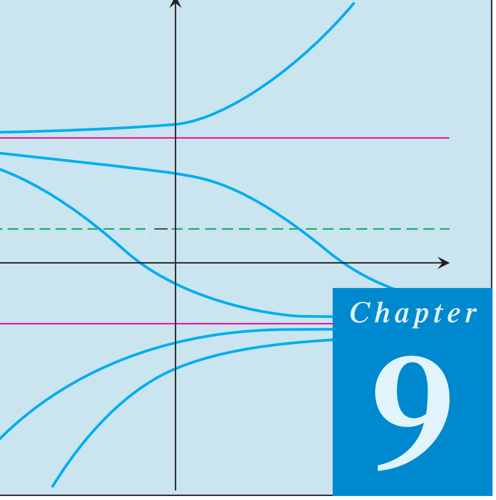

Chapter 9

Further Applications of Integration

Historical Biography

Carl Friedrich Gauss (1777–1855)

Biography History

OVERVIEW In Section 4.8 we introduced differential equations of the form $dy/dx = f(x)$, where y is an unknown function being differentiated. For a continuous function f, we found the general solution $y(x)$ by integration: $y(x) = \int f(x)\,dx$. (Remember that the indefinite integral represents *all* the antiderivatives of f, so it contains an arbitrary constant $+C$ which must be shown once an antiderivative is found.) Many applications in the sciences, engineering, and economics involve a model formulated by even more general differential equations. In Section 7.5, for example, we found that exponential growth and decay is modeled by a differential equation of the form $dy/dx = ky$, for some constant $k \neq 0$. We have not yet considered differential equations such as $dy/dx = y - x$, yet such equations arise frequently in applications. In this chapter, we study several differential equations having the form $dy/dx = f(x, y)$, where f is a function of *both* the independent and dependent variables. We use the theory of indefinite integration to solve these differential equations, and investigate analytic, graphical, and numerical solution methods.

9.1 Slope Fields and Separable Differential Equations

Historical Biography

Jules Henri Poincaré (1854–1912)

Biography

In calculating derivatives by implicit differentiation (Section 3.6), we found that the expression for the derivative dy/dx often contained both variables x and y, not just the independent variable x. We begin this section by considering the general differential equation $dy/dx = f(x, y)$ and what is meant by a solution to it. Then we investigate equations having a special form for which the function f can be expressed as a product of a function of x and a function of y.

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first-order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \quad \text{and} \quad \frac{d}{dx}y = f(x, y),$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution. Establishing that a solution *is* the general solution may require deeper results from the theory of differential equations and is best studied in a more advanced course.

EXAMPLE 1 Verifying Solution Functions

Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

Solution Differentiating $y = C/x + 2$ gives

$$\frac{dy}{dx} = C\frac{d}{dx}\left(\frac{1}{x}\right) + 0 = -\frac{C}{x^2}.$$

Thus we need only verify that for all $x \in (0, \infty)$,

$$-\frac{C}{x^2} = \frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right].$$

This last equation follows immediately by expanding the expression on the right side:

$$\frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right] = \frac{1}{x}\left(-\frac{C}{x}\right) = -\frac{C}{x^2}.$$

Therefore, for every value of C, the function $y = C/x + 2$ is a solution of the differential equation. ■

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

EXAMPLE 2 Verifying That a Function Is a Particular Solution

Video Video

Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

On the right:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

The graph of the function is shown in Figure 9.1. ■

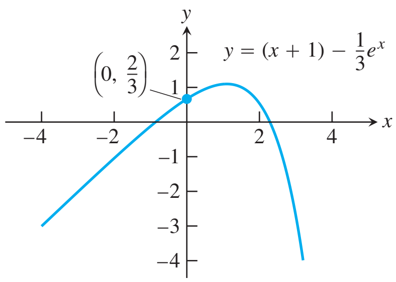

FIGURE 9.1 Graph of the solution $y = (x + 1) - \frac{1}{3}e^x$ to the differential equation $dy/dx = y - x$, with initial condition $y(0) = \frac{2}{3}$ (Example 2).

Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 9.2a shows a slope field, with a particular solution sketched into it in Figure 9.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.

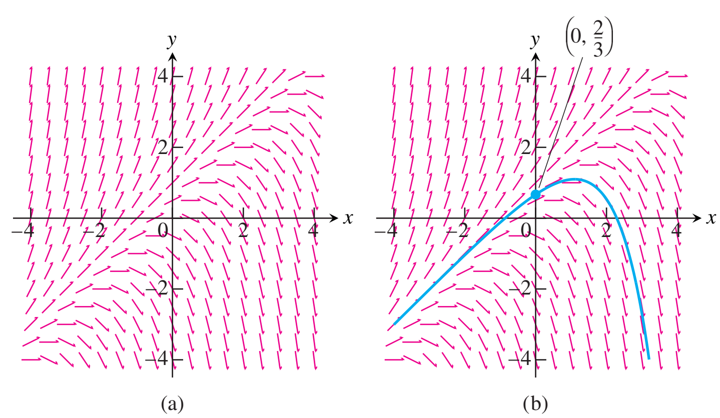

FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

Figure 9.3 shows three slope fields and we see how the solution curves behave by following the tangent line segments in these fields.

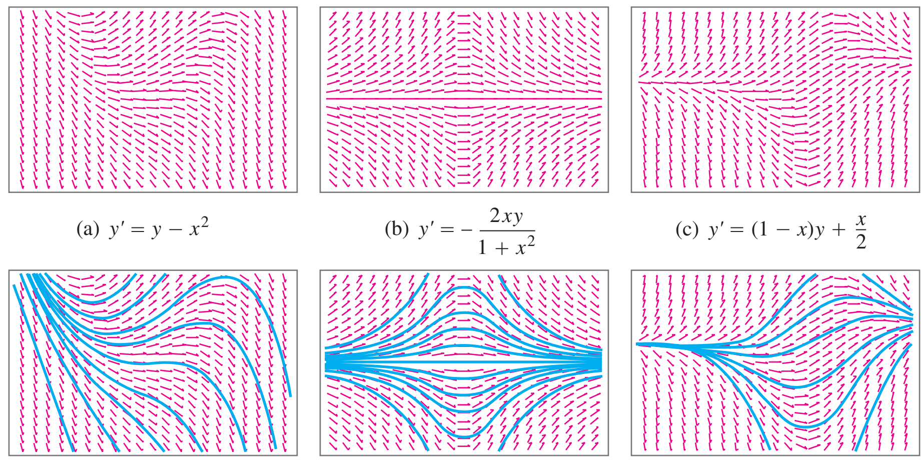

FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.

Constructing a slope field with pencil and paper can be quite tedious. All our examples were generated by a computer.

While general differential equations are difficult to solve, many important equations that arise in science and applications have special forms that make them solvable by special techniques. One such class is the separable equations.

Separable Equations

The equation $y' = f(x, y)$ is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \qquad H(y) = \frac{1}{h(y)}$$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\, dy = g(x)\, dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\, dy = \int g(x)\, dx. \tag{2}$$

After completing the integrations we obtain the solution y defined implicitly as a function of x.

The justification that we can simply integrate both sides in Equation (2) is based on the Substitution Rule (Section 5.5):

$$\int h(y)\,dy = \int h(y(x))\frac{dy}{dx}\,dx$$

$$= \int h(y(x))\frac{g(x)}{h(y(x))}\,dx \qquad \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$= \int g(x)\,dx.$$

EXAMPLE 3 Solving a Separable Equation

Solve the differential equation

$$\frac{dy}{dx} = (1 + y^2)e^x.$$

Solution Since $1 + y^2$ is never zero, we can solve the equation by separating the variables.

$$\frac{dy}{dx} = (1 + y^2)e^x$$

$$dy = (1 + y^2)e^x\,dx \qquad \text{Treat } dy/dx \text{ as a quotient of differentials and multiply both sides by } dx.$$

$$\frac{dy}{1 + y^2} = e^x\,dx \qquad \text{Divide by } (1 + y^2).$$

$$\int \frac{dy}{1 + y^2} = \int e^x\,dx \qquad \text{Integrate both sides.}$$

$$\tan^{-1} y = e^x + C \qquad C \text{ represents the combined constants of integration.}$$

The equation $\tan^{-1} y = e^x + C$ gives y as an implicit function of x. When $-\pi/2 < e^x + C < \pi/2$, we can solve for y as an explicit function of x by taking the tangent of both sides:

$$\tan(\tan^{-1} y) = \tan(e^x + C)$$

$$y = \tan(e^x + C).$$

■

EXAMPLE 4 Solve the equation

$$(x + 1)\frac{dy}{dx} = x(y^2 + 1).$$

Solution We change to differential form, separate the variables, and integrate:

$$(x + 1)\,dy = x(y^2 + 1)\,dx$$

$$\frac{dy}{y^2 + 1} = \frac{x\,dx}{x + 1} \qquad x \neq -1$$

$$\int \frac{dy}{1 + y^2} = \int \left(1 - \frac{1}{x + 1}\right)dx$$

$$\tan^{-1} y = x - \ln|x + 1| + C.$$

■

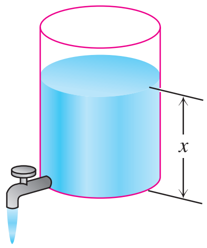

FIGURE 9.4 The rate at which water runs out is $k\sqrt{x}$, where k is a positive constant. In Example 5, $k = 1/2$ and x is measured in feet.

HISTORICAL BIOGRAPHY

Evangelista Torricelli (1608–1647)

Biography

The initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

involves a separable differential equation, and the solution $y = y_0 e^{kt}$ gives the Law of Exponential Change (Section 7.5). We found this initial value problem to be a model for such phenomena as population growth, radioactive decay, and heat transfer. We now present an application involving a different separable first-order equation.

Torricelli's Law

Torricelli's Law says that if you drain a tank like the one in Figure 9.4, the rate at which the water runs out is a constant times the square root of the water's depth x. The constant depends on the size of the drainage hole. In Example 5, we assume that the constant is $1/2$.

EXAMPLE 5 Draining a Tank

A right circular cylindrical tank with radius 5 ft and height 16 ft that was initially full of water is being drained at the rate of $0.5\sqrt{x}\ \text{ft}^3/\text{min}$. Find a formula for the depth and the amount of water in the tank at any time t. How long will it take to empty the tank?

Solution The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$, so the volume of water in the tank (Figure 9.4) is

$$V = \pi r^2 h = \pi(5)^2 x = 25\pi x.$$

Diffentiation leads to

$$\frac{dV}{dt} = 25\pi \frac{dx}{dt} \qquad \text{Negative because } V \text{ is decreasing and } dx/dt < 0$$

$$-0.5\sqrt{x} = 25\pi \frac{dx}{dt} \qquad \text{Torricelli's Law}$$

Thus we have the initial value problem

$$\frac{dx}{dt} = -\frac{\sqrt{x}}{50\pi},$$

$$x(0) = 16 \qquad \text{The water is 16 ft deep when } t = 0.$$

We solve the differential equation by separating the variables.

$$x^{-1/2}\,dx = -\frac{1}{50\pi}\,dt$$

$$\int x^{-1/2}\,dx = -\int \frac{1}{50\pi}\,dt \qquad \text{Integrate both sides.}$$

$$2x^{1/2} = -\frac{1}{50\pi}t + C \qquad \text{Constants combined}$$

The initial condition $x(0) = 16$ determines the value of C.

$$2(16)^{1/2} = -\frac{1}{50\pi}(0) + C$$

$$C = 8$$

With $C = 8$, we have

$$2x^{1/2} = -\frac{1}{50\pi}t + 8 \qquad \text{or} \qquad x^{1/2} = 4 - \frac{t}{100\pi}.$$

The formulas we seek are

$$x = \left(4 - \frac{t}{100\pi}\right)^2 \qquad \text{and} \qquad V = 25\pi x = 25\pi\left(4 - \frac{t}{100\pi}\right)^2.$$

At any time t, the water in the tank is $(4 - t/(100\pi))^2$ ft deep and the amount of water is $25\pi(4 - t/(100\pi))^2$ ft^3. At $t = 0$, we have $x = 16$ ft and $V = 400\pi$ ft^3, as required. The tank will empty $(V = 0)$ in $t = 400\pi$ minutes, which is about 21 hours. ■