9.5 Applications of First-Order Differential Equations

We now look at three applications of the differential equations we have been studying. The first application analyzes an object moving along a straight line while subject to a force opposing its motion. The second is a model of population growth which takes into account factors in the environment placing limits on growth, such as the availability of food or other vital resources. The last application considers a curve or curves intersecting each curve in a second family of curves *orthogonally* (that is, at right angles).

Resistance Proportional to Velocity

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. To describe this in mathematical terms, we picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

We can express the assumption that the resisting force is proportional to velocity by writing

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 7.5)

$$v = v_0 e^{-(k/m)t}. \tag{1}$$

What can we learn from Equation (1)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero (because t must be large in the exponent of the equation in order to make kt/m large enough for v to be small). We can learn even more if we integrate Equation (1) to find the position s as a function of time t.

Suppose that a body is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

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The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)\\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

This is an ideal figure, of course. Only in mathematics can time stretch to infinity. The number $v_0 m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least: if m is large, it will take a lot of energy to stop the body. That is why ocean liners have to be docked by tugboats. Any liner of conventional design entering a slip with enough speed to steer would smash into the pier before it could stop.

EXAMPLE 1 A Coasting Ice Skater

In the English system, where weight is measured in pounds, mass is measured in **slugs**. Thus,

$$\text{Pounds} = \text{slugs} \times 32,$$

assuming the gravitational constant is 32 ft/sec^2.

For a 192-lb ice skater, the k in Equation (1) is about 1/3 slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

You Try It

Solution We answer the first question by solving Equation (1) for t:

$$\begin{aligned}11e^{-t/18} &= 1 && \text{Eq. (1) with } k = 1/3,\ m = 6,\ v_0 = 11,\ v = 1\\ e^{-t/18} &= 1/11\\ -t/18 &= \ln(1/11) = -\ln 11\\ t &= 18\ln 11 \approx 43 \text{ sec}.\end{aligned}$$

We answer the second question with Equation (3):

$$\begin{aligned}\text{Distance coasted} &= \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3}\\ &= 198 \text{ ft}.\end{aligned}$$

■

Modeling Population Growth

In Section 7.5 we modeled population growth with the Law of Exponential Change:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

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where P is the population at time t, $k > 0$ is a constant growth rate, and P_0 is the size of the population at time $t = 0$. In Section 7.5 we found the solution $P = P_0 e^{kt}$ to this model. However, an issue to be addressed is "how good is the model?"

To begin an assessment of the model, notice that the exponential growth differential equation says that

$$\frac{dP/dt}{P} = k \tag{4}$$

is constant. This rate is called the **relative growth rate**. Now, Table 9.4 gives the world population at midyear for the years 1980 to 1989. Taking $dt = 1$ and $dP \approx \Delta P$, we see from the table that the relative growth rate in Equation (4) is approximately the constant 0.017. Thus, based on the tabled data with $t = 0$ representing 1980, $t = 1$ representing 1981, and so forth, the world population could be modeled by

$$\text{Differential equation:} \quad \frac{dP}{dt} = 0.017P$$

$$\text{Initial condition:} \quad P(0) = 4454.$$

TABLE 9.4 World population (midyear)

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 1999): www.census.gov/ipc/www/worldpop.html.

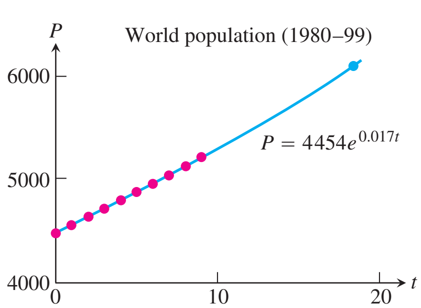

FIGURE 9.23 Notice that the value of the solution $P = 4454e^{0.017t}$ is 6152.16 when $t = 19$, which is slightly higher than the actual population in 1999.

The solution to this initial value problem gives the population function $P = 4454e^{0.017t}$. In year 1999 (so $t = 19$), the solution predicts the world population in midyear to be about 6152 million, or 6.15 billion (Figure 9.23), which is more than the actual population of 6001 million given by the U.S. Bureau of the Census (Table 9.5). Let's examine more recent data to see if there is a change in the growth rate.

Table 9.5 shows the world population for the years 1990 to 2002. From the table we see that the relative growth rate is positive but decreases as the population increases due to

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environmental, economic, and other factors. On average, the growth rate decreases by about 0.0003 per year over the years 1990 to 2002. That is, the graph of k in Equation (4) is closer to being a line with a negative slope $-r = -0.0003$. In Example 5 of Section 9.4 we proposed the more realistic **logistic growth model**

$$\frac{dP}{dt} = r(M - P)P, \tag{5}$$

where M is the maximum population, or **carrying capacity**, that the environment is capable of sustaining in the long run. Comparing Equation (5) with the exponential model, we see that $k = r(M - P)$ is a linearly decreasing function of the population rather than a constant. The graphical solution curves to the logistic model of Equation (5) were obtained in Section 9.4 and are displayed (again) in Figure 9.24. Notice from the graphs that if $P < M$, the population grows toward M; if $P > M$, the growth rate will be negative (as $r > 0$, $M > 0$) and the population decreasing.

TABLE 9.5 Recent world population

Year	Population (millions)	$\Delta P/P$
1990	5275	$84/5275 \approx 0.0159$
1991	5359	$84/5359 \approx 0.0157$
1992	5443	$81/5443 \approx 0.0149$
1993	5524	$81/5524 \approx 0.0147$
1994	5605	$80/5605 \approx 0.0143$
1995	5685	$79/5685 \approx 0.0139$
1996	5764	$80/5764 \approx 0.0139$
1997	5844	$79/5844 \approx 0.0135$
1998	5923	$78/5923 \approx 0.0132$
1999	6001	$78/6001 \approx 0.0130$
2000	6079	$73/6079 \approx 0.0120$
2001	6152	$76/6152 \approx 0.0124$
2002	6228	?
2003	?	

Source: U.S. Bureau of the Census (Sept., 2003): www.census.gov/ipc/www/worldpop.html.

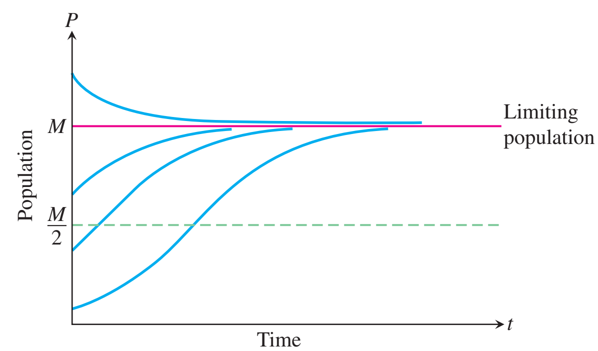

FIGURE 9.24 Solution curves to the logistic population model $dP/dt = r(M - P)P$.

EXAMPLE 2 Modeling a Bear Population

A national park is known to be capable of supporting 100 grizzly bears, but no more. Ten bears are in the park at present. We model the population with a logistic differential equation with $r = 0.001$ (although the model may not give reliable results for very small population levels).

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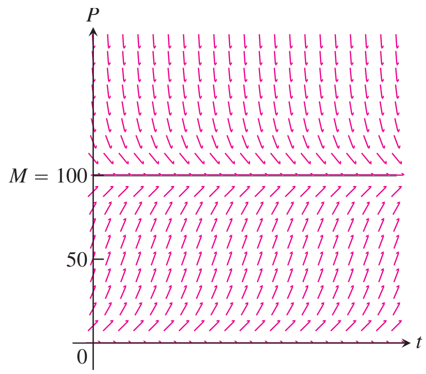

FIGURE 9.25 A slope field for the logistic differential equation $dP/dt = 0.001(100 - P)P$ (Example 2).

(a) Draw and describe a slope field for the differential equation.

(b) Use Euler's method with step size $dt = 1$ to estimate the population size in 20 years.

(c) Find a logistic growth analytic solution $P(t)$ for the population and draw its graph.

(d) When will the bear population reach 50?

Solution

(a) *Slope field*. The carrying capacity is 100, so $M = 100$. The solution we seek is a solution to the following differential equation.

$$\frac{dP}{dt} = 0.001(100 - P)P$$

Figure 9.25 shows a slope field for this differential equation. There appears to be a horizontal asymptote at $P = 100$. The solution curves fall toward this level from above and rise toward it from below.

(b) *Euler's method*. With step size $dt = 1$, $t_0 = 0$, $P(0) = 10$, and

$$\frac{dP}{dt} = f(t, P) = 0.001(100 - P)P,$$

we obtain the approximations in Table 9.6, using the iteration formula

$$P_n = P_{n-1} + 0.001(100 - P_{n-1})P_{n-1}.$$

TABLE 9.6 Euler solution of $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$

t	P (Euler)	t	P (Euler)
0	10		
1	10.9	11	24.3629
2	11.8712	12	26.2056
3	12.9174	13	28.1395
4	14.0423	14	30.1616
5	15.2493	15	32.2680
6	16.5417	16	34.4536
7	17.9222	17	36.7119
8	19.3933	18	39.0353
9	20.9565	19	41.4151
10	22.6130	20	43.8414

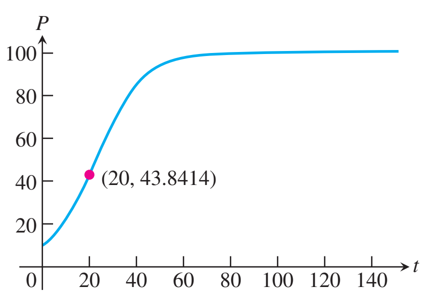

FIGURE 9.26 Euler approximations of the solution to $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$.

There are approximately 44 grizzly bears after 20 years. Figure 9.26 shows a graph of the Euler approximation over the interval $0 \le t \le 150$ with step size $dt = 1$. It looks like the lower curves we sketched in Figure 9.24.

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(c) *Analytic solution*. We can assume that $t = 0$ when the bear population is 10, so $P(0) = 10$. The logistic growth model we seek is the solution to the following initial value problem.

Differential equation: $\frac{dP}{dt} = 0.001(100 - P)P$

Initial condition: $P(0) = 10$

To prepare for integration, we rewrite the differential equation in the form

$$\frac{1}{P(100 - P)} \frac{dP}{dt} = 0.001.$$

Using partial fraction decomposition on the left-hand side and multiplying both sides by 100, we get

$$\left(\frac{1}{P} + \frac{1}{100 - P}\right) \frac{dP}{dt} = 0.1$$

$$\ln|P| - \ln|100 - P| = 0.1t + C \qquad \text{Integrate with respect to } t.$$

$$\ln\left|\frac{P}{100 - P}\right| = 0.1t + C$$

$$\ln\left|\frac{100 - P}{P}\right| = -0.1t - C \qquad \ln\frac{a}{b} = -\ln\frac{b}{a}$$

$$\left|\frac{100 - P}{P}\right| = e^{-0.1t-C} \qquad \text{Exponentiate.}$$

$$\frac{100 - P}{P} = (\pm e^{-C})e^{-0.1t}$$

$$\frac{100}{P} - 1 = Ae^{-0.1t} \qquad \text{Let } A = \pm e^{-c}.$$

$$P = \frac{100}{1 + Ae^{-0.1t}}. \qquad \text{Solve for } P.$$

This is the general solution to the differential equation. When $t = 0$, $P = 10$, and we obtain

$$10 = \frac{100}{1 + Ae^{0}}$$

$$1 + A = 10$$

$$A = 9.$$

Thus, the logistic growth model is

$$P = \frac{100}{1 + 9e^{-0.1t}}.$$

Its graph (Figure 9.27) is superimposed on the slope field from Figure 9.25.

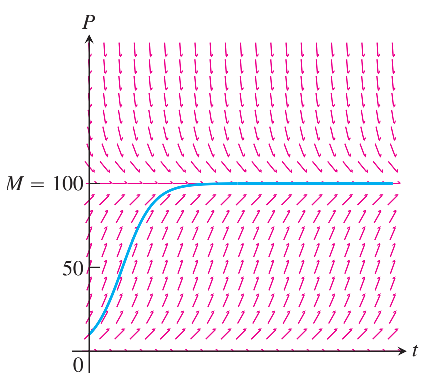

FIGURE 9.27 The graph of

$$P = \frac{100}{1 + 9e^{-0.1t}}$$

superimposed on the slope field in Figure 9.25 (Example 2).

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(d) When will the bear population reach 50? For this model,

$$\begin{aligned} 50 &= \frac{100}{1 + 9e^{-0.1t}} \\ 1 + 9e^{-0.1t} &= 2 \\ e^{-0.1t} &= \frac{1}{9} \\ e^{0.1t} &= 9 \\ t &= \frac{\ln 9}{0.1} \approx 22 \text{ years.} \end{aligned}$$

The solution of the general logistic differential equation

$$\frac{dP}{dt} = r(M - P)P$$

can be obtained as in Example 2. In Exercise 10, we ask you to show that the solution is

$$P = \frac{M}{1 + Ae^{-rMt}}.$$

The value of A is determined by an appropriate initial condition.

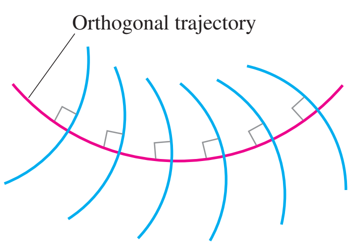

FIGURE 9.28 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally* (Figure 9.28). For instance, each straight line through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$, centered at the origin (Figure 9.29). Such mutually orthogonal systems of curves are of particular importance in physical problems related to electrical potential, where the curves in one family correspond to flow of electric current and those in the other family correspond to curves of constant potential. They also occur in hydrodynamics and heat-flow problems.

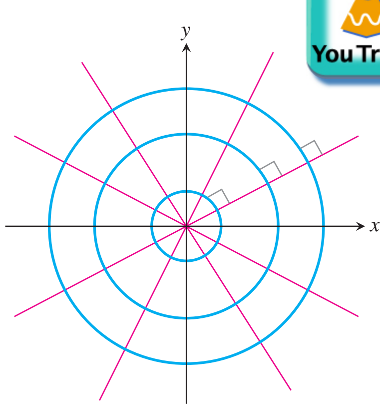

FIGURE 9.29 Every straight line through the origin is orthogonal to the family of circles centered at the origin.

EXAMPLE 3 Finding Orthogonal Trajectories

Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas with asymptotes $y = \pm x$. First we find the slopes of each curve in this family, or their dy/dx values. Differentiating $xy = a$ implicitly gives

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Thus the slope of the tangent line at any point (x, y) on one of the hyperbolas $xy = a$ is $y' = -y/x$. On an orthogonal trajectory the slope of the tangent line at this same point must be the negative reciprocal, or x/y. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

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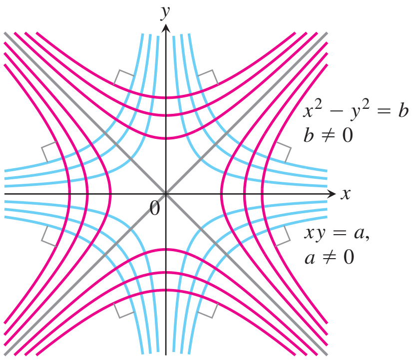

FIGURE 9.30 Each curve is orthogonal to every curve it meets in the other family (Example 3).

This differential equation is separable and we solve it as in Section 9.1:

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 - x^2 = b, \tag{6}$$

where $b = 2C$ is an arbitrary constant. The orthogonal trajectories are the family of hyperbolas given by Equation (6) and sketched in Figure 9.30. ■

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