

DERIVATIVES

In the following formulas u, v, w represent functions of x , while a, c, n represent fixed real numbers. All arguments in the trigonometric functions are measured in radians, and all inverse trigonometric and hyperbolic functions represent principal values

1. $\frac{d}{dx}(a) = 0$
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(au) = a \frac{du}{dx}$
4. $\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$
5. $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$
6. $\frac{d}{dx}(uvw) = uv \frac{dw}{dx} + vw \frac{du}{dx} + uw \frac{dv}{dx}$
7. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{1}{v} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx}$
8. $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$
9. $\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}} \frac{du}{dx}$
10. $\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2} \frac{du}{dx}$
11. $\frac{d}{dx}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}} \frac{du}{dx}$
12. $\frac{d}{dx}\left(\frac{u^n}{v^m}\right) = \frac{u^{n-1}}{v^{m+1}} \left(n v \frac{du}{dx} - m u \frac{dv}{dx} \right)$
13. $\frac{d}{dx}(u^n v^m) = u^{n-1} v^{m-1} \left(n v \frac{du}{dx} + m u \frac{dv}{dx} \right)$
14. $\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$

*Let $y = f(x)$ and $\frac{dy}{dx} = \frac{d[f(x)]}{dx} = f'(x)$ define respectively a function and its derivative for any value x in their common domain. The differential for the function at such a value x is accordingly defined as

$$dy = d[f(x)] = \frac{dy}{dx} dx = \frac{d[f(x)]}{dx} dx = f'(x) dx$$

Each derivative formula has an associated differential formula. For example, formula 6 above has the differential formula

$$d(uvw) = uv dw + vw du + uw dv$$

15. $\frac{d^2}{dx^2}[f(u)] = \frac{df(u)}{du} \cdot \frac{d^2u}{dx^2} + \frac{d^2f(u)}{du^2} \cdot \left(\frac{du}{dx}\right)^2$
16. $\frac{d^n}{dx^n}[uv] = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1} u}{dx^{n-1}} + \binom{n}{2} \frac{d^2 v}{dx^2} \frac{d^{n-2} u}{dx^{n-2}} + \cdots + \binom{n}{k} \frac{d^k v}{dx^k} \frac{d^{n-k} u}{dx^{n-k}} + \cdots + \binom{n}{n} u \frac{d^n v}{dx^n}$
 where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the binomial coefficient, n non-negative integer, and $\binom{n}{0} = 1$.
17. $\frac{du}{dx} = \frac{1}{\frac{dx}{du}}$ if $\frac{dx}{du} \neq 0$
18. $\frac{d}{dx}(\log_a u) = (\log_a e) \frac{1}{u} \frac{du}{dx}$

19. $\frac{d}{dx}(\log_e u) = \frac{1}{u} \frac{du}{dx}$
20. $\frac{d}{dx}(a^u) = a^u (\log_e a) \frac{du}{dx}$
21. $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$
22. $\frac{d}{dx}(u^v) = v u^{v-1} \frac{du}{dx} + (\log_e u) u^v \frac{dv}{dx}$
23. $\frac{d}{dx}(\sin u) = (\cos u) \frac{du}{dx}$
24. $\frac{d}{dx}(\cos u) = -(\sin u) \frac{du}{dx}$
25. $\frac{d}{dx}(\tan u) = (\sec^2 u) \frac{du}{dx}$
26. $\frac{d}{dx}(\cot u) = -(\csc^2 u) \frac{du}{dx}$
27. $\frac{d}{dx}(\sec u) = \sec u \cdot \tan u \frac{du}{dx}$
28. $\frac{d}{dx}(\csc u) = -\csc u \cdot \cot u \frac{du}{dx}$
29. $\frac{d}{dx}(\text{vers } u) = \sin u \frac{du}{dx}$
30. $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad \left(-\frac{\pi}{2} \leq \arcsin u \leq \frac{\pi}{2}\right)$
31. $\frac{d}{dx}(\arccos u) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad (0 \leq \arccos u \leq \pi)$
32. $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2} \frac{du}{dx}, \quad \left(-\frac{\pi}{2} < \arctan u < \frac{\pi}{2}\right)$
33. $\frac{d}{dx}(\text{arc cot } u) = -\frac{1}{1+u^2} \frac{du}{dx}, \quad (0 \leq \text{arc cot } u \leq \pi)$
34. $\frac{d}{dx}(\text{arc sec } u) = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}, \quad \left(0 \leq \text{arc sec } u < \frac{\pi}{2}, -\pi \leq \text{arc sec } u < -\frac{\pi}{2}\right)$
35. $\frac{d}{dx}(\text{arc csc } u) = -\frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}, \quad \left(0 < \text{arc csc } u \leq \frac{\pi}{2}, -\pi < \text{arc csc } u \leq -\frac{\pi}{2}\right)$
36. $\frac{d}{dx}(\text{arc vers } u) = \frac{1}{\sqrt{2u-u^2}} \frac{du}{dx}, \quad (0 \leq \text{arc vers } u \leq \pi)$
37. $\frac{d}{dx}(\sinh u) = (\cosh u) \frac{du}{dx}$
38. $\frac{d}{dx}(\cosh u) = (\sinh u) \frac{du}{dx}$
39. $\frac{d}{dx}(\tanh u) = (\text{sech}^2 u) \frac{du}{dx}$
40. $\frac{d}{dx}(\text{coth } u) = -(\text{csch}^2 u) \frac{du}{dx}$
41. $\frac{d}{dx}(\text{sech } u) = -(\text{sech } u \cdot \tanh u) \frac{du}{dx}$
42. $\frac{d}{dx}(\text{csch } u) = -(\text{csch } u \cdot \text{coth } u) \frac{du}{dx}$
43. $\frac{d}{dx}(\sinh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2 + 1})] = \frac{1}{\sqrt{u^2 + 1}} \frac{du}{dx}$
44. $\frac{d}{dx}(\cosh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2 - 1})] = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \quad (u > 1, \cosh^{-1} u > 0)$

45. $\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx} \left[\frac{1}{2} \log \frac{1+u}{1-u} \right] = \frac{1}{1-u^2} \frac{du}{dx}, \quad (u^2 < 1)$
46. $\frac{d}{dx}(\coth^{-1} u) = \frac{d}{dx} \left[\frac{1}{2} \log \frac{u+1}{u-1} \right] = \frac{1}{1-u^2} \frac{du}{dx}, \quad (u^2 > 1)$
47. $\frac{d}{dx}(\operatorname{sech}^{-1} u) = \frac{d}{dx} \left[\log \frac{1+\sqrt{1-u^2}}{u} \right] = -\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad (0 < u < 1, \operatorname{sech}^{-1} u > 0)$
48. $\frac{d}{dx}(\operatorname{csch}^{-1} u) = \frac{d}{dx} \left[\log \frac{1+\sqrt{1+u^2}}{u} \right] = -\frac{1}{|u|\sqrt{1+u^2}} \frac{du}{dx}$
49. $\frac{d}{dq} \int_p^q f(x) dx = f(q), \quad [p \text{ constant}]$
50. $\frac{d}{dp} \int_p^q f(x) dx = -f(p), \quad [q \text{ constant}]$
51. $\frac{d}{da} \int_p^q f(x, a) dx = \int_p^q \frac{\partial}{\partial a} [f(x, a)] dx + f(q, a) \frac{dq}{da} - f(p, a) \frac{dp}{da}$

INTEGRATION

The following is a brief discussion of some integration techniques. A more complete discussion can be found in a number of good text books. However, the purpose of this introduction is simply to discuss a few of the important techniques which may be used, in conjunction with the integral table which follows, to integrate particular functions.

No matter how extensive the integral table, it is a fairly uncommon occurrence to find in the table the exact integral desired. Usually some form of transformation will have to be made. The simplest type of transformation, and yet the most general, is substitution. Simple forms of substitution, such as $y = ax$, are employed almost unconsciously by experienced users of integral tables. Other substitutions may require more thought. In some sections of the tables, appropriate substitutions are suggested for integrals which are similar to, but not exactly like, integrals in the table. Finding the right substitution is largely a matter of intuition and experience.

Several precautions must be observed when using substitutions:

1. Be sure to make the substitution in the dx term, as well as everywhere else in the integral.
2. Be sure that the function substituted is one-to-one and continuous. If this is not the case, the integral must be restricted in such a way as to make it true. See the example following.
3. With definite integrals, the limits should also be expressed in terms of the new dependent variable. With indefinite integrals, it is necessary to perform the reverse substitution to obtain the answer in terms of the original independent variable. This may also be done for definite integrals, but it is usually easier to change the limits.

Example:

$$\int \frac{x^4}{\sqrt{a^2 - x^2}} dx$$

Here we make the substitution $x = |a| \sin \theta$. Then $dx = |a| \cos \theta d\theta$, and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = |a| \sqrt{1 - \sin^2 \theta} = |a \cos \theta|$$

Notice the absolute value signs. It is very important to keep in mind that a square root radical always denotes the positive square root, and to assure the sign is always kept positive. Thus $\sqrt{x^2} = |x|$. Failure to observe this is a common cause of errors in integration.

Notice also that the indicated substitution is not a one-to-one function, that is, it does not have a unique inverse. Thus we must restrict the range of θ in such a way as to make the function one-to-one. Fortunately, this is easily done by solving for θ

$$\theta = \sin^{-1} \frac{x}{|a|}$$

and restricting the inverse sine to the principal values, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.