# 8.6 Integral Tables and Computer Algebra Systems

As we have studied, the basic techniques of integration are substitution and integration by parts. We apply these techniques to transform unfamiliar integrals into integrals whose forms we recognize or can find in a table. But where do the integrals in the tables come from? They come from applying substitutions and integration by parts, or by differentiating important functions that arise in practice or applications and tabling the results (as we did in creating Table 8.1). When an integral matches an integral in the table or can be changed into one of the tabulated integrals with some appropriate combination of algebra, trigonometry, substitution, and calculus, the matched result can be used to solve the integration problem at hand.

Computer Algebra Systems (CAS) can also be used to evaluate an integral, if such a system is available. However, beware that there are many relatively simple functions, like $\sin(x^2)$ or $1/\ln x$ for which even the most powerful computer algebra systems cannot find explicit antiderivative formulas because no such formulas exist.

In this section we discuss how to use tables and computer algebra systems to evaluate integrals.

## Integral Tables

A Brief Table of Integrals is provided at the back of the book, after the index. (More extensive tables appear in compilations such as *CRC Mathematical Tables*, which contain thousands of integrals.) The integration formulas are stated in terms of constants $a$, $b$, $c$, $m$, $n$, and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 5 requires $n \neq -1$, for example, and Formula 11 requires $n \neq 2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 8 assumes that $a \neq 0$, and Formulas 13(a) and (b) cannot be used unless $b$ is positive.

The integrals in Examples 1–5 of this section can be evaluated using algebraic manipulation, substitution, or integration by parts. Here we illustrate how the integrals are found using the Brief Table of Integrals.

**EXAMPLE 1** Find

$$\int x(2x+5)^{-1}\,dx.$$

**Solution** We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C.$$ ■

**EXAMPLE 2** Find

$$\int \frac{dx}{x\sqrt{2x+4}}.$$

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**Solution** We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}} \right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\begin{aligned} \int \frac{dx}{x\sqrt{2x+4}} &= \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{\sqrt{2x+4}-2}{\sqrt{2x+4}+2} \right| + C. \end{aligned}$$

Formula 13(a), which would require $b < 0$ here, is not appropriate in Example 2. It *is* appropriate, however, in the next example.

**EXAMPLE 3** Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

**Solution** We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x-4}{4}} + C = \tan^{-1} \sqrt{\frac{x-2}{2}} + C.$$

You Try It

**EXAMPLE 4** Find

$$\int \frac{dx}{x^2\sqrt{2x-4}}.$$

**Solution** We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2 \cdot 4} \int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4} \tan^{-1} \sqrt{\frac{x-2}{2}} + C.$$

You Try It

**EXAMPLE 5** Find

$$\int x \sin^{-1} x \, dx.$$

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**Solution** We use Formula 99:

$$\int x^n \sin^{-1} ax\, dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\, dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x\, dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2\, dx}{\sqrt{1-x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2\, dx}{\sqrt{1-x^2}} = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned}\int x \sin^{-1} x\, dx &= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C\right)\\ &= \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C'.\end{aligned}$$

■

## Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying formulas like

$$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx \tag{1}$$

$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx \tag{2}$$

$$\int \sin^n x \cos^m x\, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\, dx \qquad (n \neq -m). \tag{3}$$

Formulas like these are called **reduction formulas** because they replace an integral containing some power of a function with an integral of the same form with the power reduced. By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly.

You Try It

**EXAMPLE 6** Using a Reduction Formula

Find

$$\int \tan^5 x\, dx.$$

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**Solution** We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x\, dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\, dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\, dx = \frac{1}{2}\tan^2 x - \int \tan x\, dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\, dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$ ■

As their form suggests, reduction formulas are derived by integration by parts.

### EXAMPLE 7 Deriving a Reduction Formula

Show that for any positive integer $n$,

$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx.$$

**Solution** We use the integration by parts formula

$$\int u\, dv = uv - \int v\, du$$

with

$$u = (\ln x)^n, \qquad du = n(\ln x)^{n-1}\frac{dx}{x}, \qquad dv = dx, \qquad v = x,$$

to obtain

$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx.$$ ■

Sometimes two reduction formulas come into play.

### EXAMPLE 8 Find

$$\int \sin^2 x \cos^3 x\, dx.$$

**Solution 1** We apply Equation (3) with $n = 2$ and $m = 3$ to get

$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= -\frac{\sin x \cos^4 x}{2+3} + \frac{1}{2+3}\int \sin^0 x \cos^3 x\, dx\\ &= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\int \cos^3 x\, dx.\end{aligned}$$

We can evaluate the remaining integral with Formula 61 (another reduction formula):

$$\int \cos^n ax\, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\, dx.$$

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With $n = 3$ and $a = 1$, we have

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C.\end{aligned}$$

The combined result is

$$\begin{aligned}\int \sin^2 x \cos^3 x\,dx &= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C\right) \\ &= -\frac{\sin x \cos^4 x}{5} + \frac{\cos^2 x \sin x}{15} + \frac{2}{15}\sin x + C'.\end{aligned}$$

**Solution 2** Equation (3) corresponds to Formula 68 in the table, but there is another formula we might use, namely Formula 69. With $a = 1$, Formula 69 gives

$$\int \sin^n x \cos^m x\,dx = \frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^n x \cos^{m-2} x\,dx.$$

In our case, $n = 2$ and $m = 3$, so that

$$\begin{aligned}\int \sin^2 x \cos^3 x\,dx &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\int \sin^2 x \cos x\,dx \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\left(\frac{\sin^3 x}{3}\right) + C \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{15}\sin^3 x + C.\end{aligned}$$

As you can see, it is faster to use Formula 69, but we often cannot tell beforehand how things will work out. Do not spend a lot of time looking for the "best" formula. Just find one that will work and forge ahead.

Notice also that Formulas 68 (Solution 1) and 69 (Solution 2) lead to different-looking answers. That is often the case with trigonometric integrals and is no cause for concern. The results are equivalent, and we may use whichever one we please. ■

## Nonelementary Integrals

The development of computers and calculators that find antiderivatives by symbolic manipulation has led to a renewed interest in determining which antiderivatives can be expressed as finite combinations of elementary functions (the functions we have been studying) and which cannot. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. They require infinite series (Chapter 11) or numerical methods for their evaluation. Examples of the latter include the error function (which measures the probability of random errors)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

and integrals such as

$$\int \sin x^2\,dx \qquad \text{and} \qquad \int \sqrt{1 + x^4}\,dx$$

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that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$
$$\int \sqrt{1 - k^2 \sin^2 x}\,dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The integrands all have antiderivatives, as a consequence of the Fundamental Theorem of the Calculus, Part 1, because they are continuous. However, none of the antiderivatives is elementary.

None of the integrals you are asked to evaluate in the present chapter falls into this category, but you may encounter nonelementary integrals in your other work.

## Integration with a CAS

A powerful capability of computer algebra systems is their ability to integrate symbolically. This is performed with the **integrate command** specified by the particular system (for example, **int** in Maple, **Integrate** in Mathematica).

### EXAMPLE 9 Using a CAS with a Named Function

Suppose that you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2 + x^2}.$$

Using Maple, you first define or name the function:

$$> f := x\hat{\ }2 * \text{sqrt}(a\hat{\ }2 + x\hat{\ }2);$$

Then you use the integrate command on $f$, identifying the variable of integration:

$$> \text{int}(f, x);$$

Maple returns the answer

$$\frac{1}{4}x(a^2 + x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2 + x^2} - \frac{1}{8}a^4\ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want to see if the answer can be simplified, enter

$$> \text{simplify}(\%);$$

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2 + x^2} + \frac{1}{4}x^3\sqrt{a^2 + x^2} - \frac{1}{8}a^4\ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

$$> \text{int}(f, x = 0..\text{Pi}/2);$$

Maple (Version 5.1) will return the expression

$$\frac{1}{64}\pi(4a^2 + \pi^2)^{(3/2)} - \frac{1}{32}a^2\pi\sqrt{4a^2 + \pi^2} + \frac{1}{8}a^4\ln(2)$$
$$- \frac{1}{8}a^4\ln\left(\pi + \sqrt{4a^2 + \pi^2}\right) + \frac{1}{16}a^4\ln(a^2).$$

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You can also find the definite integral for a particular value of the constant $a$:

$$> a := 1;$$

$$> \operatorname{int}(f, x = 0..1);$$

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} + \frac{1}{8}\ln\left(\sqrt{2} - 1\right).$$ ■

## EXAMPLE 10 Using a CAS Without Naming the Function

Use a CAS to find

$$\int \sin^2 x \cos^3 x \, dx.$$

**Solution** With Maple, we have the entry

$$> \operatorname{int}((\sin\text{^}2)(x) * (\cos\text{^}3)(x), x);$$

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x).$$ ■

## EXAMPLE 11 A CAS May Not Return a Closed Form Solution

Use a CAS to find

$$\int (\cos^{-1} ax)^2 \, dx.$$

**Solution** Using Maple, we enter

$$> \operatorname{int}((\arccos(a * x))\text{^}2, x);$$

and Maple returns the expression

$$\int \arccos(ax)^2 \, dx,$$

indicating that it does not have a closed form solution known by Maple. In Chapter 11, you will see how series expansion may help to evaluate such an integral. ■

Computer algebra systems vary in how they process integrations. We used Maple 5.1 in Examples 9–11. Mathematica 4 would have returned somewhat different results:

**1.** In Example 9, given

$$\textit{In [1]:=} \text{ Integrate } [x\text{^}2 * \text{Sqrt } [a\text{^}2 + x\text{^}2], x]$$

Mathematica returns

$$\textit{Out [1]=} \sqrt{a^2 + x^2}\left(\frac{a^2 x}{8} + \frac{x^3}{4}\right) - \frac{1}{8}a^4 \operatorname{Log}\left[x + \sqrt{a^2 + x^2}\right]$$

without having to simplify an intermediate result. The answer is close to Formula 22 in the integral tables.

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**2.** The Mathematica answer to the integral

$$\textit{In [2]:=}\ \text{Integrate [Sin [}x\text{]\^{}2 * Cos [}x\text{]\^{}3, }x\text{]}$$

in Example 10 is

$$\textit{Out [2]=}\ \frac{\text{Sin}\,[x]}{8} - \frac{1}{48}\,\text{Sin}\,[3\,x] - \frac{1}{80}\,\text{Sin}\,[5\,x]$$

differing from both the Maple answer and the answers in Example 8.

**3.** Mathematica does give a result for the integration

$$\textit{In [3]:=}\ \text{Integrate [ArcCos [}a * x\text{]\^{}2, }x\text{]}$$

in Example 11, provided $a \neq 0$:

$$\textit{Out [3]=}\ -2x - \frac{2\sqrt{1 - a^2 x^2}\ \text{ArcCos}\,[a\,x]}{a} + x\ \text{ArcCos}\,[a\,x]^2$$

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). Note, too, that neither Maple nor Mathematica return an arbitrary constant $+C$. On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 111.

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