

8.5

Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. They come from the reference right triangles in Figure 8.2.

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

With $x = a \sec \theta$,

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

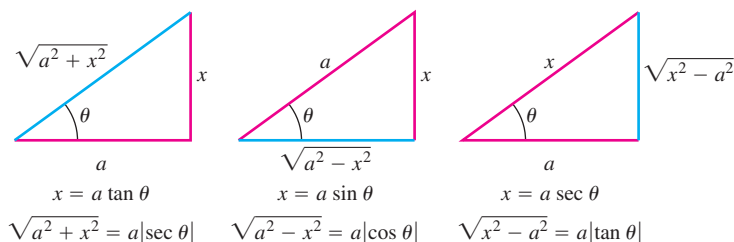


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a \tan \theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a \sin \theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a \sec \theta$.

As we know from Section 7.7, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a \tan \theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$x = a \sec \theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \geq 1, \\ \frac{\pi}{2} < \theta \leq \pi & \text{if } \frac{x}{a} \leq -1. \end{cases}$$

To simplify calculations with the substitution $x = a \sec \theta$, we will restrict its use to integrals in which $x/a \geq 1$. This will place θ in $[0, \pi/2)$ and make $\tan \theta \geq 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta| = a \tan \theta$, free of absolute values, provided $a > 0$.

EXAMPLE 1 Using the Substitution $x = a \tan \theta$

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

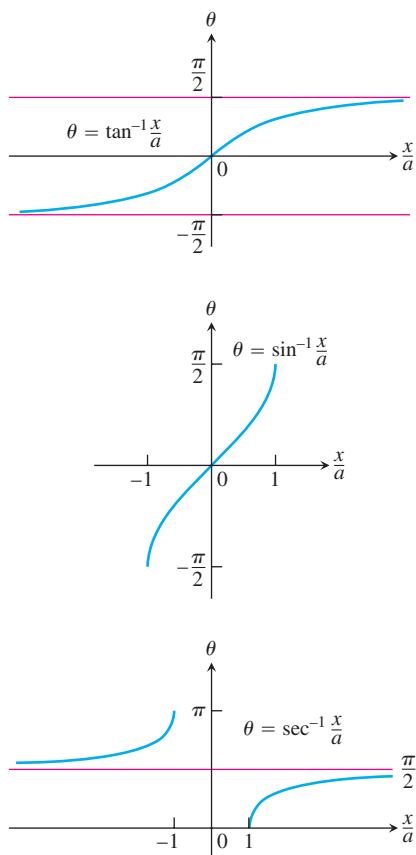


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a , graphed as functions of x/a .

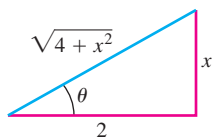


FIGURE 8.4 Reference triangle for $x = 2 \tan \theta$ (Example 1):

$$\tan \theta = \frac{x}{2}$$

and

$$\sec \theta = \frac{\sqrt{4 + x^2}}{2}.$$

Solution We set

$$x = 2 \tan \theta, \quad dx = 2 \sec^2 \theta \, d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{4 + x^2}} &= \int \frac{2 \sec^2 \theta \, d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta \, d\theta}{|\sec \theta|} && \sqrt{\sec^2 \theta} = |\sec \theta| \\ &= \int \sec \theta \, d\theta && \sec \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C && \text{From Fig. 8.4} \\ &= \ln |\sqrt{4 + x^2} + x| + C'. && \text{Taking } C' = C - \ln 2 \end{aligned}$$

Notice how we expressed $\ln |\sec \theta + \tan \theta|$ in terms of x : We drew a reference triangle for the original substitution $x = 2 \tan \theta$ (Figure 8.4) and read the ratios from the triangle. ■

EXAMPLE 2 Using the Substitution $x = a \sin \theta$

Evaluate

$$\int \frac{x^2 \, dx}{\sqrt{9 - x^2}}.$$

Solution We set

$$x = 3 \sin \theta, \quad dx = 3 \cos \theta \, d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta.$$

Then

$$\begin{aligned} \int \frac{x^2 \, dx}{\sqrt{9 - x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta \, d\theta}{|3 \cos \theta|} \\ &= 9 \int \sin^2 \theta \, d\theta && \cos \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= 9 \int \frac{1 - \cos 2\theta}{2} \, d\theta \\ &= \frac{9}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C \\ &= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C && \sin 2\theta = 2 \sin \theta \cos \theta \\ &= \frac{9}{2} \left(\sin^{-1} \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \right) + C && \text{Fig. 8.5} \\ &= \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2} \sqrt{9 - x^2} + C. \end{aligned}$$

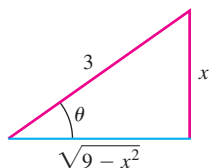


FIGURE 8.5 Reference triangle for $x = 3 \sin \theta$ (Example 2):

$$\sin \theta = \frac{x}{3}$$

and

$$\cos \theta = \frac{\sqrt{9 - x^2}}{3}.$$

EXAMPLE 3 Using the Substitution $x = a \sec \theta$

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\begin{aligned}\sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\ &= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}\end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5} \sec \theta, \quad dx = \frac{2}{5} \sec \theta \tan \theta \, d\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\begin{aligned}x^2 - \left(\frac{2}{5}\right)^2 &= \frac{4}{25} \sec^2 \theta - \frac{4}{25} \\ &= \frac{4}{25} (\sec^2 \theta - 1) = \frac{4}{25} \tan^2 \theta\end{aligned}$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5} |\tan \theta| = \frac{2}{5} \tan \theta. \quad \begin{array}{l} \tan \theta > 0 \text{ for} \\ 0 < \theta < \pi/2 \end{array}$$

With these substitutions, we have

$$\begin{aligned}\int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5) \sec \theta \tan \theta \, d\theta}{5 \cdot (2/5) \tan \theta} \\ &= \frac{1}{5} \int \sec \theta \, d\theta = \frac{1}{5} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{5} \ln \left| \frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2} \right| + C.\end{aligned}$$

Fig. 8.6

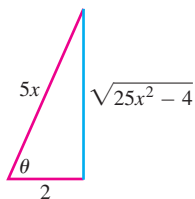


FIGURE 8.6 If $x = (2/5)\sec \theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 3).

A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.

EXAMPLE 4 Finding the Volume of a Solid of Revolution

Find the volume of the solid generated by revolving about the x -axis the region bounded by the curve $y = 4/(x^2 + 4)$, the x -axis, and the lines $x = 0$ and $x = 2$.

Solution We sketch the region (Figure 8.7) and use the disk method:

$$V = \int_0^2 \pi [R(x)]^2 \, dx = 16\pi \int_0^2 \frac{dx}{(x^2 + 4)^2}. \quad R(x) = \frac{4}{x^2 + 4}$$

To evaluate the integral, we set

$$\begin{aligned}x &= 2 \tan \theta, & dx &= 2 \sec^2 \theta \, d\theta, & \theta &= \tan^{-1} \frac{x}{2}, \\ x^2 + 4 &= 4 \tan^2 \theta + 4 = 4(\tan^2 \theta + 1) = 4 \sec^2 \theta\end{aligned}$$

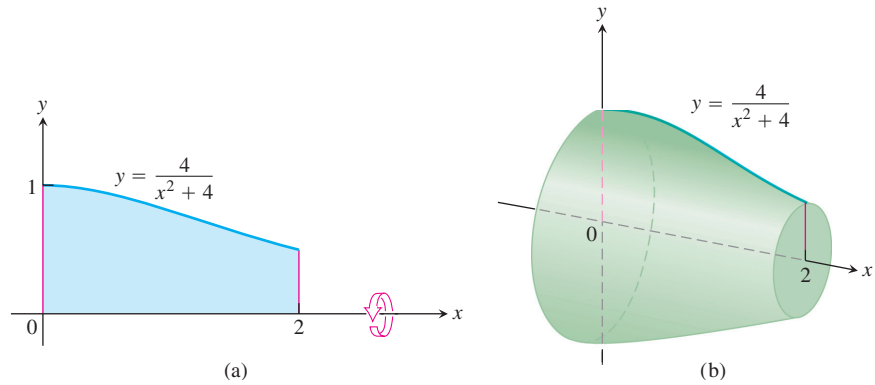


FIGURE 8.7 The region (a) and solid (b) in Example 4.

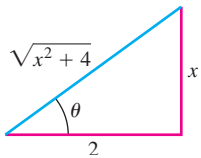


FIGURE 8.8 Reference triangle for $x = 2 \tan \theta$ (Example 4).

(Figure 8.8). With these substitutions,

$$\begin{aligned}
 V &= 16\pi \int_0^2 \frac{dx}{(x^2 + 4)^2} \\
 &= 16\pi \int_0^{\pi/4} \frac{2 \sec^2 \theta d\theta}{(4 \sec^2 \theta)^2} && \theta = 0 \text{ when } x = 0; \\
 &= 16\pi \int_0^{\pi/4} \frac{2 \sec^2 \theta d\theta}{16 \sec^4 \theta} = \pi \int_0^{\pi/4} 2 \cos^2 \theta d\theta && \theta = \pi/4 \text{ when } x = 2 \\
 &= \pi \int_0^{\pi/4} (1 + \cos 2\theta) d\theta = \pi \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4} && 2 \cos^2 \theta = 1 + \cos 2\theta \\
 &= \pi \left[\frac{\pi}{4} + \frac{1}{2} \right] \approx 4.04.
 \end{aligned}$$

EXAMPLE 5 Finding the Area of an Ellipse

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (Figure 8.9). Solving the equation of the ellipse for $y \geq 0$, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2},$$

or

$$y = \frac{b}{a} \sqrt{a^2 - x^2} \quad 0 \leq x \leq a$$

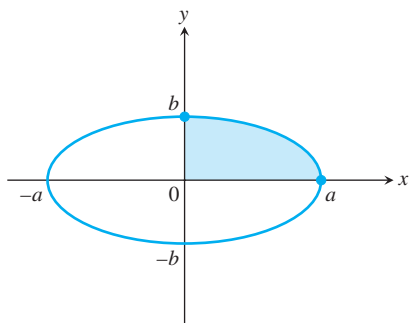


FIGURE 8.9 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in Example 5.

The area of the ellipse is

$$\begin{aligned} A &= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx \\ &= 4 \frac{b}{a} \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta d\theta && \begin{array}{l} x = a \sin \theta, dx = a \cos \theta d\theta, \\ \theta = 0 \text{ when } x = 0; \\ \theta = \pi/2 \text{ when } x = a \end{array} \\ &= 4ab \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 4ab \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta \\ &= 2ab \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\ &= 2ab \left[\frac{\pi}{2} + 0 - 0 \right] = \pi ab. \end{aligned}$$

If $a = b = r$ we get that the area of a circle with radius r is πr^2 . ■