THE FOURIER TRANSFORMS

For a piecewise continuous function $F(x)$ over a finite interval $0 \le x \le \pi$; the *finite Fourier cosine transform* of $F(x)$ is

$$f_c(n) = \int_0^{\pi} F(x) \cos nx \, dx \quad (n = 0,\ 1,\ 2, \ldots)$$

If x ranges over the interval $0 \le x \le L$, the substitution $x' = \pi x/L$ allows the use of this definition, also. The inverse transform is written.

$$\overline{F}(x) = \frac{1}{\pi} f_c(0) - \frac{2}{\pi} \sum_{n=1}^{x} f_c(n) \cos nx \quad (0 < x < \pi)$$

where $F(x) = \frac{F(x+\epsilon)+F(x-\epsilon)}{2}$. We observe that $F(x+) = F(x-) = F(x)$ at points of continuity. The formula

$$\begin{aligned} f_c^{(2)}(n) &= \int_0^{\pi} F''(x) \cos nx \, dx \\ &= -n^2 f_c(n) - F'(0) + (-1)^n F'(\pi) \end{aligned} \tag{1}$$

makes the finite Fourier cosine transform useful in certain boundary value problems. Analogously, the *finite Fourier sine transform* of $F(x)$ is

$$f_s(n) = \int_0^{\pi} F(x) \sin nx \, dx \quad (n = 1, 2, 3, \ldots)$$

and

$$\overline{F}(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx \quad (0 < x < \pi)$$

Corresponding to (1) we have

$$\begin{aligned} f_s^{(2)}(n) &= \int_0^{\pi} F''(x) \sin nx \, dx \\ &= -n^2 f_s(n) - n\, F(0) - n(-1)^n F(\pi) \end{aligned} \tag{2}$$

If $F(x)$ is defined for $x \le 0$ and is piecewise continuous over any finite interval, and if $\int_0^x F(x)\, dx$ is absolutely convergent, then

$$f_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{x} F(x) \cos(\alpha x) \, dx$$

is the *Fourier cosine transform* of $F(x)$. Furthermore,

$$\overline{F}(x) = \sqrt{\frac{2}{\pi}} \int_0^x f_c(\alpha) \cos(\alpha x)\, d\alpha.$$

If $\lim_{x\to\infty} d^n F/dx^n = 0$, then an important property of the Fourier cosine transform is

$$\begin{aligned} f_c^{(2r)}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^x \left(\frac{d^{2r}F}{dx^{2r}}\right) \cos(\alpha x)\, dx \\ &= -\sqrt{\frac{2}{\pi}} \sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \alpha^{2n} + (-1)^r \alpha^{2r} f_c(\alpha) \end{aligned} \tag{3}$$

where $\lim_{x\to\infty} d^r F/dx^r = a_r$, makes it useful in the solution of many problems.

Under the same conditions.

$$f_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^x F(x) \sin(\alpha x)\, dx$$

defines the *Fourier sine transform* of $F(x)$, and

$$\overline{F}(x) = \sqrt{\frac{2}{\pi}} \int_0^x f_s(\alpha) \sin(\alpha x)\, d\alpha$$

Corresponding to (3) we have

$$\begin{aligned} f_s^{(2r)}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^{2r}F}{dx^{2r}} \sin(\alpha x)\, dx \\ &= -\sqrt{\frac{2}{\pi}} \sum_{n=1}^{r} (-1)^n \alpha^{2n-1} a_{2r-2n} + (-1)^{r-1} \alpha^{2r} f_s(\alpha) \end{aligned} \tag{4}$$

Similarly, if $F(x)$ is defined for $-\infty < x < \infty$, and if $\int_{-\infty}^{\infty} F(x) dx$ is absolutely convergent, then

$$f(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{iax}\, dx$$

is the *Fourier transform* of $F(x)$, and

$$\overline{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha) e^{-iax}\, d\alpha$$

Also, if

$$\lim_{|x|\to\infty} \left| \frac{d^n F}{dx^n} \right| = 0 \quad (n = 1, 2, \ldots, r-1)$$

then

$$f^{(r)}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F^{(r)}(x) e^{i\alpha x} dx = (-i\alpha)^r f(\alpha)$$

Finite Sine Transforms

	$f_s(n)$	$F(x)$
1	$f_s(n) = \int_0^\pi F(x)\sin nx\,dx\ (n = 1, 2, \ldots)$	$F(x)$
2	$(-1)^{n+1} f_s(n)$	$F(\pi - x)$
3	$\frac{1}{n}$	$\frac{\pi - x}{\pi}$
4	$\frac{(-1)^{n+1}}{n}$	$\frac{x}{\pi}$
5	$\frac{1-(-1)^n}{n}$	1
6	$\frac{2}{n^2}\sin\frac{n\pi}{2}$	$\begin{cases} x & \text{when } 0 < x < \pi/2 \\ \pi - x & \text{when } \pi/2 < x < \pi \end{cases}$
7	$\frac{(-1)^{n+1}}{n^3}$	$\frac{x(\pi^2 - x^2)}{6\pi}$
8	$\frac{1-(-1)^n}{n^3}$	$\frac{x(\pi - x)}{2}$
9	$\frac{\pi^2(-1)^{n-1}}{n} - \frac{2[1-(-1)^n]}{n^3}$	x^2
10	$\pi(-1)^n\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)$	x^3
11	$\frac{n}{n^2+c^2}[1 - (-1)^n e^{c\pi}]$	e^{cx}
12	$\frac{n}{n^2+c^2}$	$\frac{\sinh c(\pi - x)}{\sinh c\pi}$

	$f_s(n)$	$F(x)$
13	$\frac{n}{n^2-k^2}\quad (k \neq 0, 1, 2, \ldots)$	$\frac{\sin k(\pi - x)}{\sin k\pi}$
14	$\begin{cases} \frac{\pi}{2} & \text{when } n = m \\ 0 & \text{when } n \neq m \end{cases}\quad (m = 1, 2, \ldots)$	$\sin mx$
15	$\frac{n}{n^2 - k^2}[1 - (-1)^n \cos k\pi]$ $(k \neq 1, 2, \ldots)$	$\cos kx$
16	$\begin{cases} \frac{n}{n^2-m^2}[1 - (-1)^{n+m}] & \text{when } n \neq m = 1, 2, \ldots \\ 0 & \text{when } n = m \end{cases}$	$\cos mx$
17	$\frac{n}{(n^2-k^2)^2}(k \neq 0, 1, 2, \ldots)$	$\frac{\pi \sin kx}{2k\sin^2 k\pi} - \frac{x\cos k(\pi - x)}{2k\sin k\pi}$
18	$\frac{b^n}{n}\ (\lvert b\rvert \leq 1)$	$\frac{2}{\pi}\arctan\frac{b\sin x}{1 - b\cos x}$
19	$\frac{1-(-1)^n}{n}b^n\quad (\lvert b\rvert \leq 1)$	$\frac{2}{\pi}\arctan\frac{2b\sin x}{1 - b^2}$

Finite Cosine Transforms

	$f_c(n)$	$F(x)$
1	$f_c(n) = \int_0^\pi F(x)\cos nx\,dx \quad (n = 0, 1, 2, \ldots)$	$F(x)$
2	$(-1)^n f_c(n)$	$F(\pi - x)$
3	0 when $n = 1, 2, \cdots; \quad f_c(0) = \pi$	1
4	$\frac{2}{n}\sin\frac{n\pi}{2}; \quad f_c(0) = 0$	$\begin{cases} 1 & \text{when } 0 < x < \pi/2 \\ -1 & \text{when } \pi/2 < x < \pi \end{cases}$
5	$-\frac{1-(-1)^n}{n^2}; \quad f_c(0) = \frac{\pi^2}{2}$	x
6	$\frac{(-1)^n}{n^2}; \quad f_c(0) = \frac{\pi^2}{6}$	$\frac{x^2}{2\pi}$
7	$\frac{1}{n^2}; \quad f_c(0) = 0$	$\frac{(\pi-x)^2}{2\pi} - \frac{\pi}{6}$
8	$3\pi^2\frac{(-1)^n}{n^2} - 6\frac{1-(-1)^n}{n^4}; \quad f_c(0) = \frac{\pi^4}{4}$	x^3
9	$\frac{(-1)^n e^{c}\pi - 1}{n^2+c^2}$	$\frac{1}{c}e^{cx}$
10	$\frac{1}{n^2+c^2}$	$\frac{\cosh c(\pi-x)}{c\sinh c\pi}$
11	$\frac{k}{n^2-k^2}[(-1)^n\cos\pi k - 1](k \neq 0, 1, 2, \cdots)$	$\sin kx$
12	$\frac{(-1)^{n+m}-1}{n^2-m^2}; \; f_c(m) = 0 \quad (m = 1, 2, \cdots)$	$\frac{1}{m}\sin mx$
13	$\frac{1}{n^2-k^2} \quad (k \neq 0, 1, 2, \ldots)$	$-\frac{\cos k(\pi-x)}{k\sin k\pi}$
14	$\begin{cases} 0 & \text{for } n = 1, 2, \cdots; \; n \neq m \\ \frac{\pi}{2} & \text{for } n = m \end{cases}$	$\cos mx \quad$ for $m = 1, 2, 3, \ldots$

Fourier Sine Transforms

	$F(x)$	$f_s(\alpha)$
1	$\begin{cases} 1 & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos\alpha}{\alpha}\right]$
2	$x^{p-1}(0 < p < 1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(p)}{\alpha^p}\sin\frac{p\pi}{2}$
3	$\begin{cases} \sin x & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin[a(1-\alpha)]}{1-\alpha} - \frac{\sin[a(1+\alpha)]}{1+\alpha}\right]$
4	e^{-x}	$\sqrt{\frac{2}{\pi}}\left[\frac{\alpha}{1+\alpha^2}\right]$
5	$xe^{-x^2/2}$	$\alpha e^{-\alpha^2/2}$
6	$\cos\frac{x^2}{2}$	$\sqrt{2}\left[\sin\frac{\alpha^2}{2}C\left(\frac{\alpha^2}{2}\right) - \cos\frac{\alpha^2}{2}S\left(\frac{\alpha^2}{2}\right)\right]^*$
7	$\sin\frac{x^2}{2}$	$\sqrt{2}\left[\cos\frac{\alpha^2}{2}C\left(\frac{\alpha^2}{2}\right) + \sin\frac{\alpha^2}{2}S\left(\frac{\alpha^2}{2}\right)\right]^*$

Here $C(y)$ and $S(y)$ are the Fresnel integrals:

$$C(y) = \frac{1}{\sqrt{2\pi}}\int_0^y \frac{1}{\sqrt{t}}\cos t\,dt, \qquad S(y) = \frac{1}{\sqrt{2\pi}}\int_0^y \frac{1}{\sqrt{t}}\sin t\,dt$$

*More extensive tables of the Fourier sine and cosine transforms can be found in Fritz Oberhettinger, *Tabellen zur-Fourier Transformation*, Springer, 1957.

Fourier Cosine Transforms

	$F(x)$	$f_c(\alpha)$
1	$\begin{cases} 1 & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin a\alpha}{\alpha}$
2	$x^{p-1} \quad (0 < p < 1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(p)}{\alpha^p}\cos\frac{p\pi}{2}$
3	$\begin{cases} \cos x & (0 < x < a) \\ 0 & (x > a) \end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin[a(1-\alpha)]}{1-\alpha} + \frac{\sin[a(1+\alpha)]}{1+\alpha}\right]$
4	e^{-x}	$\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\alpha^2}\right)$
5	$e^{-x^2/2}$	$e^{-\alpha^1/2}$
6	$\cos\frac{x^2}{2}$	$\cos\left(\frac{\alpha^2}{2} - \frac{\pi}{4}\right)$
7	$\sin\frac{x^2}{2}$	$\cos\left(\frac{\alpha^2}{2} + \frac{\pi}{4}\right)$

Fourier Transforms

	$F(x)$	$f(\alpha)$
1	$\frac{\sin ax}{x}$	$\begin{cases} \sqrt{\frac{\pi}{2}} & \lvert\alpha\rvert < a \\ 0 & \lvert\alpha\rvert > a \end{cases}$
2	$\begin{cases} e^{iwx} & (p < x < q) \\ 0 & (x < p,\ x > q) \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ip(w+\alpha)} - e^{iq(w+\alpha)}}{(w+\alpha)}$
3	$\begin{cases} e^{-cx+iwx} & (x > 0) \\ 0 & (x < 0) \end{cases} \quad (c > 0)$	$\frac{i}{\sqrt{2\pi}(w+\alpha+ic)}$
4	$e^{-px^2}\ R(p) > 0$	$\frac{1}{\sqrt{2p}} e^{-\alpha^2/4p}$
5	$\cos px^2$	$\frac{1}{\sqrt{2p}} \cos\left[\frac{\alpha^2}{4p} - \frac{\pi}{4}\right]$
6	$\sin px^2$	$\frac{1}{\sqrt{2p}} \cos\left[\frac{\alpha^2}{4p} + \frac{\pi}{4}\right]$
7	$\lvert x\rvert^{-p} \quad (0 < p < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(1-p) \sin \frac{p\pi}{2}}{\lvert\alpha\rvert^{(1-p)}}$
8	$\frac{e^{-a\lvert x\rvert}}{\sqrt{\lvert x\rvert}}$	$\frac{\sqrt{\sqrt{(a^2+\alpha^2)}+a}}{\sqrt{a^2+\alpha^2}}$
9	$\frac{\cosh ax}{\cosh \pi x} \quad (-\pi < a < \pi)$	$\sqrt{\frac{2}{\pi}} \frac{\cos \frac{a}{2} \cosh \frac{\alpha}{2}}{\cosh \alpha + \cos a}$
10	$\frac{\sinh ax}{\sinh \pi x} \quad (-\pi < a < \pi)$	$\frac{1}{\sqrt{2\pi}} \frac{\sin a}{\cosh \alpha + \cos a}$
11	$\begin{cases} \frac{1}{\sqrt{a^2-x^2}} & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$\sqrt{\frac{\pi}{2}} J_0(a\alpha)$
12	$\frac{\sin[b\sqrt{a^2+x^2}]}{\sqrt{a^2+x^2}}$	$\begin{cases} 0 & (\lvert\alpha\rvert > b) \\ \sqrt{\frac{\pi}{2}} J_0(a\sqrt{b^2-\alpha^2}) & (\lvert\alpha\rvert < b) \end{cases}$
13	$\begin{cases} p_{n(x)} & (\lvert x\rvert < 1) \\ 0 & (\lvert x\rvert > 1) \end{cases}$	$\frac{i^n}{\sqrt{\alpha}} J_{n+\frac{1}{2}}(\alpha)$
14	$\begin{cases} \frac{\cos[b\sqrt{a^2-x^2}]}{\sqrt{a^2-x^2}} & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$\sqrt{\frac{\pi}{2}} J_0(a\sqrt{a^2+b^2})$
15	$\begin{cases} \frac{\cosh[b\sqrt{a^2-x^2}]}{\sqrt{a^2-x^2}} & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$\sqrt{\frac{\pi}{2}} J_0(a\sqrt{\alpha^2-b^2})$

*More extensive tables of Fourier transforms can be found in *W*. Magnus and *F*. Oberhettinger, *Formulas and Theorems of the Special Functions of Mathematical Physics*. Chelsea, 1949, 116–120.