EXERCISES 8.5

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\int \frac{dy}{\sqrt{9 + y^2}}$

2. $\int \frac{3\,dy}{\sqrt{1 + 9y^2}}$

3. $\int_{-2}^{2} \frac{dx}{4 + x^2}$

4. $\int_{0}^{2} \frac{dx}{8 + 2x^2}$

5. $\int_{0}^{3/2} \frac{dx}{\sqrt{9 - x^2}}$

6. $\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1 - 4x^2}}$

7. $\int \sqrt{25 - t^2}\,dt$

8. $\int \sqrt{1 - 9t^2}\,dt$

9. $\int \frac{dx}{\sqrt{4x^2 - 49}}, \quad x > \frac{7}{2}$

10. $\int \frac{5\,dx}{\sqrt{25x^2 - 9}}, \quad x > \frac{3}{5}$

11. $\int \frac{\sqrt{y^2 - 49}}{y}\,dy, \quad y > 7$

12. $\int \frac{\sqrt{y^2 - 25}}{y^3}\,dy, \quad y > 5$

13. $\int \frac{dx}{x^2\sqrt{x^2 - 1}}, \quad x > 1$

14. $\int \frac{2\,dx}{x^3\sqrt{x^2 - 1}}, \quad x > 1$

15. $\int \frac{x^3\,dx}{\sqrt{x^2 + 4}}$

16. $\int \frac{dx}{x^2\sqrt{x^2 + 1}}$

17. $\int \frac{8\,dw}{w^2\sqrt{4 - w^2}}$

18. $\int \frac{\sqrt{9 - w^2}}{w^2}\,dw$

19. $\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1 - x^2)^{3/2}}$

20. $\int_{0}^{1} \frac{dx}{(4 - x^2)^{3/2}}$

21. $\int \frac{dx}{(x^2 - 1)^{3/2}}, \quad x > 1$

22. $\int \frac{x^2\,dx}{(x^2 - 1)^{5/2}}, \quad x > 1$

23. $\int \frac{(1 - x^2)^{3/2}}{x^6}\,dx$

24. $\int \frac{(1 - x^2)^{1/2}}{x^4}\,dx$

25. $\int \frac{8\,dx}{(4x^2 + 1)^2}$

26. $\int \frac{6\,dt}{(9t^2 + 1)^2}$

27. $\int \frac{v^2\,dv}{(1 - v^2)^{5/2}}$

28. $\int \frac{(1 - r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t} + 9}}$

30. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1 + e^{2t})^{3/2}}$

31. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t} + 4t\sqrt{t}}$

32. $\int_{1}^{e} \frac{dy}{y\sqrt{1 + (\ln y)^2}}$

33. $\int \frac{dx}{x\sqrt{x^2 - 1}}$

34. $\int \frac{dx}{1 + x^2}$

35. $\int \frac{x\,dx}{\sqrt{x^2 - 1}}$

36. $\int \frac{dx}{\sqrt{1 - x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $x\frac{dy}{dx} = \sqrt{x^2 - 4}, \quad x \ge 2, \quad y(2) = 0$

38. $\sqrt{x^2 - 9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $(x^2 + 4)\frac{dy}{dx} = 3, \quad y(2) = 0$

40. $(x^2 + 1)^2\frac{dy}{dx} = \sqrt{x^2 + 1}, \quad y(0) = 1$

Applications

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2/3}$.

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1 + x^2)$, and the line $x = 1$.

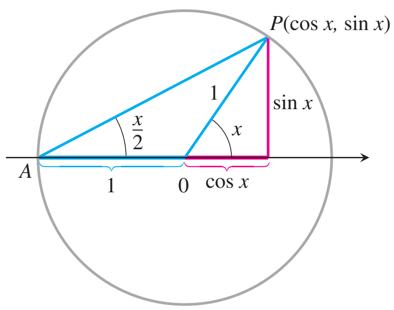
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{1}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1$$

$$\cos x = \frac{1 - z^2}{1 + z^2}, \tag{2}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)} \cdot \cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2} \cdot \frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2z}{1 + z^2}. \tag{3}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{4}$$

Examples

a.
$$\int \frac{1}{1 + \cos x}\,dx = \int \frac{1 + z^2}{2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int dz = z + C$$

$$= \tan\left(\frac{x}{2}\right) + C$$

b.
$$\int \frac{1}{2 + \sin x}\,dx = \int \frac{1 + z^2}{2 + 2z + 2z^2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4}$$

$$= \int \frac{du}{u^2 + a^2}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z + 1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1 + 2\tan(x/2)}{\sqrt{3}} + C$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 43–50. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

43. $\displaystyle\int \frac{dx}{1 - \sin x}$

44. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

46. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$

47. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}$

48. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta + \sin\theta}$

49. $\displaystyle\int \frac{dt}{\sin t - \cos t}$

50. $\displaystyle\int \frac{\cos t\,dt}{1 - \cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \sec\theta\,d\theta$

52. $\displaystyle\int \csc\theta\,d\theta$