SERIES EXPANSION

The expression in parentheses following certain of the series indicates the region of convergence. If not otherwise indicated it is to be understood that the series converges for all finite values of x.

Binomial Series

$$
\begin{aligned}
(x+y)^n &= x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots (y^2 < x^2) \\
(1 \pm x)^n &= 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots (x^2 < 1) \\
(1 \pm x)^{-n} &= 1 \mp nx + \frac{n(n+1)x^2}{2!} \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots (x^2 < 1) \\
(1 \pm x)^{-1} &= 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \quad (x^2 < 1) \\
(1 \pm x)^{-2} &= 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots \qquad (x^2 < 1)
\end{aligned}
$$

Reversion of Series

Let a series be represented by

$$y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

with $a_1 \neq 0$. The coefficients of the series

$$x = A_1y + A_2y^2 + A_3y^3 + A_4y^4 + \cdots$$

are

$$A_1 = \frac{1}{a_1} \qquad A_2 = -\frac{a_2}{a_1^3} \qquad A_3 = \frac{1}{a_1^5}(2a_2^2 - a_1 a_3)$$

$$A_4 = \frac{1}{a_1^7}(5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3)$$

$$\begin{aligned}
A_5 &= \frac{1}{a_1^9}(6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_2^4 - a_1^3 a_5 - 21a_1 a_2^2 a_3) \\
A_6 &= \frac{1}{a_1^{11}}(7a_1^3 a_2 a_5 + 7a_1^3 a_3 a_4 + 84a_1 a_2^3 a_3 - a_1^4 a_6 - 28a_1^2 a_2^2 a_4 - 28a_1^2 a_2 a_3^2 - 42a_2^5) \\
A_7 &= \frac{1}{a_1^{13}}(8a_1^4 a_2 a_6 + 8a_1^4 a_3 a_5 + 4a_1^4 a_4^2 + 120a_1^2 a_2^3 a_4 + 180a_1^2 a_2^2 a_3^2 + 132a_2^6 - a_1^5 a_7 \\
&\quad -36a_1^3 a_2^2 a_5 - 72a_1^3 a_2 a_3 a_4 - 12a_1^3 a_3^3 - 330a_1 a_2^4 a_3)
\end{aligned}$$

Taylor Series

1. $$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \frac{(x-a)^3}{3!}\,f'''(a) + \cdots + \frac{(x-a)^n}{n!}\,f^{(n)}(a) + \cdots \qquad \text{(Taylor's Series)}$$

(Increment form)

2. $$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}\,f''(x) + \frac{h^3}{3!}\,f'''(x) + \cdots \\ &= f(h) + xf'(h) + \frac{x^2}{2!}\,f''(h) + \frac{x^3}{3!}\,f'''(h) + \cdots \end{aligned}$$

3. If $f(x)$ is a function possessing derivatives of all orders throughout the interval $a \le x \le b$, then there is a value X, with $a < X < b$, such that

$$f(b) = f(a) + (b-a)\,f'(a) + \frac{(b-a)^2}{2!}\,f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}\,f^{(n-1)}(a) + \frac{(b-a)^n}{n!}\,f^{(n)}(X)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}\,f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}\,f^{(n-1)}(a) + \frac{h^n}{n!}\,f^{(n)}(a+\theta h)$$

where $b = a + h$ and $0 < \theta < 1$. Or

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \cdots + (x-a)^{n-1}\,\frac{f^{(n-1)}(a)}{(n-1)!} + R_n,$$

where

$$R_n = \frac{f^{(n)}[a + \theta \cdot (x-a)]}{n!}(x-a)^n,\ 0 < \theta < 1.$$

The above forms are known as Taylor's series with the remainder term.

4. Taylor's series for a function of two variables

If $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) = h\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y}$;

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}$$

etc., and if $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{x=a}^{y=b}$ where the bar and subscripts means that after differentiation we are to replace x by a and y by b, then

$$f(a+h,\ b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y)\Big|_{x=a}^{y=b} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{x=a}^{y=b} + \cdots$$

Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + x^{n-1} \frac{f^{(n-1)}(0)}{(n-1)!} + R_n,$$

where

$$R_n = \frac{x^n f^{(n)}(\theta x)}{n!}, \quad 0 < \theta < 1.$$

Exponential Series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$a^x = 1 + x \log_e a + \frac{(x \log_e a)^2}{2!} + \frac{(x \log_e a)^3}{3!} + \cdots$$

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \cdots \right]$$

Logarithmic Series

$$\log_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad (x > \tfrac{1}{2})$$

$$\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \qquad (2 \geq x > 0)$$

$$\log_e x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \qquad (x > 0)$$

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad (-1 < x \leq 1)$$

$$\log_e(n+1) - \log_e(n-1) = 2\left[\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right]$$

$$\log_e(a+x) = \log_e a + 2\left[\frac{x}{2a+x} + \frac{1}{3}\left(\frac{x}{2a+x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a+x}\right)^5 + \cdots\right] \qquad (a > 0, -a < x < +\infty)$$

$$\log_e \frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right] \qquad -1 < x < 1$$

$$\log_e x = \log_e a + \frac{(x-a)}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - + \cdots \qquad 0 < x \leq 2a$$

Trigonometric Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{(all real values of } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{(all real values of } x\text{)}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots + \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1} + \cdots,$$

$$\left[x^2 < \frac{\pi^2}{4} \quad \text{and } B_n \text{ represents the } n^{\text{th}} \text{ Bernoulli number}\right]$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \cdots - \frac{(-1)^{n+1} 2^{2n}}{(2n)!} B_{2n} x^{2n-1} - \cdots,$$

$$\left[x^2 < \pi^2 \quad \text{and } B_n \text{ represents the } n^{\text{th}} \text{ Bernoulli number}\right]$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots + \frac{(-1)^n}{(2n)!} E_{2n} x^{2n} + \cdots,$$

$$\left[x^2 < \frac{\pi^2}{4} \quad \text{and } E_n \text{ represents the } n^{\text{th}} \text{ Euler number}\right]$$

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15{,}120}x^5 + \frac{127}{604{,}800}x^7 + \cdots + \frac{(-1)^{n+1} 2(2^{2n-1}-1)}{(2n)!} B_{2n} x^{2n-1} + \cdots,$$

$$\left[x^2 < \pi^2 \quad \text{and } B_n \text{ represents the } n^{\text{th}} \text{ Bernoulli number}\right]$$

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots \qquad (x^2<\infty)$$

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\cdots \qquad (x^2<\infty)$$

$$\sin^{-1} x = x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}x^5+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7+\cdots \qquad \left(x^2<1,\ -\frac{\pi}{2}<\sin^{-1}x<\frac{\pi}{2}\right)$$

$$\cos^{-1} x = \frac{\pi}{2}-\left(x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}x^5+\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots\right) \qquad (x^2<1,\ 0<\cos^{-1}x<\pi)$$

$$\tan^{-1} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \qquad (x^2<1)$$

$$\tan^{-1} x = \frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\frac{1}{7x^7}-\cdots \qquad (x>1)$$

$$\tan^{-1} x = -\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\frac{1}{7x^7}-\cdots \qquad (x<-1)$$

$$\cot^{-1} x = \frac{\pi}{2}-x+\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}-\cdots \qquad (x^2<1)$$

$$\log_e \sin x = \log_e x-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\cdots \qquad (x^2<\pi^2)$$

$$\log_e \cos x = -\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\frac{17x^8}{2520}-\cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$$

$$\log_e \tan x = \log_e x+\frac{x^2}{3}+\frac{7x^4}{90}+\frac{62x^6}{2835}+\cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$$

$$e^{\sin x} = 1+x+\frac{x^2}{2!}-\frac{3x^4}{4!}-\frac{8x^5}{5!}-\frac{3x^6}{6!}+\frac{56x^7}{7!}+\cdots$$

$$e^{\cos x} = e\left(1-\frac{x^2}{2!}+\frac{4x^4}{4!}-\frac{31x^6}{6!}+\cdots\right)$$

$$e^{\tan x} = 1+x+\frac{x^2}{2!}+\frac{3x^3}{3!}+\frac{9x^4}{4!}+\frac{37x^5}{5!}+\cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$$

$$\sin x = \sin a+(x-a)\cos a-\frac{(x-a)^2}{2!}\sin a - \frac{(x-a)^3}{3!}\cos a+\frac{(x-a)^4}{4!}\sin a+\cdots$$