EXERCISES 8.7

Estimating Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

b. Evaluate the integral directly and find $|E_T|$.

c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

II. Using Simpson's Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

b. Evaluate the integral directly and find $|E_S|$.

c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

1. $\int_1^2 x\,dx$

2. $\int_1^3 (2x - 1)\,dx$

3. $\int_{-1}^1 (x^2 + 1)\,dx$

4. $\int_{-2}^0 (x^2 - 1)\,dx$

5. $\int_0^2 (t^3 + t)\,dt$

6. $\int_{-1}^1 (t^3 + 1)\,dt$

7. $\int_1^2 \frac{1}{s^2}\,ds$

8. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

9. $\int_0^{\pi} \sin t\,dt$

10. $\int_0^1 \sin \pi t\,dt$

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In Exercises 11–14, use the tabulated values of the integrand to estimate the integral with **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule with $n = 8$ steps. Round your answers to five decimal places. Then **(c)** find the integral's exact value and the approximation error E_T or E_S, as appropriate.

11. $\int_0^1 x\sqrt{1 - x^2}\,dx$

x	$x\sqrt{1-x^2}$
0	0.0
0.125	0.12402
0.25	0.24206
0.375	0.34763
0.5	0.43301
0.625	0.48789
0.75	0.49608
0.875	0.42361
1.0	0

12. $\int_0^3 \frac{\theta}{\sqrt{16 + \theta^2}}\,d\theta$

θ	$\theta/\sqrt{16+\theta^2}$
0	0.0
0.375	0.09334
0.75	0.18429
1.125	0.27075
1.5	0.35112
1.875	0.42443
2.25	0.49026
2.625	0.58466
3.0	0.6

13. $\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2 + \sin t)^2}\,dt$

t	$(3\cos t)/(2 + \sin t)^2$
−1.57080	0.0
−1.17810	0.99138
−0.78540	1.26906
−0.39270	1.05961
0	0.75
0.39270	0.48821
0.78540	0.28946
1.17810	0.13429
1.57080	0

14. $\int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\,dy$

y	$(\csc^2 y)\sqrt{\cot y}$
0.78540	2.0
0.88357	1.51606
0.98175	1.18237
1.07992	0.93998
1.17810	0.75402
1.27627	0.60145
1.37445	0.46364
1.47262	0.31688
1.57080	0

The Minimum Number of Subintervals

In Exercises 15–26, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 15–22 are the integrals from Exercises 1–8.)

15. $\int_1^2 x\,dx$

16. $\int_1^3 (2x - 1)\,dx$

17. $\int_{-1}^1 (x^2 + 1)\,dx$

18. $\int_{-2}^0 (x^2 - 1)\,dx$

19. $\int_0^2 (t^3 + t)\,dt$

20. $\int_{-1}^1 (t^3 + 1)\,dt$

21. $\int_1^2 \frac{1}{s^2}\,ds$

22. $\int_2^4 \frac{1}{(s - 1)^2}\,ds$

23. $\int_0^3 \sqrt{x + 1}\,dx$

24. $\int_0^3 \frac{1}{\sqrt{x + 1}}\,dx$

25. $\int_0^2 \sin(x + 1)\,dx$

26. $\int_{-1}^1 \cos(x + \pi)\,dx$

Applications

27. Volume of water in a swimming pool A rectangular swimming pool is 30 ft wide and 50 ft long. The table shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

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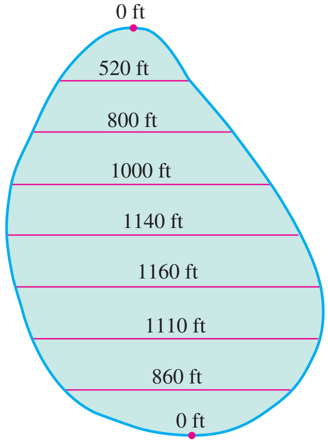
Exercises

28. Stocking a fish pond As the fish and game warden of your township, you are responsible for stocking the town pond with fish before the fishing season. The average depth of the pond is 20 ft. Using a scaled map, you measure distances across the pond at 200-ft intervals, as shown in the accompanying diagram.

a. Use the Trapezoidal Rule to estimate the volume of the pond.

b. You plan to start the season with one fish per 1000 cubic feet. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sell if the average seasonal catch is 20 fish per license?



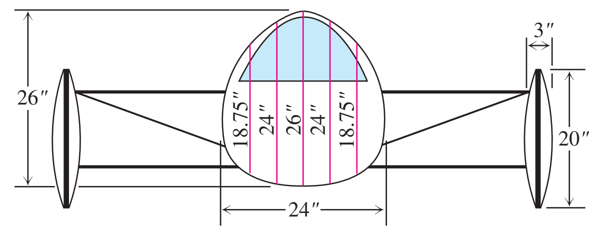
Vertical spacing = 200 ft

29. Ford® Mustang Cobra™ The accompanying table shows time-to-speed data for a 1994 Ford Mustang Cobra accelerating from rest to 130 mph. How far had the Mustang traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

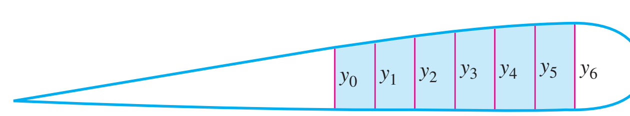
Speed change	Time (sec)
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

Source: Car and Driver, April 1994.

30. Aerodynamic drag A vehicle's aerodynamic drag is determined in part by its cross-sectional area, so, all other things being equal, engineers try to make this area as small as possible. Use Simpson's Rule to estimate the cross-sectional area of the body of James Worden's solar-powered Solectria® automobile at MIT from the diagram.



31. Wing design The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

32. Oil consumption on Pathfinder Island A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced.

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Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	Oil consumption rate (liters/h)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

Theory and Examples

33. Usable values of the sine-integral function *The sine-integral function,*

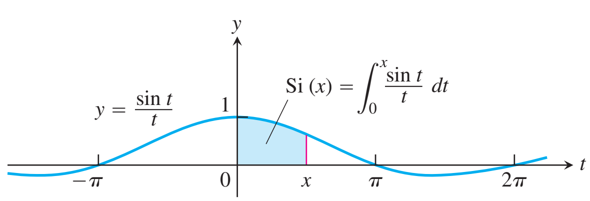
$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\mathrm{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.



a. Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\mathrm{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's Rule with $n = 4$.

b. Estimate $\mathrm{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.

c. Express the error bound you found in part (a) as a percentage of the value you found in part (b).

34. The error function *The error function,*

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2}.

a. Use Simpson's Rule with $n = 10$ to estimate erf (1).

b. In [0, 1],

$$\left|\frac{d^4}{dt^4}\left(e^{-t^2}\right)\right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

35. (*Continuation of Example 3.*) The error bounds for E_T and E_S are "worst case" estimates, and the Trapezoidal and Simpson Rules are often more accurate than the bounds suggest. The Trapezoidal Rule estimate of

$$\int_0^\pi x \sin x\,dx$$

in Example 3 is a case in point.

a. Use the Trapezoidal Rule with $n = 10$ to approximate the value of the integral. The table to the right gives the necessary y-values.

x	$x \sin x$
0	0
$(0.1)\pi$	0.09708
$(0.2)\pi$	0.36932
$(0.3)\pi$	0.76248
$(0.4)\pi$	1.19513
$(0.5)\pi$	1.57080
$(0.6)\pi$	1.79270
$(0.7)\pi$	1.77912
$(0.8)\pi$	1.47727
$(0.9)\pi$	0.87372
π	0

b. Find the magnitude of the difference between π, the integral's value, and your approximation in part (a). You will find the difference to be considerably less than the upper bound of 0.133 calculated with $n = 10$ in Example 3.

T **c.** The upper bound of 0.133 for $|E_T|$ in Example 3 could have been improved somewhat by having a better bound for

$$|f''(x)| = |2 \cos x - x \sin x|$$

on $[0, \pi]$. The upper bound we used was $2 + \pi$. Graph f'' over $[0, \pi]$ and use Trace or Zoom to improve this upper bound.

Use the improved upper bound as M to make an improved estimate of $|E_T|$. Notice that the Trapezoidal Rule approximation in part (a) is also better than this improved estimate would suggest.

T **36.** (*Continuation of Exercise 35.*)

a. Show that the fourth derivative of $f(x) = x \sin x$ is

$$f^{(4)}(x) = -4 \cos x + x \sin x.$$

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Use Trace or Zoom to find an upper bound M for the values of $|f^{(4)}|$ on $[0, \pi]$.

b. Use the value of M from part (a) to obtain an upper bound for the magnitude of the error in estimating the value of

$$\int_0^{\pi} x \sin x \, dx$$

with Simpson's Rule with $n = 10$ steps.

c. Use the data in the table in Exercise 35 to estimate $\int_0^{\pi} x \sin x \, dx$ with Simpson's Rule with $n = 10$ steps.

d. To six decimal places, find the magnitude of the difference between your estimate in part (c) and the integral's true value, π. You will find the error estimate obtained in part (b) to be quite good.

37. Prove that the sum T in the Trapezoidal Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (*Hint:* Use the Intermediate Value Theorem to show the existence of c_k in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)

38. Prove that the sum S in Simpson's Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (See Exercise 37.)

T Numerical Integration

As we mentioned at the beginning of the section, the definite integrals of many continuous functions cannot be evaluated with the Fundamental Theorem of Calculus because their antiderivatives lack elementary formulas. Numerical integration offers a practical way to estimate the values of these so-called *nonelementary integrals*. If your calculator or computer has a numerical integration routine, try it on the integrals in Exercises 39–42.

39. $\displaystyle\int_0^1 \sqrt{1 + x^4}\, dx$ — A nonelementary integral that came up in Newton's research

40. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$ — The integral from Exercise 33. To avoid division by zero, you may have to start the integration at a small positive number like 10^{-6} instead of 0.

41. $\displaystyle\int_0^{\pi/2} \sin(x^2)\, dx$ — An integral associated with the diffraction of light

42. $\displaystyle\int_0^{\pi/2} 40\sqrt{1 - 0.64\cos^2 t}\, dt$ — The length of the ellipse $(x^2/25) + (y^2/9) = 1$

T 43. Consider the integral $\int_0^{\pi} \sin x \, dx$.

a. Find the Trapezoidal Rule approximations for $n = 10$, 100, and 1000.

b. Record the errors with as many decimal places of accuracy as you can.

c. What pattern do you see?

d. Explain how the error bound for E_T accounts for the pattern.

T 44. (*Continuation of Exercise 43.*) Repeat Exercise 43 with Simpson's Rule and E_S.

45. Consider the integral $\int_{-1}^{1} \sin(x^2)\, dx$.

a. Find f'' for $f(x) = \sin(x^2)$.

b. Graph $y = f''(x)$ in the viewing window $[-1, 1]$ by $[-3, 3]$.

c. Explain why the graph in part (b) suggests that $|f''(x)| \le 3$ for $-1 \le x \le 1$.

d. Show that the error estimate for the Trapezoidal Rule in this case becomes

$$|E_T| \le \frac{(\Delta x)^2}{2}.$$

e. Show that the Trapezoidal Rule error will be less than or equal to 0.01 in magnitude if $\Delta x \le 0.1$.

f. How large must n be for $\Delta x \le 0.1$?

46. Consider the integral $\int_{-1}^{1} \sin(x^2)\, dx$.

a. Find $f^{(4)}$ for $f(x) = \sin(x^2)$. (You may want to check your work with a CAS if you have one available.)

b. Graph $y = f^{(4)}(x)$ in the viewing window $[-1, 1]$ by $[-30, 10]$.

c. Explain why the graph in part (b) suggests that $|f^{(4)}(x)| \le 30$ for $-1 \le x \le 1$.

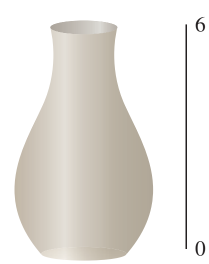
d. Show that the error estimate for Simpson's Rule in this case becomes

$$|E_S| \le \frac{(\Delta x)^4}{3}.$$

e. Show that the Simpson's Rule error will be less than or equal to 0.01 in magnitude if $\Delta x \le 0.4$.

f. How large must n be for $\Delta x \le 0.4$?

T 47. A vase We wish to estimate the volume of a flower vase using only a calculator, a string, and a ruler. We measure the height of the vase to be 6 in. We then use the string and the ruler to find circumferences of the vase (in inches) at half-inch intervals. (We list them from the top down to correspond with the picture of the vase.)



Circumferences	
5.4	10.8
4.5	11.6
4.4	11.6
5.1	10.8
6.3	9.0
7.8	6.3
9.4	

a. Find the areas of the cross-sections that correspond to the given circumferences.

b. Express the volume of the vase as an integral with respect to y over the interval $[0, 6]$.

c. Approximate the integral using the Trapezoidal Rule with $n = 12$.

d. Approximate the integral using Simpson's Rule with $n = 12$. Which result do you think is more accurate? Give reasons for your answer.

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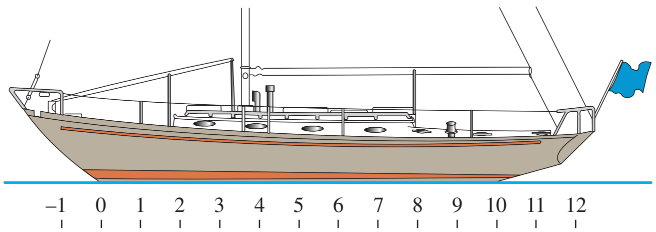
48. A sailboat's displacement To find the volume of water displaced by a sailboat, the common practice is to partition the waterline into 10 subintervals of equal length, measure the cross-sectional area $A(x)$ of the submerged portion of the hull at each partition point, and then use Simpson's Rule to estimate the integral of $A(x)$ from one end of the waterline to the other. The table here lists the area measurements at "Stations" 0 through 10, as the partition points are called, for the cruising sloop *Pipedream*, shown here. The common subinterval length (distance between consecutive stations) is $\Delta x = 2.54$ ft (about 2 ft 6-1/2 in., chosen for the convenience of the builder).



a. Estimate *Pipedream*'s displacement volume to the nearest cubic foot.

Station	Submerged area (ft^2)
0	0
1	1.07
2	3.84
3	7.82
4	12.20
5	15.18
6	16.14
7	14.00
8	9.21
9	3.24
10	0

b. The figures in the table are for seawater, which weighs 64 lb/ft^3. How many pounds of water does *Pipedream* displace? (Displacement is given in pounds for small craft and in long tons (1 long ton = 2240 lb) for larger vessels.) (Data from *Skene's Elements of Yacht Design* by Francis S. Kinney (Dodd, Mead, 1962.)

c. Prismatic coefficients A boat's prismatic coefficient is the ratio of the displacement volume to the volume of a prism whose height equals the boat's waterline length and whose base equals the area of the boat's largest submerged cross-section. The best sailboats have prismatic coefficients between 0.51 and 0.54. Find *Pipedream*'s prismatic coefficient, given a waterline length of 25.4 ft and a largest submerged cross-sectional area of 16.14 ft^2 (at Station 6).

49. Elliptic integrals The length of the ellipse

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$$

turns out to be

$$\text{Length} = 4a\int_0^{\pi/2}\sqrt{1 - e^2\cos^2 t}\,dt,$$

where e is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1.

a. Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

b. Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2\cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

50. The length of one arch of the curve $y = \sin x$ is given by

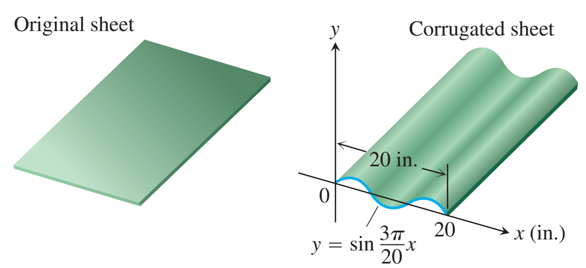
$$L = \int_0^{\pi}\sqrt{1 + \cos^2 x}\,dx.$$

Estimate L by Simpson's Rule with $n = 8$.

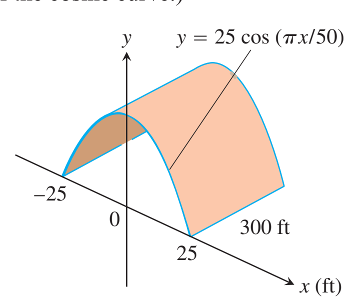
51. Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin\frac{3\pi}{20}x, \quad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.



52. Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross-section is shaped like one arch of the curve $y = 25\cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



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Surface Area

Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 53–56 about the x-axis.

53. $y = \sin x, \quad 0 \le x \le \pi$

54. $y = x^2/4, \quad 0 \le x \le 2$

55. $y = x + \sin 2x, \quad -2\pi/3 \le x \le 2\pi/3$ (the curve in Section 4.4, Exercise 5)

56. $y = \frac{x}{12}\sqrt{36 - x^2}, 0 \le x \le 6$ (the surface of the plumb bob in Section 6.1, Exercise 56)

Estimating Function Values

57. Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1 - x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to five decimal places.

58. Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\, dx.$$

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