8.7 Numerical Integration

As we have seen, the ideal way to evaluate a definite integral $\int_a^b f(x)\,dx$ is to find a formula $F(x)$ for one of the antiderivatives of $f(x)$ and calculate the number $F(b) - F(a)$. But some antiderivatives require considerable work to find, and still others, like the antiderivatives of $\sin(x^2)$, $1/\ln x$, and $\sqrt{1 + x^4}$, have no elementary formulas.

Another situation arises when a function is defined by a table whose entries were obtained experimentally through instrument readings. In this case a formula for the function may not even exist.

Whatever the reason, when we cannot evaluate a definite integral with an antiderivative, we turn to numerical methods such as the *Trapezoidal Rule* and *Simpson's Rule* developed in this section. These rules usually require far fewer subdivisions of the integration interval to get accurate results compared to the various rectangle rules presented in Sections 5.1 and 5.2. We also estimate the error obtained when using these approximation methods.

Trapezoidal Approximations

When we cannot find a workable antiderivative for a function f that we have to integrate, we partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f. In our presentation we assume that f is positive, but the only requirement is for f to be continuous over the interval of integration $[a, b]$.

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles, as in Figure 8.10. It is not necessary for the subdivision points $x_0, x_1, x_2, \ldots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b - a}{n}.$$

The length $\Delta x = (b - a)/n$ is called the **step size** or **mesh size**. The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{y_{i-1} + y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1} + y_i),$$

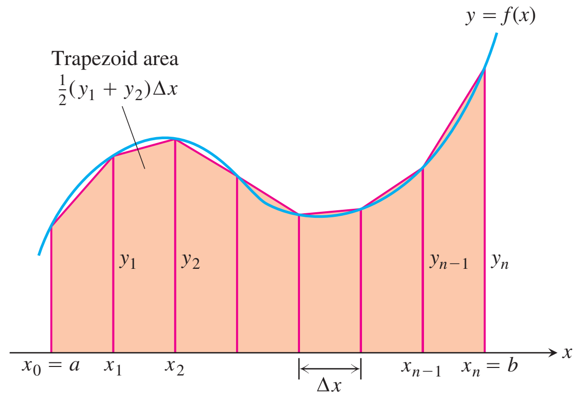

FIGURE 8.10 The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the areas of the trapezoids made by joining the ends of the segments to the x-axis.

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. This area is the length Δx of the trapezoid's horizontal "altitude" times the average of its two vertical "bases." (See Figure 8.10.) The area below the curve $y = f(x)$ and above the x-axis is then approximated by adding the areas of all the trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\ &\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \ldots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

The Trapezoidal Rule says: Use T to estimate the integral of f from a to b.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

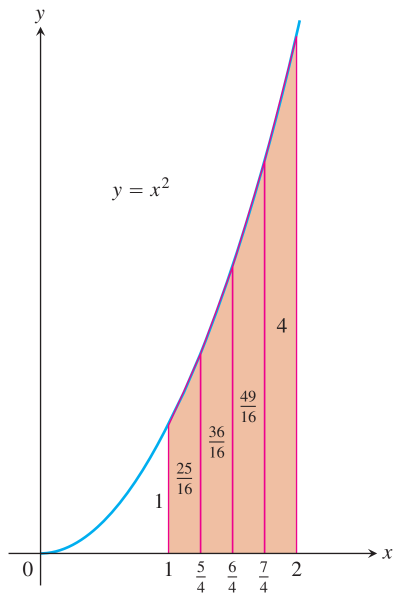

FIGURE 8.11 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

TABLE 8.3

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

EXAMPLE 1 Applying the Trapezoidal Rule

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Solution Partition [1, 2] into four subintervals of equal length (Figure 8.11). Then evaluate $y = x^2$ at each partition point (Table 8.3).

Using these y values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

We could have predicted that the Trapezoidal Rule would overestimate the integral in Example 1 by considering the geometry of the graph in Figure 8.11. Since the parabola is concave *up*, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. In Figure 8.10, we see that the straight segments lie *under* the curve on those intervals where the curve is concave *down*, causing the Trapezoidal Rule to *underestimate* the integral on those intervals.

EXAMPLE 2 Averaging Temperatures

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table.

Time	N	1	2	3	4	5	6	7	8	9	10	11	M
Temp	63	65	66	68	70	69	68	68	65	64	62	58	55

What was the average temperature for the 12-hour period?

Solution We are looking for the average value of a continuous function (temperature) for which we know values at discrete times that are one unit apart. We need to find

$$\text{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx,$$

without having a formula for $f(x)$. The integral, however, can be approximated by the Trapezoidal Rule, using the temperatures in the table as function values at the points of a 12-subinterval partition of the 12-hour interval (making $\Delta x = 1$).

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{11} + y_{12}\right) \\ &= \frac{1}{2}\left(63 + 2\cdot 65 + 2\cdot 66 + \cdots + 2\cdot 58 + 55\right) \\ &= 782 \end{aligned}$$

Using T to approximate $\int_a^b f(x)\,dx$, we have

$$\mathrm{av}(f) \approx \frac{1}{b-a} \cdot T = \frac{1}{12} \cdot 782 \approx 65.17.$$

Rounding to the accuracy of the given data, we estimate the average temperature as 65 degrees. ■

Error Estimates for the Trapezoidal Rule

As n increases and the step size $\Delta x = (b-a)/n$ approaches zero, T approaches the exact value of $\int_a^b f(x)\,dx$. To see why, write

$$\begin{aligned} T &= \Delta x \left(\frac{1}{2} y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2} y_n \right) \\ &= (y_1 + y_2 + \cdots + y_n)\Delta x + \frac{1}{2}(y_0 - y_n)\,\Delta x \\ &= \sum_{k=1}^{n} f(x_k)\Delta x + \frac{1}{2}[f(a) - f(b)]\,\Delta x. \end{aligned}$$

As $n \to \infty$ and $\Delta x \to 0$,

$$\sum_{k=1}^{n} f(x_k)\Delta x \to \int_a^b f(x)\,dx \qquad \text{and} \qquad \frac{1}{2}[f(a) - f(b)]\Delta x \to 0.$$

Therefore,

$$\lim_{n\to\infty} T = \int_a^b f(x)\,dx + 0 = \int_a^b f(x)\,dx.$$

This means that in theory we can make the difference between T and the integral as small as we want by taking n large enough, assuming only that f is integrable. In practice, though, how do we tell how large n should be for a given tolerance?

We answer this question with a result from advanced calculus, which says that if f'' is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = T - \frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

for some number c between a and b. Thus, as Δx approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches zero as the *square* of Δx.

The inequality

$$|E_T| \le \frac{b-a}{12} \max |f''(x)| (\Delta x)^2,$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. In practice, we usually cannot find the exact value of $\max |f''(x)|$ and have to estimate an upper bound or "worst case" value for it instead. If M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \le M$ on $[a, b]$, then

$$|E_T| \le \frac{b-a}{12} M(\Delta x)^2.$$

If we substitute $(b - a)/n$ for Δx, we get

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

This is the inequality we normally use in estimating $|E_T|$. We find the best M we can and go on to estimate $|E_T|$ from there. This may sound careless, but it works. To make $|E_T|$ small for a given M, we just make n large.

The Error Estimate for the Trapezoidal Rule

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

EXAMPLE 3 Bounding the Trapezoidal Rule Error

Find an upper bound for the error incurred in estimating

$$\int_0^{\pi} x \sin x \, dx$$

with the Trapezoidal Rule with $n = 10$ steps (Figure 8.12).

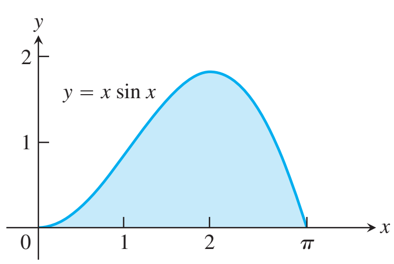

FIGURE 8.12 Graph of the integrand in Example 3.

Solution With $a = 0$, $b = \pi$, and $n = 10$, the error estimate gives

$$|E_T| \leq \frac{M(b-a)^3}{12n^2} = \frac{\pi^3}{1200} M.$$

The number M can be any upper bound for the magnitude of the second derivative of $f(x) = x \sin x$ on $[0, \pi]$. A routine calculation gives

$$f''(x) = 2\cos x - x \sin x,$$

so

$$\begin{aligned} |f''(x)| &= |2\cos x - x\sin x| \\ &\leq 2|\cos x| + |x||\sin x| \\ &\leq 2 \cdot 1 + \pi \cdot 1 = 2 + \pi. \end{aligned}$$

$|\cos x|$ and $|\sin x|$ never exceed 1, and $0 \leq x \leq \pi$.

We can safely take $M = 2 + \pi$. Therefore,

$$|E_T| \leq \frac{\pi^3}{1200} M = \frac{\pi^3(2+\pi)}{1200} < 0.133.$$

Rounded up to be safe

The absolute error is no greater than 0.133.

For greater accuracy, we would not try to improve M but would take more steps. With $n = 100$ steps, for example, we get

$$|E_T| \leq \frac{(2+\pi)\pi^3}{120{,}000} < 0.00133 = 1.33 \times 10^{-3}.$$

■

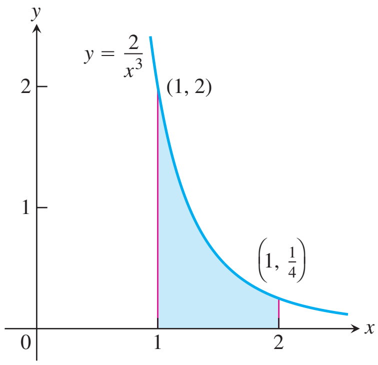

FIGURE 8.13 The continuous function $y = 2/x^3$ has its maximum value on [1, 2] at $x = 1$.

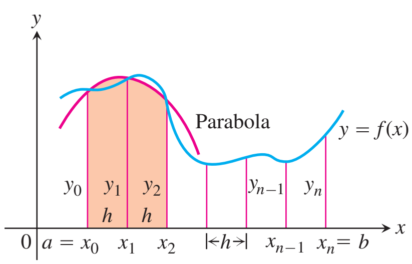

FIGURE 8.14 Simpson's Rule approximates short stretches of the curve with parabolas.

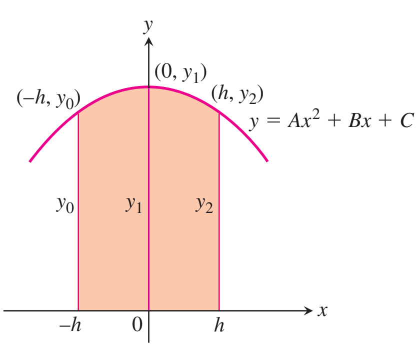

FIGURE 8.15 By integrating from $-h$ to h, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

EXAMPLE 4 Finding How Many Steps Are Needed for a Specific Accuracy

How many subdivisions should be used in the Trapezoidal Rule to approximate

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

with an error whose absolute value is less than 10^{-4}?

Solution With $a = 1$ and $b = 2$, the error estimate is

$$|E_T| \le \frac{M(2-1)^3}{12n^2} = \frac{M}{12n^2}.$$

This is one of the rare cases in which we actually can find $\max|f''|$ rather than having to settle for an upper bound M. With $f(x) = 1/x$, we find

$$f''(x) = \frac{d^2}{dx^2}(x^{-1}) = 2x^{-3} = \frac{2}{x^3}.$$

On [1, 2], $y = 2/x^3$ decreases steadily from a maximum of $y = 2$ to a minimum of $y = 1/4$ (Figure 8.13). Therefore, $M = 2$ and

$$|E_T| \le \frac{2}{12n^2} = \frac{1}{6n^2}.$$

The error's absolute value will therefore be less than 10^{-4} if

$$\frac{1}{6n^2} < 10^{-4}, \qquad \frac{10^4}{6} < n^2, \qquad \frac{100}{\sqrt{6}} < n, \qquad \text{or} \qquad 40.83 < n.$$

The first integer beyond 40.83 is $n = 41$. With $n = 41$ subdivisions we can guarantee calculating ln 2 with an error of magnitude less than 10^{-4}. Any larger n will work, too. ■

Simpson's Rule: Approximations Using Parabolas

Riemann sums and the Trapezoidal Rule both give reasonable approximations to the integral of a continuous function over a closed interval. The Trapezoidal Rule is more efficient, giving a better approximation for small values of n, which makes it a faster algorithm for numerical integration.

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight line segments which produced trapezoids. As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \ge 0$ by a parabola, as shown in Figure 8.14. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}) on the curve.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ (Figure 8.15), where $h = \Delta x = (b - a)/n$. The area under the parabola will be the same if we shift the y-axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

so the area under it from $x = -h$ to $x = h$ is

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\, dx$$

$$= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\Big]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2), we also have

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$C = y_1,$$

$$Ah^2 - Bh = y_0 - y_1,$$

$$Ah^2 + Bh = y_2 - y_1,$$

$$2Ah^2 = y_0 + y_2 - 2y_1.$$

Hence, expressing the area A_p in terms of the ordinates y_0, y_1, and y_2, we have

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.14 does not change the area under it. Thus the area under the parabola through (x_0, y_0), (x_1, y_1), and (x_2, y_2) in Figure 8.14 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points (x_2, y_2), (x_3, y_3), and (x_4, y_4) is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\int_a^b f(x)\, dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots$$

$$+ \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

Historical Biography

Thomas Simpson
(1720–1761)

The result is known as Simpson's Rule, and it is again valid for any continuous function $y = f(x)$ (Exercise 38). The function need not be positive, as in our derivation. The number n of subintervals must be even to apply the rule because each parabolic arc uses two subintervals.

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

Note the pattern of the coefficients in the above rule: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 1.

TABLE 8.4

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

EXAMPLE 5 Applying Simpson's Rule

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution Partition [0, 2] into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.4). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\ &= 32\frac{1}{12}. \end{aligned}$$

This estimate differs from the exact value (32) by only 1/12, a percentage error of less than three-tenths of one percent, and this was with just four subintervals. ■

Error Estimates for Simpson's Rule

To estimate the error in Simpson's rule, we start with a result from advanced calculus that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x)\,dx = S - \frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4$$

for some point c between a and b. Thus, as Δx approaches zero, the error,

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4,$$

approaches zero as the *fourth power* of Δx (This helps to explain why Simpson's Rule is likely to give better results than the Trapezoidal Rule.)

The inequality

$$|E_S| \le \frac{b-a}{180} \max|f^{(4)}(x)|\,(\Delta x)^4$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. As with $\max|f''|$ in the error formula for the Trapezoidal Rule, we usually cannot find the exact value of $\max|f^{(4)}(x)|$ and have to replace it with an upper bound. If M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| \leq \frac{b-a}{180} M(\Delta x)^4.$$

Substituting $(b-a)/n$ for Δx in this last expression gives

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

This is the formula we usually use in estimating the error in Simpson's Rule. We find a reasonable value for M and go on to estimate $|E_S|$ from there.

The Error Estimate for Simpson's Rule

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

As with the Trapezoidal Rule, we often cannot find the smallest possible value of M. We just find the best value we can and go on from there.

EXAMPLE 6 Bounding the Error in Simpson's Rule

Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 5).

Solution To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \leq x \leq 2$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$, the error estimate for Simpson's Rule gives

$$|E_S| \leq \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$

■

EXAMPLE 7 Comparing the Trapezoidal Rule and Simpson's Rule Approximations

As we saw in Chapter 7, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

Table 8.5 shows T and S values for approximations of $\int_1^2 (1/x)\,dx$ using various values of n. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule. In particular, notice that when we double the value of n (thereby halving the value of $h = \Delta x$), the T error is divided by 2 *squared*, whereas the S error is divided by 2 *to the fourth*.

TABLE 8.5 Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of $\ln 2 = \int_1^2 (1/x)\,dx$

n	T_n	|Error| less than . . .	S_n	|Error| less than . . .
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ agrees to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than four, then its fourth derivative is zero, and

$$E_S = -\frac{b-a}{180} f^{(4)}(c)(\Delta x)^4 = -\frac{b-a}{180}(0)(\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of f. In other words, if f is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of f exactly, whatever the number of subdivisions. Similarly, if f is a constant or a linear function, then its second derivative is zero and

$$E_T = -\frac{b-a}{12} f''(c)(\Delta x)^2 = -\frac{b-a}{12}(0)(\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of f. This is no surprise, for the trapezoids fit the graph perfectly. Although decreasing the step size Δx reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice.

When Δx is very small, say $\Delta x = 10^{-5}$, computer or calculator round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking Δx below a certain size can actually make things worse. Although this is not an issue in this book, you should consult a text on numerical analysis for alternative methods if you are having problems with round-off.

EXAMPLE 8 Estimate

$$\int_0^2 x^3\,dx$$

with Simpson's Rule.

Solution The fourth derivative of $f(x) = x^3$ is zero, so we expect Simpson's Rule to give the integral's exact value with any (even) number of steps. Indeed, with $n = 2$ and $\Delta x = (2 - 0)/2 = 1$,

$$\begin{aligned} S &= \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) \\ &= \frac{1}{3}((0)^3 + 4(1)^3 + (2)^3) = \frac{12}{3} = 4, \end{aligned}$$

while

$$\int_0^2 x^3\,dx = \frac{x^4}{4}\Big]_0^2 = \frac{16}{4} - 0 = 4.$$

EXAMPLE 9 Draining a Swamp

A town wants to drain and fill a small polluted swamp (Figure 8.16). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

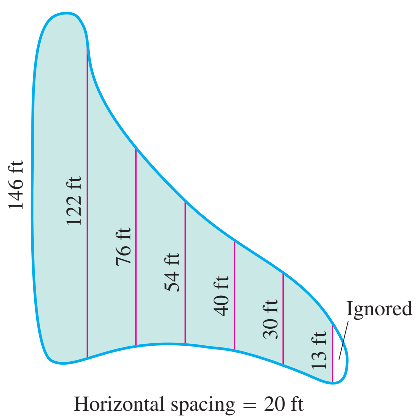

FIGURE 8.16 The dimensions of the swamp in Example 9.

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's Rule with $\Delta x = 20$ ft and the y's equal to the distances measured across the swamp, as shown in Figure 8.16.

$$\begin{aligned} S &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100 \end{aligned}$$

The volume is about $(8100)(5) = 40{,}500\ \text{ft}^3$ or $1500\ \text{yd}^3$.