45. $\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{1+u}{1-u}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2 < 1)$

46. $\frac{d}{dx}(\coth^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{u+1}{u-1}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2 > 1)$

47. $\frac{d}{dx}(\text{sech}^{-1}\, u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1-u^2}}{u}\right] = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad (0 < u < 1,\ \text{sech}^{-1}\, u > 0)$

48. $\frac{d}{dx}(\text{csch}^{-1}\, u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1+u^2}}{u}\right] = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$

49. $\frac{d}{dq}\int_p^q f(x)\,dx = f(q), \quad [p \text{ constant}]$

50. $\frac{d}{dp}\int_p^q f(x)\,dx = -f(p), \quad [q \text{ constant}]$

51. $\frac{d}{da}\int_p^q f(x,a)\,dx = \int_p^q \frac{\partial}{\partial a}[f(x,a)]\,dx + f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da}$

INTEGRATION

The following is a brief discussion of some integration techniques. A more complete discussion can be found in a number of good text books. However, the purpose of this introduction is simply to discuss a few of the important techniques which may be used, in conjunction with the integral table which follows, to integrate particular functions.

No matter how extensive the integral table, it is a fairly uncommon occurrence to find in the table the exact integral desired. Usually some form of transformation will have to be made. The simplest type of transformation, and yet the most general, is substitution. Simple forms of substitution, such as $y = ax$, are employed almost unconsciously by experienced users of integral tables. Other substitutions may require more thought. In some sections of the tables, appropriate substitutions are suggested for integrals which are similar to, but not exactly like, integrals in the table. Finding the right substitution is largely a matter of intuition and experience.

Several precautions must be observed when using substitutions:

1. Be sure to make the substitution in the *dx* term, as well as everywhere else in the integral.
2. Be sure that the function substituted is one-to-one and continuous. If this is not the case, the integral must be restricted in such a way as to make it true. See the example following.
3. With definite integrals, the limits should also be expressed in terms of the new dependent variable. With indefinite integrals, it is necessary to perform the reverse substitution to obtain the answer in terms of the original independent variable. This may also be done for definite integrals, but it is usually easier to change the limits.

Example:

$$\int \frac{x^4}{\sqrt{a^2 - x^2}}\,dx$$

Here we make the substitution $x = |a|\sin\theta$. Then $dx = |a|\cos\theta\,d\theta$, and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = |a|\sqrt{1 - \sin^2\theta} = |a\cos\theta|$$

Notice the absolute value signs. It is very important to keep in mind that a square root radical always denotes the positive square root, and to assure the sign is always kept positive. Thus $\sqrt{x^2} = |x|$. Failure to observe this is a common cause of errors in integration.

Notice also that the indicated substitution is not a one-to-one function, that is, it does not have a unique inverse. Thus we must restrict the range of θ in such a way as to make the function one-to-one. Fortunately, this is easily done by solving for θ

$$\theta = \sin^{-1}\frac{x}{|a|}$$

and restricting the inverse sine to the principal values, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Thus the integral becomes

$$\int \frac{a^4 \sin^4\theta |a| \cos\theta \, d\theta}{|a|\,|\cos\theta|}$$

Now, however, in the range of values chosen for θ, $\cos\theta$ is always positive. Thus we may remove the absolute value signs from $\cos\theta$ in the denominator. (This is one of the reasons that the principal values of the inverse trigonometric functions are defined as they are.) Then the $\cos\theta$ terms cancel, and the integral becomes

$$a^4 \int \sin^4\theta \, d\theta$$

By application of integral formulas 299 and 296, we integrate this to

$$-a^4 \frac{\sin^3\theta\cos\theta}{4} - \frac{3a^4}{8}\cos\theta\sin\theta + \frac{3a^4}{8}\theta + C$$

We now must perform the inverse substitution to get the result in terms of x. We have

$$\theta = \sin^{-1}\frac{x}{|a|}$$

$$\sin\theta = \frac{x}{|a|}$$

Then

$$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-\frac{x^2}{a^2}} = \pm\frac{\sqrt{a^2-x^2}}{|a|}.$$

Because of the previously mentioned fact that $\cos\theta$ is positive, we may omit the $\pm$ sign. The reverse substitution then produces the final answer

$$\int \frac{x^4}{\sqrt{a^2-x^2}}\,dx = -\frac{1}{4}x^3\sqrt{a^2-x^2} - \frac{3}{8}a^2x\sqrt{a^2-x^2} + \frac{3}{8}a^4\sin^{-1}\frac{x}{|a|} + C.$$

Any rational function of x may be integrated, if the denominator is factored into linear and irreducible quadratic factors. The function may then be broken into partial fractions, and the individual partial fractions integrated by use of the appropriate formula from the integral table. See the section on partial fractions for further information.

Many integrals may be reduced to rational functions by proper substitutions. For example,

$$z = \tan\frac{x}{2}$$

will reduce any rational function of the six trigonometric functions of x to a rational function of z. (Frequently there are other substitutions which are simpler to use, but this one will always work. See integral formula number 484.)

Any rational function of x and $\sqrt{ax+b}$ may be reduced to a rational function of z by making the substitution

$$z = \sqrt{ax+b}.$$

Other likely substitutions will be suggested by looking at the form of the integrand.

The other main method of transforming integrals is integration by parts. This involves applying formula number 5 or 6 in the accompanying integral table. The critical factor in this method is the choice of the functions u and v. In order for the method to be successful, $v = \int dv$ and $\int v\,du$ must be easier to integrate than the original integral. Again, this choice is largely a matter of intuition and experience.

Example:

$$\int x\sin x\,dx$$

Two obvious choices are $u = x$, $dv = \sin x\,dx$, or $u = \sin x$, $dv = x\,dx$. Since a preliminary mental calculation indicates that $\int v\,du$ in the second choice would be more, rather than less, complicated than the original integral (it would contain x^2), we use the first choice.

$$u = x \qquad du = dx$$

$$dv = \sin x\,dx \qquad v = -\cos x$$

$$\int x\sin x\,dx = \int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x\,dx$$

$$= \sin x - x\cos x$$

Of course, this result could have been obtained directly from the integral table, but it provides a simple example of the method. In more complicated examples the choice of u and v may not be so obvious, and several different choices may have to be tried. Of course, there is no guarantee that any of them will work.

Integration by parts may be applied more than once, or combined with substitution. A fairly common case is illustrated by the following example.

Example:

$$\int e^x \sin x \, dx$$

Let

$$u = e^x \qquad \text{Then } du = e^x dx$$

$$dv = \sin x \, dx \qquad v = -\cos x$$

$$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = -e^x \cos x + \int e^x \cos x \, dx$$

In this latter integral,

$$\text{Let } u = e^x \qquad \text{Then } du = e^x dx$$

$$dv = \cos x \, dx \qquad v = \sin x$$

$$\begin{aligned}\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx &= -e^x \cos x + \int u \, dv \\ &= -e^x \cos x + uv - \int v \, du \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx\end{aligned}$$

This looks as if a circular transformation has taken place, since we are back at the same integral we started from. However, the above equation can be solved algebraically for the required integral:

$$\int e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x$$

In the second integration by parts, if the parts had been chosen as $u = \cos x$, $dv = e^x dx$, we would indeed have made a circular transformation, and returned to the starting place.
In general, when doing repeated integration by parts, one should never choose the function u at any stage to be the same as the function v at the previous stage, or a constant times the previous v

The following rule is called the extended rule for integration by parts. It is the result of $n+1$ successive applications of integration by parts. If

$$\begin{aligned} g_1(x) &= \int g(x) \, dx, \qquad g_2(x) = \int g_1(x) \, dx, \\ g_3(x) &= \int g_2(x) \, dx, \ldots, g_m(x) = \int g_{m-1}(x) \, dx, \ldots, \end{aligned}$$

then

$$\begin{aligned} \int f(x) \cdot g(x) \, dx &= f(x) \cdot g_1(x) - f'(x) \cdot g_2(x) + f''(x) \cdot g_3(x) - + \cdots \\ &\quad + (-1)^n f^{(n)}(x) g_{n+1}(x) + (-1)^{n+1} \int f^{(n+1)}(x) g_{n+1}(x) \, dx. \end{aligned}$$

A useful special case of the above rule is when $f(x)$ is a polynomial of degree n. Then $f^{(n+1)}(x) = 0$, and

$$\int f(x) \cdot g(x) \, dx = f(x) \cdot g_1(x) - f'(x) \cdot g_2(x) + f''(x) \cdot g_3(x) - + \cdots + (-1)^n f^{(n)}(x) g_{n+1}(x) + C$$

Example: If $f(x) = x^2$, $g(x) = \sin x$

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Another application of this formula occurs if

$$f''(x) = af(x) \qquad \text{and} \qquad g''(x) = bg(x),$$

where a and b are unequal constants. In this case, by a process similar to that used in the above example for $\int e^x \sin x\, dx$, we get the formula

$$\int f(x)g(x)\, dx = \frac{f(x) \cdot g'(x) - f'(x) \cdot g(x)}{b - a} + C$$

This formula could have been used in the example mentioned. Here is another example.

Example: If $f(x) = e^{2x}$, $g(x) = \sin 3x$, then $a = 4$, $b = -9$, and

$$\int e^{2x} \sin 3x\, dx = \frac{3\, e^{2x} \cos 3x - 2\, e^{2x} \sin 3x}{-9 - 4} + C = \frac{e^{2x}}{13}(2 \sin 3x - 3 \cos 3x) + C$$

The following additional points should be observed when using this table.

1. A constant of integration is to be supplied with the answers for indefinite integrals.
2. Logarithmic expressions are to base $e = 2.71828\ldots$, unless otherwise specified, and are to be evaluated for the absolute value of the arguments involved therein.
3. All angles are measured in radians, and inverse trigonometric and hyperbolic functions represent principal values, unless otherwise indicated.
4. If the application of a formula produces either a zero denominator or the square root of a negative number in the result, there is usually available another form of the answer which avoids this difficulty. In many of the results, the excluded values are specified, but when such are omitted it is presumed that one can tell what these should be, especially when difficulties of the type herein mentioned are obtained.
5. When inverse trigonometric functions occur in the integrals, be sure that any replacements made for them are strictly in accordance with the rules for such functions. This causes little difficulty when the argument of the inverse trigonometric function is positive, since then all angles involved are in the first quadrant. However, if the argument is negative, special care must be used. Thus if $u > 0$,

   $$\sin^{-1} u = \cos^{-1} \sqrt{1 - u^2} = \csc^{-1} \frac{1}{u}, \text{ etc.}$$

   However, if $u < 0$,

   $$\sin^{-1} u = -\cos^{-1} \sqrt{1 - u^2} = -\pi - \csc^{-1} \frac{1}{u}, \text{ etc.}$$

   See the section on inverse trigonometric functions for a full treatment of the allowable substitutions.
6. In integrals 340–345 and some others, the right side includes expressions of the form

   $$A \tan^{-1}[B + C \tan f(x)].$$

   In these formulas, the $\tan^{-1}$ does not necessarily represent the principal value. Instead of always employing the principal branch of the inverse tangent function, one must instead use that branch of the inverse tangent function upon which $f(x)$ lies for any particular choice of x.

   Example:

   $$\begin{aligned} \int_0^{4\pi} \frac{dx}{2 + \sin x} &= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2 \tan(x/2 + 1)}{\sqrt{3}} \Bigg]_0^{4\pi} \\ &= \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{2 \tan 2\pi + 1}{\sqrt{3}} - \tan^{-1} \frac{2 \tan 0 + 1}{\sqrt{3}} \right] \\ &= \frac{2}{\sqrt{3}} \left[\frac{13\pi}{6} - \frac{\pi}{6} \right] = \frac{4\pi}{\sqrt{3}} = \frac{4\sqrt{3}\pi}{3} \end{aligned}$$

Here

$$\tan^{-1}\frac{2\tan 2\pi+1}{\sqrt{3}}=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{13\pi}{6},$$

since $f(x)=2\pi$; and

$$\tan^{-1}\frac{2\tan 0+1}{\sqrt{3}}=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6},$$

since $f(x)=0$.

7. B_n and E_n where used in Integrals represents the Bernoulli and Euler numbers as defined in tables of Bernoulli and Euler polynomials contained in certain mathematics reference and hand-books.

INTEGRALS

ELEMENTARY FORMS

1. $\int a\,dx = ax$

2. $\int a\cdot f(x)\,dx = a\int f(x)\,dx$

3. $\int \phi(y)\,dx = \int \frac{\phi(y)}{y'}\,dy, \quad \text{where } y'=\frac{dy}{dx}$

4 $\int (u+v)\,dx = \int u\,dx + \int v\,dx$, where u and v are any functions of x

5. $\int u\,dv = u\int dv - \int v\,du = uv - \int v\,du$

6. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

7. $\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad \text{except } n=-1$

8. $\int \frac{f'(x)\,dx}{f(x)} = \log f(x), \quad (df(x)=f'(x)\,dx)$

9. $\int \frac{dx}{x} = \log x$

10. $\int \frac{f'(x)\,dx}{2\sqrt{f(x)}} = \sqrt{f(x)}, \quad (df(x)=f'(x)\,dx)$

11. $\int e^x\,dx = e^x$

12. $\int e^{ax}\,dx = e^{ax}/a$

13. $\int b^{ax}\,dx = \frac{b^{ax}}{a\log b}, \quad (b>0)$

14. $\int \log x\,dx = x\log x - x$

15. $\int a^x\log a\,dx = a^x, \quad (a>0)$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

17. $$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{a+x}{a-x}, \quad (a^2>x^2) \end{cases}$$

18. $$\int \frac{dx}{x^2-a^2} = \begin{cases} -\frac{1}{a}\coth^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{x-a}{x+a}, \quad (x^2>a^2) \end{cases}$$