8.8 Improper Integrals

Project Project

Up to now, definite integrals have been required to have two properties. First, that the domain of integration $[a, b]$ be finite. Second, that the range of the integrand be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.17a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.17b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see that improper integrals play an important role when investigating the convergence of certain infinite series in Chapter 11.

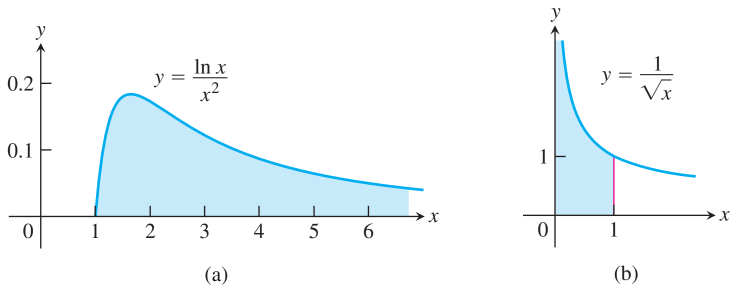


FIGURE 8.17 Are the areas under these infinite curves finite?

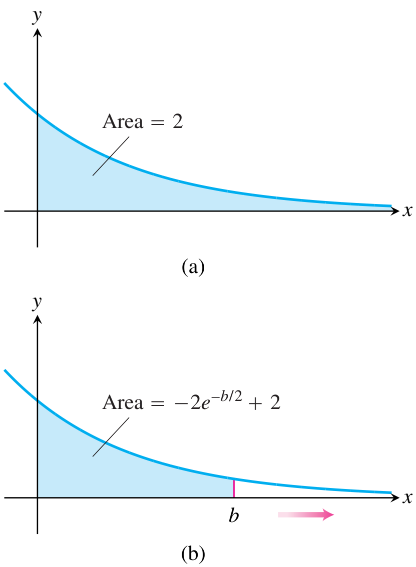


FIGURE 8.18 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) an improper integral of the first type.

Infinite Limits of Integration

Consider the infinite region that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.18a). You might think this region has infinite area, but we will see that the natural value to assign is finite. Here is how to assign a value to the area. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.18b).

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

It can be shown that the choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x)\,dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.18 as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

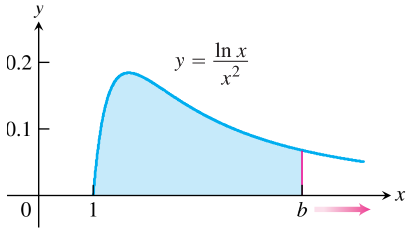


FIGURE 8.19 The area under this curve is an improper integral (Example 1).

EXAMPLE 1 Evaluating an Improper Integral on $[1, \infty)$

Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.19). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$.

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx \\
&= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] \\
&= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1 \\
&= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1. \qquad \text{l'Hôpital's Rule}
\end{aligned}$$

Thus, the improper integral converges and the area has finite value 1. ■

EXAMPLE 2 Evaluating an Integral on $(-\infty, \infty)$

Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Solution According to the definition (Part 3), we can write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0 \\
&= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \\
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} \\
&= \lim_{b\to\infty} \tan^{-1} x\Big]_0^b \\
&= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

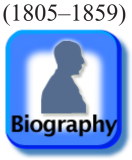
Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.20). ■

HISTORICAL BIOGRAPHY

Lejeune Dirichlet
(1805–1859)



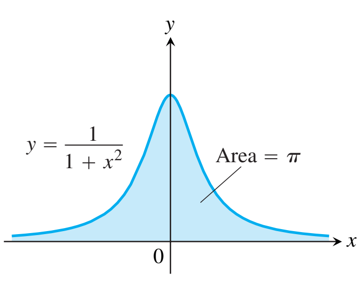


FIGURE 8.20 The area under this curve is finite (Example 2).

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 3 Determining Convergence

For what values of p does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}\left(b^{-p+1} - 1\right) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b \to \infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b \to \infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x}$$

$$= \lim_{b \to \infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b \to \infty} \ln x\Big]_1^b$$

$$= \lim_{b \to \infty} (\ln b - \ln 1) = \infty.$$

■

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

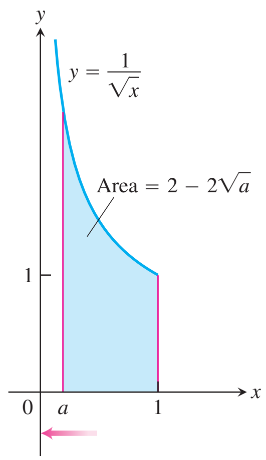


FIGURE 8.21 The area under this curve is

$$\lim_{a \to 0^+} \int_a^1 \left(\frac{1}{\sqrt{x}}\right) dx = 2,$$

an improper integral of the second kind.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.17b). First we find the area of the portion from a to 1 (Figure 8.21).

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

The area under the curve from 0 to 1 is finite and equals

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION **Type II Improper Integrals**

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then
$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$
3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

EXAMPLE 4 A Divergent Improper Integral

Investigate the convergence of

$$\int_0^1 \frac{1}{1 - x}\,dx.$$

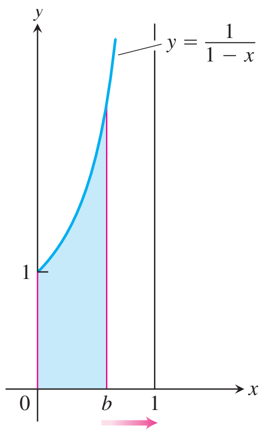


FIGURE 8.22 The limit does not exist:

$$\int_0^1 \left(\frac{1}{1-x}\right) dx = \lim_{b \to 1^-} \int_0^b \frac{1}{1-x}\, dx = \infty.$$

The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.22). We evaluate the integral as

$$\lim_{b \to 1^-} \int_0^b \frac{1}{1-x}\, dx = \lim_{b \to 1^-} \Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b \to 1^-} [-\ln(1-b) + 0] = \infty.$$

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Vertical Asympote at an Interior Point

Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.23). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b \to 1^-} 3(x-1)^{1/3}\Big]_0^b$$

$$= \lim_{b \to 1^-} [3(b-1)^{1/3} + 3] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c \to 1^+} 3(x-1)^{1/3}\Big]_c^3$$

$$= \lim_{c \to 1^+} \left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$ ■

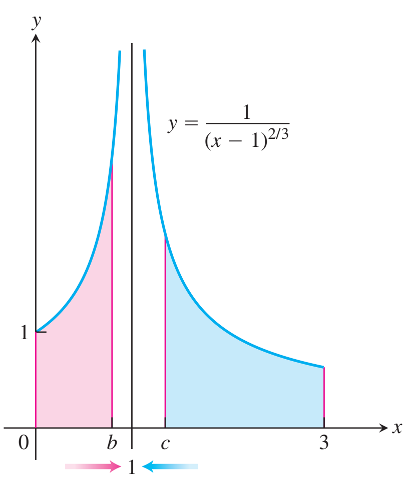


FIGURE 8.23 Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

EXAMPLE 6 A Convergent Improper Integral

Evaluate

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\, dx.$$

Solution

$$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{x+3}{(x-1)(x^2+1)}\,dx$$

$$= \lim_{b\to\infty}\int_2^{b} \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right)dx \qquad \text{Partial fractions}$$

$$= \lim_{b\to\infty}\left[2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x\right]_2^b$$

$$= \lim_{b\to\infty}\left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1}x\right]_2^b \qquad \text{Combine the logarithms.}$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1}b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1}2$$

$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1}2 \approx 1.1458$$

Notice that we combined the logarithms in the antiderivative *before* we calculated the limit as $b\to\infty$. Had we not done so, we would have encountered the indeterminate form

$$\lim_{b\to\infty}(2\ln(b-1) - \ln(b^2+1)) = \infty - \infty.$$

The way to evaluate the indeterminate form, of course, is to combine the logarithms, so we would have arrived at the same answer in the end. ■

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral in Example 6 using Maple, enter

$$> f := (x+3)/((x-1)*(x\hat{\ }2+1));$$

Then use the integration command

$$> \text{int}(f, x = 2..\text{infinity});$$

Maple returns the answer

$$-\frac{1}{2}\pi + \ln(5) + \arctan(2).$$

To obtain a numerical result, use the evaluation command **evalf** and specify the number of digits, as follows:

$$> \text{evalf}(\%, 6);$$

The symbol % instructs the computer to evaluate the last expression on the screen, in this case $(-1/2)\pi + \ln(5) + \arctan(2)$. Maple returns 1.14579.

Using Mathematica, entering

$$\textit{In [1]:=}\ \text{Integrate}\,[(x+3)/((x-1)(x\hat{\ }2+1)), \{x, 2, \text{Infinity}\}]$$

returns

$$\textit{Out [1]=}\ \frac{-\text{Pi}}{2} + \text{ArcTan}\,[2] + \text{Log}\,[5].$$

To obtain a numerical result with six digits, use the command "N[%, 6]"; it also yields 1.14579.

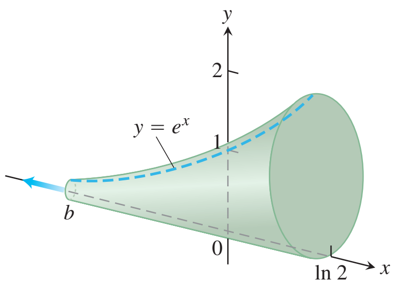


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

EXAMPLE 7 Finding the Volume of an Infinite Solid

The cross-sections of the solid horn in Figure 8.24 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x \le \ln 2$. Find the volume of the horn.

Solution The area of a typical cross-section is

$$A(x) = \pi(\text{radius})^2 = \pi\left(\frac{1}{2}y\right)^2 = \frac{\pi}{4}e^{2x}.$$

We define the volume of the horn to be the limit as $b \to -\infty$ of the volume of the portion from b to ln 2. As in Section 6.1 (the method of slicing), the volume of this portion is

$$V = \int_b^{\ln 2} A(x)\,dx = \int_b^{\ln 2} \frac{\pi}{4}e^{2x}\,dx = \frac{\pi}{8}e^{2x}\Big]_b^{\ln 2}$$
$$= \frac{\pi}{8}(e^{\ln 4} - e^{2b}) = \frac{\pi}{8}(4 - e^{2b}).$$

As $b \to -\infty$, $e^{2b} \to 0$ and $V \to (\pi/8)(4 - 0) = \pi/2$. The volume of the horn is $\pi/2$. ■

EXAMPLE 8 An Incorrect Calculation

Evaluate

$$\int_0^3 \frac{dx}{x-1}.$$

Solution Suppose we fail to notice the discontinuity of the integrand at $x = 1$, interior to the interval of integration. If we evaluate the integral as an ordinary integral we get

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

This result is *wrong* because the integral is improper. The correct evaluation uses limits:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-} \ln|x-1|\Big]_0^b$$
$$= \lim_{b\to 1^-}(\ln|b-1| - \ln|-1|)$$
$$= \lim_{b\to 1^-}\ln(1-b) = -\infty. \qquad 1 - b \to 0^+ \text{ as } b \to 1^-$$

Since $\int_0^1 dx/(x-1)$ is divergent, the original integral $\int_0^3 dx/(x-1)$ is divergent. ■

Example 8 illustrates what can go wrong if you mistake an improper integral for an ordinary integral. Whenever you encounter an integral $\int_a^b f(x)\,dx$ you must examine the function f on $[a, b]$ and then decide if the integral is improper. If f is continuous on $[a, b]$, it will be proper, an ordinary integral.

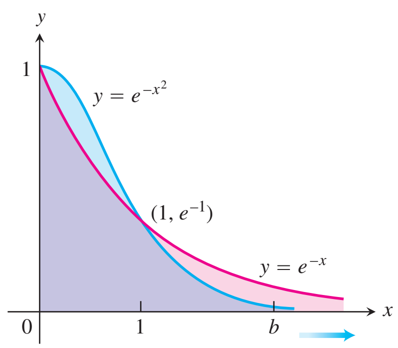


FIGURE 8.25 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 9).

Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

EXAMPLE 9 Investigating Convergence

Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

Solution By definition,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$$

We cannot evaluate the latter integral directly because it is nonelementary. But we *can* show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of b. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: For every value of $x \geq 1$ we have $e^{-x^2} \leq e^{-x}$ (Figure 8.25), so that

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence

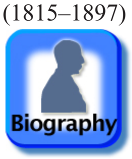
$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers, discussed in Appendix 4. ■

The comparison of e^{-x^2} and e^{-x} in Example 9 is a special case of the following test.

HISTORICAL BIOGRAPHY

Karl Weierstrass (1815–1897)



THEOREM 1 Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

The reasoning behind the argument establishing Theorem 1 is similar to that in Example 9.

If $0 \leq f(x) \leq g(x)$ for $x \geq a$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx, \qquad b > a.$$

From this it can be argued, as in Example 9, that

$$\int_a^{\infty} f(x)\,dx \text{ converges if } \int_a^{\infty} g(x)\,dx \text{ converges.}$$

Turning this around says that

$$\int_a^{\infty} g(x)\,dx \text{ diverges if } \int_a^{\infty} f(x)\,dx \text{ diverges.}$$

EXAMPLE 10 Using the Direct Comparison Test

(a) $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2}\,dx \text{ converges.}$$ Example 3

(b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^{\infty} \frac{1}{x}\,dx \text{ diverges.}$$ Example 3

■

THEOREM 2 Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^{\infty} f(x)\,dx \qquad \text{and} \qquad \int_a^{\infty} g(x)\,dx$$

both converge or both diverge.

A proof of Theorem 2 is given in advanced calculus.

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

EXAMPLE 11 Using the Limit Comparison Test

Show that

$$\int_1^\infty \frac{dx}{1+x^2}$$

converges by comparison with $\int_1^\infty (1/x^2)\,dx$. Find and compare the two integral values.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1+x^2)} = \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$

a positive finite limit (Figure 8.26). Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

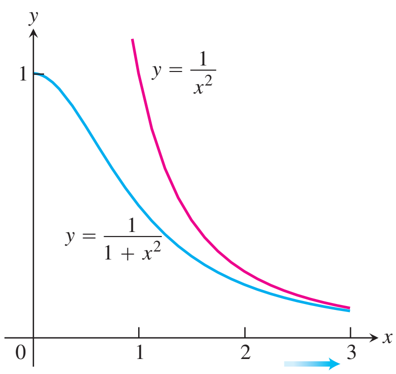


FIGURE 8.26 The functions in Example 11.

The integrals converge to different values, however.

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

■

EXAMPLE 12 Using the Limit Comparison Test

Show that

$$\int_1^\infty \frac{3}{e^x + 5}\,dx$$

converges.

Solution From Example 9, it is easy to see that $\int_1^\infty e^{-x}\,dx = \int_1^\infty (1/e^x)\,dx$ converges. Moreover, we have

$$\lim_{x\to\infty} \frac{1/e^x}{3/(e^x+5)} = \lim_{x\to\infty} \frac{e^x+5}{3e^x} = \lim_{x\to\infty} \left(\frac{1}{3} + \frac{5}{3e^x}\right) = \frac{1}{3},$$

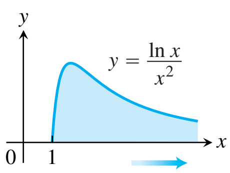
a positive finite limit. As far as the convergence of the improper integral is concerned, $3/(e^x + 5)$ behaves like $1/e^x$. ■

Types of Improper Integrals Discussed in This Section

INFINITE LIMITS OF INTEGRATION: TYPE I

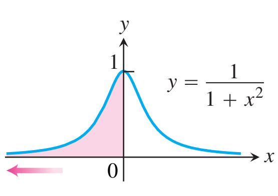
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$



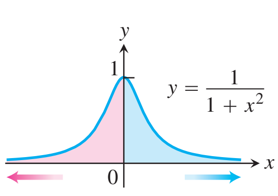
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}$$



3. Both limits

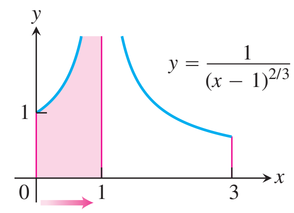
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^c \frac{dx}{1+x^2}$$



INTEGRAND BECOMES INFINITE: TYPE II

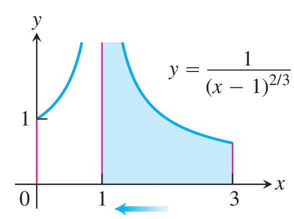
4. Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



5. Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to 1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$