{-\pi}^{0} \sqrt{1 - \cos^2 \theta}\, d\theta$ **68.** $\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta}\, d\theta$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 y}\, dy$ **70.** $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy$

Assorted Integrations

Evaluate each integral in Exercises 71–82 by using any technique you think is appropriate.

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx$ **72.** $\int_{0}^{\pi/4} (\sec x + 4 \cos x)^2\, dx$

73. $\int \cos\theta \csc(\sin\theta)\, d\theta$ **74.** $\int \left(1 + \frac{1}{x}\right) \cot(x + \ln x)\, dx$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

76. $\int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx$

77. $\int \frac{6\, dy}{\sqrt{y}(1 + y)}$ **78.** $\int \frac{dx}{x\sqrt{4x^2 - 1}}$

79. $\int \frac{7\, dx}{(x - 1)\sqrt{x^2 - 2x - 48}}$ **80.** $\int \frac{dx}{(2x + 1)\sqrt{4x^2 + 4x}}$

81. $\int \sec^2 t \tan(\tan t)\, dt$ **82.** $\int \frac{dx}{x\sqrt{3 + x^2}}$

Trigonometric Powers

83. a. Evaluate $\int \cos^3\theta\, d\theta$. (*Hint:* $\cos^2\theta = 1 - \sin^2\theta$.)

b. Evaluate $\int \cos^5\theta\, d\theta$.

c. Without actually evaluating the integral, explain how you would evaluate $\int \cos^9\theta\, d\theta$.

84. a. Evaluate $\int \sin^3\theta\, d\theta$. (*Hint:* $\sin^2\theta = 1 - \cos^2\theta$.)

b. Evaluate $\int \sin^5\theta\, d\theta$.

c. Evaluate $\int \sin^7\theta\, d\theta$.

d. Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13}\theta\, d\theta$.

85. a. Express $\int \tan^3\theta\, d\theta$ in terms of $\int \tan\theta\, d\theta$. Then evaluate $\int \tan^3\theta\, d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta - 1$.)

b. Express $\int \tan^5\theta\, d\theta$ in terms of $\int \tan^3\theta\, d\theta$.

c. Express $\int \tan^7\theta\, d\theta$ in terms of $\int \tan^5\theta\, d\theta$.

d. Express $\int \tan^{2k+1}\theta\, d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1}\theta\, d\theta$.

86. a. Express $\int \cot^3\theta\, d\theta$ in terms of $\int \cot\theta\, d\theta$. Then evaluate $\int \cot^3\theta\, d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta - 1$.)

b. Express $\int \cot^5\theta\, d\theta$ in terms of $\int \cot^3\theta\, d\theta$.

c. Express $\int \cot^7\theta\, d\theta$ in terms of $\int \cot^5\theta\, d\theta$.

d. Express $\int \cot^{2k+1}\theta\, d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1}\theta\, d\theta$.

Theory and Examples

87. Area Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

88. Area Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

89. Volume Find the volume of the solid generated by revolving the region in Exercise 87 about the x-axis.

90. Volume Find the volume of the solid generated by revolving the region in Exercise 88 about the x-axis.

91. Arc length Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

92. Arc length Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

93. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

94. Centroid Find the centroid of the region that is bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

95. The integral of csc *x* Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

96. Using different substitutions Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3}\, dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x + 1)$

b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$ **e.** $u = \tan^{-1}((x - 1)/2)$

f. $u = \cos^{-1} x$ **g.** $u = \cosh^{-1} x$

What is the value of the integral? (*Source:* "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)