9.3 Euler's Method

HISTORICAL BIOGRAPHY

Leonhard Euler
(1703–1783)

If we do not require or cannot immediately find an *exact* solution for an initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ we can often use a computer to generate a table of approximate numerical values of y for values of x in an appropriate interval. Such a table is called a **numerical solution** of the problem, and the method by which we generate the table is called a **numerical method**. Numerical methods are generally fast and accurate, and they are often the methods of choice when exact formulas are unnecessary, unavailable, or overly complicated. In this section, we study one such method, called Euler's method, upon which many other numerical methods are based.

Euler's Method

Given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \qquad \text{or} \qquad L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

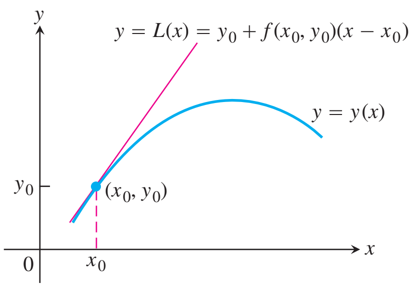

FIGURE 9.8 The linearization $L(x)$ of $y = y(x)$ at $x = x_0$.

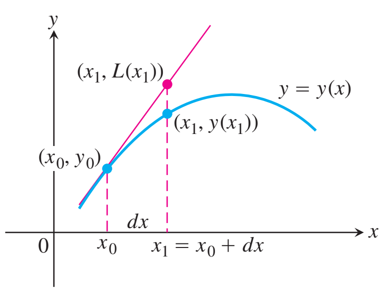

FIGURE 9.9 The first Euler step approximates $y(x_1)$ with $y_1 = L(x_1)$.

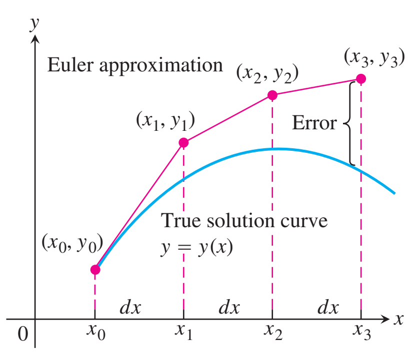

FIGURE 9.10 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

The function $L(x)$ gives a good approximation to the solution $y(x)$ in a short interval about x_0 (Figure 9.8). The basis of Euler's method is to patch together a string of linearizations to approximate the curve over a longer stretch. Here is how the method works.

We know the point (x_0, y_0) lies on the solution curve. Suppose that we specify a new value for the independent variable to be $x_1 = x_0 + dx$. (Recall that $dx = \Delta x$ in the definition of differentials.) If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)\, dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0), which lies *exactly* on the solution curve, we have obtained the point (x_1, y_1), which lies very close to the point $(x_1, y(x_1))$ on the solution curve (Figure 9.9).

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1), we take a second step. Setting $x_2 = x_1 + dx$, we use the linearization of the solution curve through (x_1, y_1) to calculate

$$y_2 = y_1 + f(x_1, y_1)\, dx.$$

This gives the next approximation (x_2, y_2) to values along the solution curve $y = y(x)$ (Figure 9.10). Continuing in this fashion, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the third approximation

$$y_3 = y_2 + f(x_2, y_2)\, dx,$$

and so on. We are literally building an approximation to one of the solutions by following the direction of the slope field of the differential equation.

The steps in Figure 9.10 are drawn large to illustrate the construction process, so the approximation looks crude. In practice, dx would be small enough to make the red curve hug the blue one and give a good approximation throughout.

EXAMPLE 1 Using Euler's Method

Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1,$$

starting at $x_0 = 0$ with $dx = 0.1$.

Solution We have $x_0 = 0, y_0 = 1, x_1 = x_0 + dx = 0.1, x_2 = x_0 + 2dx = 0.2$, and $x_3 = x_0 + 3dx = 0.3$.

$$\begin{aligned}
\textit{First:}\quad y_1 &= y_0 + f(x_0, y_0)\, dx \\
&= y_0 + (1 + y_0)\, dx \\
&= 1 + (1 + 1)(0.1) = 1.2 \\
\textit{Second:}\quad y_2 &= y_1 + f(x_1, y_1)\, dx \\
&= y_1 + (1 + y_1)\, dx \\
&= 1.2 + (1 + 1.2)(0.1) = 1.42 \\
\textit{Third:}\quad y_3 &= y_2 + f(x_2, y_2)\, dx \\
&= y_2 + (1 + y_2)\, dx \\
&= 1.42 + (1 + 1.42)(0.1) = 1.662
\end{aligned}$$

■

The step-by-step process used in Example 1 can be continued easily. Using equally spaced values for the independent variable in the table and generating n of them, set

$$\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx.
\end{aligned}$$

You Try It

Video

Then calculate the approximations to the solution,

$$y_1 = y_0 + f(x_0, y_0)\,dx$$
$$y_2 = y_1 + f(x_1, y_1)\,dx$$
$$\vdots$$
$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\,dx.$$

The number of steps n can be as large as we like, but errors can accumulate if n is too large.

Euler's method is easy to implement on a computer or calculator. A computer program generates a table of numerical solutions to an initial value problem, allowing us to input x_0 and y_0, the number of steps n, and the step size dx. It then calculates the approximate solution values $y_1, y_2, \ldots, y_n$ in iterative fashion, as just described.

Solving the separable equation in Example 1, we find that the exact solution to the initial value problem is $y = 2e^x - 1$. We use this information in Example 2.

You Try It

EXAMPLE 2 Investigating the Accuracy of Euler's Method

Use Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking

(a) $dx = 0.1$

(b) $dx = 0.05$.

Compare the approximations with the values of the exact solution $y = 2e^x - 1$.

Solution

(a) We used a computer to generate the approximate values in Table 9.1. The "error" column is obtained by subtracting the unrounded Euler values from the unrounded values found using the exact solution. All entries are then rounded to four decimal places.

TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

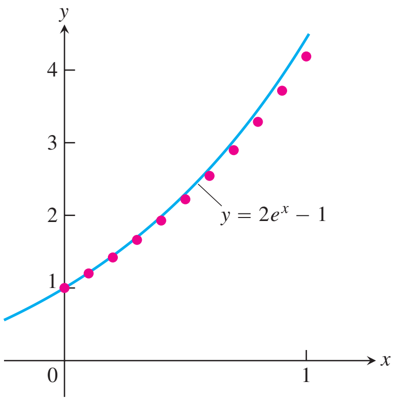

FIGURE 9.11 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1 (Example 2).

By the time we reach $x = 1$ (after 10 steps), the error is about 5.6% of the exact solution. A plot of the exact solution curve with the scatterplot of Euler solution points from Table 9.1 is shown in Figure 9.11.

(b) One way to try to reduce the error is to decrease the step size. Table 9.2 shows the results and their comparisons with the exact solutions when we decrease the step size to 0.05, doubling the number of steps to 20. As in Table 9.1, all computations are performed before rounding. This time when we reach $x = 1$, the relative error is only about 2.9%.

TABLE 9.2 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.05$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287
0.45	2.1027	2.1366	0.0340
0.50	2.2578	2.2974	0.0397
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0853
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

■

It might be tempting to reduce the step size even further in Example 2 to obtain greater accuracy. Each additional calculation, however, not only requires additional computer time but more importantly adds to the buildup of round-off errors due to the approximate representations of numbers inside the computer.

The analysis of error and the investigation of methods to reduce it when making numerical calculations are important but are appropriate for a more advanced course. There are numerical methods more accurate than Euler's method, as you can see in a further study of differential equations. We study one improvement here.

HISTORICAL BIOGRAPHY

Carl Runge
(1856–1927)

Biography

Improved Euler's Method

We can improve on Euler's method by taking an average of two slopes. We first estimate y_n as in the original Euler method, but denote it by z_n. We then take the average of $f(x_{n-1}, y_{n-1})$ and $f(x_n, z_n)$ in place of $f(x_{n-1}, y_{n-1})$ in the next step. Thus, we calculate the next approximation y_n using

$$z_n = y_{n-1} + f(x_{n-1}, y_{n-1})\, dx$$

$$y_n = y_{n-1} + \left[\frac{f(x_{n-1}, y_{n-1}) + f(x_n, z_n)}{2}\right] dx.$$

You Try It

EXAMPLE 3 Investigating the Accuracy of the Improved Euler's Method

Use the improved Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking $dx = 0.1$. Compare the approximations with the values of the exact solution $y = 2e^x - 1$.

Solution We used a computer to generate the approximate values in Table 9.3. The "error" column is obtained by subtracting the unrounded improved Euler values from the unrounded values found using the exact solution. All entries are then rounded to four decimal places.

TABLE 9.3 Improved Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (improved Euler)	y (exact)	Error
0	1	1	0
0.1	1.21	1.2103	0.0003
0.2	1.4421	1.4428	0.0008
0.3	1.6985	1.6997	0.0013
0.4	1.9818	1.9836	0.0018
0.5	2.2949	2.2974	0.0025
0.6	2.6409	2.6442	0.0034
0.7	3.0231	3.0275	0.0044
0.8	3.4456	3.4511	0.0055
0.9	3.9124	3.9192	0.0068
1.0	4.4282	4.4366	0.0084

By the time we reach $x = 1$ (after 10 steps), the relative error is about 0.19%. ■

By comparing Tables 9.1 and 9.3, we see that the improved Euler's method is considerably more accurate than the regular Euler's method, at least for the initial value problem $y' = 1 + y, \quad y(0) = 1$.

EXAMPLE 4 Oil Refinery Storage Tank Revisited

In Example 6, Section 9.2, we looked at a problem involving an additive mixture entering a 2000-gallon gasoline tank that was simultaneously being pumped. The analysis gave the initial value problem

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}, \qquad y(0) = 100$$

where $y(t)$ is the amount of additive in the tank at time t. The question was to find $y(20)$. Using Euler's method with an increment of $dt = 0.2$ (or 100 steps) gives the approximations

$$y(0.2) \approx 115.55, \qquad y(0.4) \approx 131.0298, \ldots$$

ending with $y(20) \approx 1344.3616$. The relative error from the exact solution $y(20) = 1342$ is about 0.18%. ■