EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta\, d\theta$

3. $\int t^2 \cos t\, dt$

4. $\int x^2 \sin x\, dx$

5. $\int_1^2 x \ln x\, dx$

6. $\int_1^e x^3 \ln x\, dx$

7. $\int \tan^{-1} y\, dy$

8. $\int \sin^{-1} y\, dy$

9. $\int x \sec^2 x\, dx$

10. $\int 4x \sec^2 2x\, dx$

11. $\int x^3 e^x\, dx$

12. $\int p^4 e^{-p}\, dp$

13. $\int (x^2 - 5x)e^x\, dx$

14. $\int (r^2 + r + 1)e^r\, dr$

15. $\int x^5 e^x\, dx$

16. $\int t^2 e^{4t}\, dt$

17. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$

18. $\int_0^{\pi/2} x^3 \cos 2x\, dx$

19. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$

20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx$

21. $\int e^\theta \sin \theta\, d\theta$

22. $\int e^{-y} \cos y\, dy$

23. $\int e^{2x} \cos 3x\, dx$

24. $\int e^{-2x} \sin 2x\, dx$

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$

26. $\int_0^1 x\sqrt{1-x}\,dx$

27. $\int_0^{\pi/3} x\tan^2 x\,dx$

28. $\int \ln(x+x^2)\,dx$

29. $\int \sin(\ln x)\,dx$

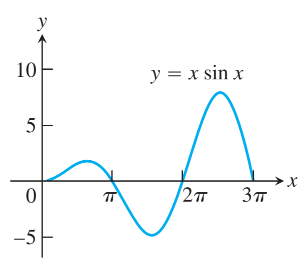
30. $\int z(\ln z)^2\,dz$

Theory and Examples

31. Finding area Find the area of the region enclosed by the curve $y = x\sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$ **b.** $\pi \le x \le 2\pi$ **c.** $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



32. Finding area Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis (see the accompanying figure) for

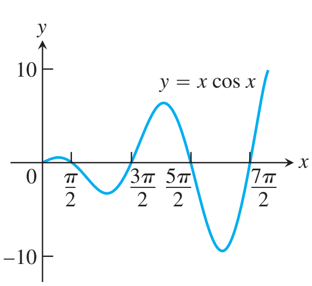
a. $\pi/2 \le x \le 3\pi/2$ **b.** $3\pi/2 \le x \le 5\pi/2$

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



33. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis. **b.** about the line $x = 1$.

35. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis. **b.** the line $x = \pi/2$.

36. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x\sin x$, $0 \le x \le \pi$, about

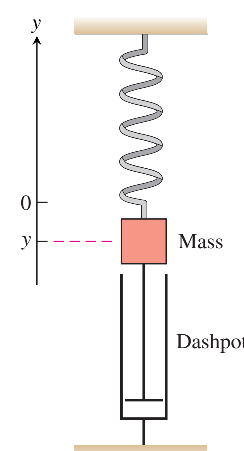
a. the y-axis. **b.** the line $x = \pi$.

(See Exercise 31 for a graph.)

37. Average value A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



38. Average value In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 39–42, use integration by parts to establish the *reduction formula*.

39. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

40. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$

41. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

42. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\int f^{-1}(x)\,dx = \int y f'(y)\,dy \qquad y = f^{-1}(x),\quad x = f(y),\quad dx = f'(y)\,dy$$

$$= yf(y) - \int f(y)\,dy \qquad \text{Integration by parts with } u = y,\ dv = f'(y)\,dy$$

$$= xf^{-1}(x) - \int f(y)\,dy$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\,dx = \int ye^y\,dy \qquad y = \ln x,\quad x = e^y,\quad dx = e^y\,dy$$

$$= ye^y - e^y + C$$

$$= x\ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \int \cos y\,dy \qquad y = \cos^{-1} x$$

$$= x\cos^{-1} x - \sin y + C$$

$$= x\cos^{-1} x - \sin(\cos^{-1} x) + C.$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \qquad (4)$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x.

43. $\int \sin^{-1} x\,dx$

44. $\int \tan^{-1} x\,dx$

45. $\int \sec^{-1} x\,dx$

46. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \qquad (5)$$

Exercises 47 and 48 compare the results of using Equations (4) and (5).

47. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

b. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

48. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (4)

b. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1 + x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with **(a)** Eq. (4) and **(b)** Eq. (5). In each case, check your work by differentiating your answer with respect to x.

49. $\int \sinh^{-1} x\,dx$

50. $\int \tanh^{-1} x\,dx$