Chapter 8 Additional and Advanced Exercises

Challenging Integrals

Evaluate the integrals in Exercises 1–10.

1. $\int (\sin^{-1} x)^2\, dx$

2. $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

3. $\int x \sin^{-1} x\, dx$

4. $\int \sin^{-1} \sqrt{y}\, dy$

5. $\int \frac{d\theta}{1 - \tan^2 \theta}$

6. $\int \ln\left(\sqrt{x} + \sqrt{1+x}\right) dx$

7. $\int \frac{dt}{t - \sqrt{1 - t^2}}$

8. $\int \frac{(2e^{2x} - e^x)\, dx}{\sqrt{3e^{2x} - 6e^x - 1}}$

9. $\int \frac{dx}{x^4 + 4}$

10. $\int \frac{dx}{x^6 - 1}$

Limits

Evaluate the limits in Exercises 11 and 12.

11. $\lim_{x \to \infty} \int_{-x}^{x} \sin t\, dt$

12. $\lim_{x \to 0^+} x \int_x^1 \frac{\cos t}{t^2}\, dt$

Evaluate the limits in Exercises 13 and 14 by identifying them with definite integrals and evaluating the integrals.

13. $\lim_{n \to \infty} \sum_{k=1}^{n} \ln \sqrt[n]{1 + \frac{k}{n}}$

14. $\lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2 - k^2}}$

Theory and Applications

15. **Finding arc length** Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt, \quad 0 \le x \le \pi/4.$$

16. **Finding arc length** Find the length of the curve $y = \ln(1 - x^2)$, $0 \le x \le 1/2$.

17. **Finding volume** The region in the first quadrant that is enclosed by the x-axis and the curve $y = 3x\sqrt{1 - x}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

18. **Finding volume** The region in the first quadrant that is enclosed by the x-axis, the curve $y = 5/\left(x\sqrt{5 - x}\right)$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

19. **Finding volume** The region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$, and the line $x = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

20. **Finding volume** The region in the first quadrant that is bounded above by the curve $y = e^x - 1$, below by the x-axis, and on the right by the line $x = \ln 2$ is revolved about the line $x = \ln 2$ to generate a solid. Find the volume of the solid.

21. **Finding volume** Let R be the "triangular" region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving R about

    a. the x-axis. **b.** the line $y = 1$.

22. **Finding volume** (*Continuation of Exercise 21.*) Find the volume of the solid generated by revolving the region R about

    a. the y-axis. **b.** the line $x = 1$.

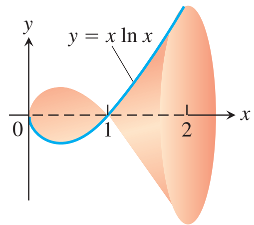
23. **Finding volume** The region between the x-axis and the curve

$$y = f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the x-axis to generate the solid shown here.

a. Show that f is continuous at $x = 0$.

b. Find the volume of the solid.



24. Finding volume The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x-axis to generate a solid. Find the volume of the solid.

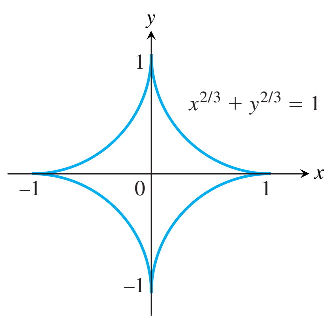
25. Centroid of a region Find the centroid of the region in the first quadrant that is bounded below by the x-axis, above by the curve $y = \ln x$, and on the right by the line $x = e$.

26. Centroid of a region Find the centroid of the region in the plane enclosed by the curves $y = \pm(1 - x^2)^{-1/2}$ and the lines $x = 0$ and $x = 1$.

27. Length of a curve Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.

28. Finding surface area Find the area of the surface generated by revolving the curve in Exercise 27 about the y-axis.

29. The length of an astroid The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see accompanying figure). Find the length of this particular astroid.



30. The surface generated by an astroid Find the area of the surface generated by revolving the curve in Exercise 29 about the x-axis.

31. Find a curve through the origin whose length is

$$\int_0^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

32. Without evaluating either integral, explain why

$$2\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}.$$

T **33. a.** Graph the function $f(x) = e^{(x - e^x)}$, $-5 \le x \le 3$.

b. Show that $\int_{-\infty}^{\infty} f(x)\, dx$ converges and find its value.

34. Find $\lim_{n\to\infty} \int_0^1 \frac{ny^{n-1}}{1 + y}\, dy$.

35. Derive the integral formula

$$\int x\left(\sqrt{x^2 - a^2}\right)^n dx = \frac{\left(\sqrt{x^2 - a^2}\right)^{n+2}}{n + 2} + C, \quad n \ne -2.$$

36. Prove that

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}.$$

(*Hint:* Observe that for $0 < x < 1$, we have $4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$, with the left-hand side becoming an equality for $x = 0$ and the right-hand side becoming an equality for $x = 1$.)

37. For what value or values of a does

$$\int_1^{\infty} \left(\frac{ax}{x^2 + 1} - \frac{1}{2x}\right) dx$$

converge? Evaluate the corresponding integral(s).

38. For each $x > 0$, let $G(x) = \int_0^{\infty} e^{-xt}\, dt$. Prove that $xG(x) = 1$ for each $x > 0$.

39. Infinite area and finite volume What values of p have the following property: The area of the region between the curve $y = x^{-p}$, $1 \le x < \infty$, and the x-axis is infinite but the volume of the solid generated by revolving the region about the x-axis is finite.

40. Infinite area and finite volume What values of p have the following property: The area of the region in the first quadrant enclosed by the curve $y = x^{-p}$, the y-axis, the line $x = 1$, and the interval $[0, 1]$ on the x-axis is infinite but the volume of the solid generated by revolving the region about one of the coordinate axes is finite.

Tabular Integration

The technique of tabular integration also applies to integrals of the form $\int f(x)g(x)\, dx$ when neither function can be differentiated repeatedly to become zero. For example, to evaluate

$$\int e^{2x} \cos x\, dx$$

we begin as before with a table listing successive derivatives of e^{2x} and integrals of $\cos x$:

e^{2x} **and its derivatives**		**cos** x **and its integrals**
e^{2x}	(+)	$\cos x$
$2e^{2x}$	(−)	$\sin x$
$4e^{2x}$	(+)	$-\cos x$ ← *Stop here:* Row is same as first row except for multiplicative constants (4 on the left, −1 on the right)

We stop differentiating and integrating as soon as we reach a row that is the same as the first row except for multiplicative constants. We interpret the table as saying

$$\int e^{2x} \cos x\, dx = +(e^{2x} \sin x) - (2e^{2x}(-\cos x)) + \int (4e^{2x})(-\cos x)\, dx.$$

We take signed products from the diagonal arrows and a signed integral for the last horizontal arrow. Transposing the integral on the right-hand side over to the left-hand side now gives

$$5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x$$

or

$$\int e^{2x}\cos x\,dx = \frac{e^{2x}\sin x + 2e^{2x}\cos x}{5} + C,$$

after dividing by 5 and adding the constant of integration.

Use tabular integration to evaluate the integrals in Exercises 41–48.

41. $\int e^{2x}\cos 3x\,dx$

42. $\int e^{3x}\sin 4x\,dx$

43. $\int \sin 3x \sin x\,dx$

44. $\int \cos 5x \sin 4x\,dx$

45. $\int e^{ax}\sin bx\,dx$

46. $\int e^{ax}\cos bx\,dx$

47. $\int \ln(ax)\,dx$

48. $\int x^2\ln(ax)\,dx$

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x"; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad x > 0.$$

For each positive x, the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞. Figure 8.27 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 31 in Chapter 15.

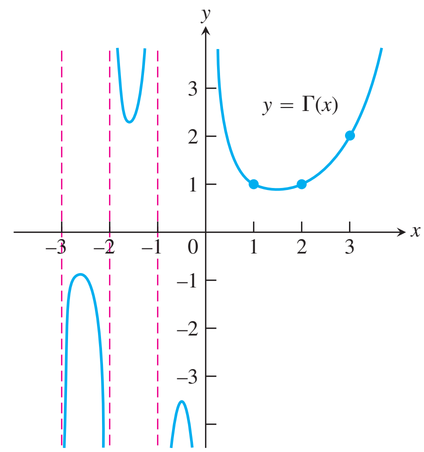


FIGURE 8.27 Euler's gamma function $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x+1))/x$, which is the subject of Exercise 49.

49. If n is a nonnegative integer, $\Gamma(n+1) = n!$

a. Show that $\Gamma(1) = 1$.

b. Then apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$. This gives

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1\\ \Gamma(3) &= 2\Gamma(2) = 2\\ \Gamma(4) &= 3\Gamma(3) = 6\\ &\vdots\\ \Gamma(n+1) &= n\Gamma(n) = n! \qquad (1)\end{aligned}$$

c. Use mathematical induction to verify Equation (1) for every nonnegative integer n.

50. Stirling's formula Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x\to\infty}\left(\frac{e}{x}\right)^x\sqrt{\frac{x}{2\pi}}\,\Gamma(x) = 1,$$

so for large x,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}(1+\epsilon(x)), \qquad \epsilon(x)\to 0 \text{ as } x\to\infty. \qquad (2)$$

Dropping $\epsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}} \qquad \textbf{(Stirling's formula)}. \qquad (3)$$

a. **Stirling's approximation for $n!$** Use Equation (3) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n\sqrt{2n\pi} \qquad \textbf{(Stirling's approximation)}. \qquad (4)$$

As you will see if you do Exercise 64 in Section 11.1, Equation (4) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \qquad (5)$$

T b. Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \dots$, as far as your calculator can go.

T c. A refinement of Equation (2) gives

$$\Gamma(x) = \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}\,e^{1/(12x)}(1+\epsilon(x)),$$

or

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\, e^{1/(12x)}$$

which tells us that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/(12n)}. \tag{6}$$

Compare the values given for 10! by your calculator, Stirling's approximation, and Equation (6).