EXERCISES 8.8

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

2. $\displaystyle\int_1^\infty \frac{dx}{x^{1.001}}$

3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\displaystyle\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\displaystyle\int_0^1 \frac{dr}{r^{0.999}}$

9. $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\displaystyle\int_{-\infty}^2 \frac{2\,dx}{x^2+4}$

11. $\displaystyle\int_2^\infty \frac{2}{v^2-v}\,dv$

12. $\displaystyle\int_2^\infty \frac{2\,dt}{t^2-1}$

13. $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$

14. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\displaystyle\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

18. $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1}v)}$

20. $\displaystyle\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx$

21. $\displaystyle\int_{-\infty}^0 \theta e^\theta\,d\theta$

22. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

23. $\displaystyle\int_{-\infty}^0 e^{-|x|}\,dx$

24. $\displaystyle\int_{-\infty}^\infty 2xe^{-x^2}\,dx$

25. $\displaystyle\int_0^1 x\ln x\,dx$

26. $\displaystyle\int_0^1 (-\ln x)\,dx$

27. $\displaystyle\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\displaystyle\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\displaystyle\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\displaystyle\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\displaystyle\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\displaystyle\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$

34. $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

36. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

37. $\displaystyle\int_0^\pi \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$

38. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$

39. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

40. $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

41. $\displaystyle\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$

42. $\displaystyle\int_0^1 \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

43. $\displaystyle\int_0^2 \frac{dx}{1-x^2}$

44. $\displaystyle\int_0^2 \frac{dx}{1-x}$

45. $\displaystyle\int_{-1}^1 \ln|x|\,dx$

46. $\displaystyle\int_{-1}^1 -x\ln|x|\,dx$

47. $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$

48. $\displaystyle\int_4^\infty \frac{dx}{\sqrt{x}-1}$

49. $\displaystyle\int_2^\infty \frac{dv}{\sqrt{v-1}}$

50. $\displaystyle\int_0^\infty \frac{d\theta}{1+e^\theta}$

51. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$

52. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

53. $\displaystyle\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

54. $\displaystyle\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$

55. $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x}\,dx$

56. $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$

57. $\displaystyle\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$

58. $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx$

59. $\displaystyle\int_1^\infty \frac{e^x}{x}\,dx$

60. $\displaystyle\int_{e^e}^\infty \ln(\ln x)\,dx$

61. $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x-x}}\,dx$

62. $\displaystyle\int_1^\infty \frac{1}{e^x-2^x}\,dx$

63. $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

64. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

Theory and Examples

65. Find the values of p for which each integral converges.

a. $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ **b.** $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

66. $\int_{-\infty}^\infty f(x)\,dx$ **may not equal** $\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$ Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^{b}\frac{2x\,dx}{x^2+1}=0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

67. Find the area of the region.

68. Find the centroid of the region.

69. Find the volume of the solid generated by revolving the region about the y-axis.

70. Find the volume of the solid generated by revolving the region about the x-axis.

71. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

72. The region in Exercise 71 is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

73. Estimating the value of a convergent improper integral whose domain is infinite

a. Show that

$$\int_{3}^{\infty}e^{-3x}\,dx=\frac{1}{3}e^{-9}<0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

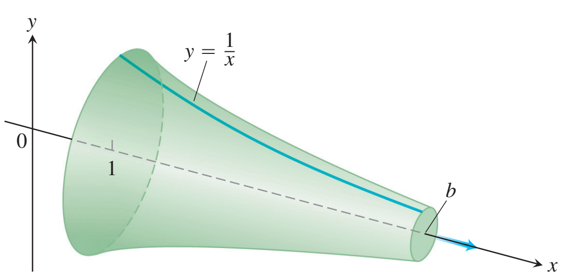
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

74. The infinite paint can or Gabriel's horn As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_{1}^{\infty}2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_{1}^{b}2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx>2\pi\int_{1}^{b}\frac{1}{x}\,dx.$$



However, the integral

$$\int_{1}^{\infty}\pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges. **(a)** Calculate it. **(b)** This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

75. Sine-integral function The integral

$$\mathrm{Si}\,(x)=\int_{0}^{x}\frac{\sin t}{t}\,dt,$$

called the *sine-integral function,* has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t > 0$. Is the Si function everywhere increasing or decreasing? Do you think $\mathrm{Si}\,(x) = 0$ for $x > 0$? Check your answers by graphing the function Si (x) for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_{0}^{\infty}\frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

76. Error function The function

$$\mathrm{erf}\,(x)=\int_{0}^{x}\frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_{0}^{\infty}\frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.3, Exercise 37.

77. Normal probability distribution function The function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean.

From the theory of probability, it is known that

$$\int_{-\infty}^{\infty}f(x)\,dx=1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

T **a.** Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

b. Evaluate

$$\int_{-n}^{n} f(x)\,dx$$

for $n = 1, 2, 3$.

c. Give a convincing argument that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_{b}^{\infty} e^{-x/2}\,dx \to 0 \quad \text{as} \quad b \to \infty.)$$

78. Here is an argument that $\ln 3$ equals $\infty - \infty$. Where does the argument go wrong? Give reasons for your answer.

$$\begin{aligned}
\ln 3 &= \ln 1 + \ln 3 = \ln 1 - \ln \frac{1}{3}\\
&= \lim_{b\to\infty} \ln\left(\frac{b-2}{b}\right) - \ln\frac{1}{3}\\
&= \lim_{b\to\infty} \left[\ln \frac{x-2}{x}\right]_3^b\\
&= \lim_{b\to\infty} \left[\ln (x-2) - \ln x\right]_3^b\\
&= \lim_{b\to\infty} \int_3^b \left(\frac{1}{x-2} - \frac{1}{x}\right) dx\\
&= \int_3^{\infty} \left(\frac{1}{x-2} - \frac{1}{x}\right) dx\\
&= \int_3^{\infty} \frac{1}{x-2}\,dx - \int_3^{\infty} \frac{1}{x}\,dx\\
&= \lim_{b\to\infty} \left[\ln (x-2)\right]_3^b - \lim_{b\to\infty} \left[\ln x\right]_3^b\\
&= \infty - \infty.
\end{aligned}$$

79. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{b}^{\infty} f(x)\,dx$ both converge.

b. $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$ when the integrals involved converge.

80. a. Show that if f is even and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx.$$

b. Show that if f is odd and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Use direct evaluation, the comparison tests, and the results in Exercise 80, as appropriate, to determine the convergence or divergence of the integrals in Exercises 81–88. If more than one method applies, use whatever method you prefer.

81. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}}$

82. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^6+1}}$

83. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

84. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{x^2+1}$

85. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$

86. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x+1)^2}$

87. $\displaystyle\int_{-\infty}^{\infty} \frac{|\sin x| + |\cos x|}{|x|+1}\,dx$

(*Hint:* $|\sin\theta| + |\cos\theta| \ge \sin^2\theta + \cos^2\theta$.)

88. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2)}$

COMPUTER EXPLORATIONS

Exploring Integrals of $x^p \ln x$

In Exercises 89–92, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

89. $\displaystyle\int_{0}^{e} x^p \ln x\,dx$

90. $\displaystyle\int_{e}^{\infty} x^p \ln x\,dx$

91. $\displaystyle\int_{0}^{\infty} x^p \ln x\,dx$

92. $\displaystyle\int_{-\infty}^{\infty} x^p \ln |x|\,dx$