8.2 Integration by Parts

Since

$$\int x\,dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx.$$

In other words, the integral of a product is generally *not* the product of the individual integrals:

$$\int f(x)g(x)\,dx \text{ is not equal to } \int f(x)\,dx \cdot \int g(x)\,dx.$$

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

leading to the **integration by parts** formula

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique.

EXAMPLE 1 Using Integration by Parts

Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x.$$

Simplest antiderivative of $\cos x$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

■

Let us examine the choices available for u and dv in Example 1.

EXAMPLE 2 Example 1 Revisited

To apply integration by parts to

$$\int x\cos x\,dx = \int u\,dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x\cos x\,dx$.
2. Let $u = x$ and $dv = \cos x\,dx$.
3. Let $u = x\cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\,dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x\cos x\,dx$ to get v.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x\cos x, & dv &= dx, \\ du &= (\cos x - x\sin x)\,dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v\,du = \int (x\cos x - x^2\sin x)\,dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x\,dx, \\ du &= -\sin x\,dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v\,du = -\int \frac{x^2}{2}\sin x\,dx.$$

This, too, is worse. ■

The goal of integration by parts is to go from an integral $\int u\,dv$ that we don't see how to evaluate to an integral $\int v\,du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. Keep in mind that integration by parts does not always work.

EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x\,dx.$$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated
$dv = dx$ Easy to integrate
$du = \frac{1}{x}\,dx,$
$v = x.$ Simplest antiderivative

Then

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$ ■

Sometimes we have to use integration by parts more than once.

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x \, dx .$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx .$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C .$$

Hence,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C . \end{aligned}$$

■

The technique of Example 4 works for any integral $\int x^n e^x \, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy. We say more about this later in this section when we discuss *tabular integration.*

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 5 Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x \, dx .$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx .$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx .$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx . \end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\, dx \qquad (3)$$

In applying Equation (3), we normally use the u and v notation from Equation (2) because it is easier to remember. Here is an example.

EXAMPLE 6 Finding Area

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

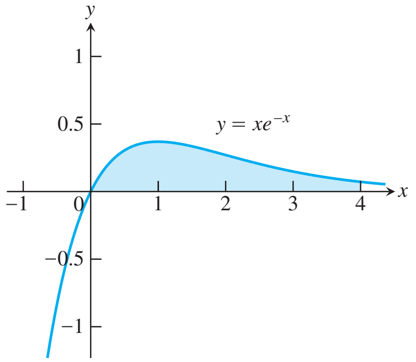

FIGURE 8.1 The region in Example 6.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\, dx.$$

Let $u = x$, $dv = e^{-x}\, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\, dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\, dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\, dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91. \qquad \blacksquare\end{aligned}$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\, dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

***f*(*x*) and its derivatives**		***g*(*x*) and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 4. ■

EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

***f*(*x*) and its derivatives**		***g*(*x*) and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

■

The Additional Exercises at the end of this chapter show how tabular integration can be used when neither function f nor g can be differentiated repeatedly to become zero.

Summary

When substitution doesn't work, try integration by parts. Start with an integral in which the integrand is the product of two functions,

$$\int f(x)g(x)\,dx.$$

(Remember that g may be the constant function 1, as in Example 3.) Match the integral with the form

$$\int u\,dv$$

by choosing dv to be part of the integrand including dx and either $f(x)$ or $g(x)$. Remember that we must be able to readily integrate dv to get v in order to obtain the right side of the formula

$$\int u\,dv = uv - \int v\,du.$$

If the new integral on the right side is more complex than the original one, try a different choice for u and dv.

EXAMPLE 9 A Reduction Formula

Obtain a "reduction" formula that expresses the integral

$$\int \cos^n x\,dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\,dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\,dx) \qquad \text{and} \qquad v = \sin x.$$

Hence

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx, \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\,dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

This allows us to reduce the exponent on $\cos x$ by 2 and is a very useful formula. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is either

$$\int \cos x\,dx = \sin x + C \qquad \text{or} \qquad \int \cos^0 x\,dx = \int dx = x + C. \quad \blacksquare$$

EXAMPLE 10 Using a Reduction Formula

Evaluate

$$\int \cos^3 x\,dx.$$

Solution From the result in Example 9,

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C. \quad \blacksquare\end{aligned}$$