BESSEL FUNCTIONS

1. Bessel's differential equation for a real variable x is

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

2. When n is not an integer, two independent solutions of the equation are $J_n(x)$, $J_{-n}(x)$, where

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

3. If n is an integer $J_n(x) = (-1)^n J_n(x)$, where

$$J_n(x) = \frac{x^n}{2^n n!}\left\{1 - \frac{x^2}{2^2 \cdot 1!(n+1)} + \frac{x^4}{2^4 \cdot 2!(n+1)(n+2)} + \frac{x^6}{2^6 \cdot 3!(n+1)(n+2)(n+3)} + \ldots\right\}$$

4. For $n = 0$ and $n = 1$, this formula becomes

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \frac{x^8}{2^8(4!)^2} - \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \cdot 1!2!} + \frac{x^5}{2^5 \cdot 2!3!} - \frac{x^7}{2^7 \cdot 3!4!} + \frac{x^9}{2^9 \cdot 4!5!} - \ldots$$

5. When x is large and positive, the following asymptotic series may be used

$$J_0(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left\{ P_0(x)\cos\left(x - \frac{\pi}{4}\right) - Q_0(x)\sin\left(x - \frac{\pi}{4}\right)\right\}$$
$$J_1(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left\{ P_1(x)\cos\left(x - \frac{3\pi}{4}\right) - Q_1(x)\sin\left(x - \frac{3\pi}{4}\right)\right\}$$

where

$$P_0(x) \sim 1 - \frac{1^2\cdot 3^2}{2!(8x)^2} + \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{4!(8x)^4} - \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdot 9^2\cdot 11^2}{6!(8x)^6} + \cdots$$
$$Q_0(x) \sim -\frac{1^2}{1!8x} + \frac{1^2\cdot 3^2\cdot 5^2}{3!(8x)^3} - \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdot 9^2}{5!(8x)^5} + - \cdots$$
$$P_1(x) \sim 1 + \frac{1^2\cdot 3\cdot 5}{2!(8x)^2} - \frac{1^2\cdot 3^2\cdot 5^2\cdot 7\cdot 9}{4!(8x)^4} + \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdot 9^2\cdot 11\cdot 13}{6!(8x)^6} - + \cdots$$
$$Q_1(x) \sim \frac{1\cdot 3}{1!8x} - \frac{1^2\cdot 3^2\cdot 5\cdot 7}{3!(8x)^3} + \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdot 9\cdot 11}{5!(8x)^5} - \cdots$$

[In $P_1(x)$ the signs alternate from + to − after the first term]

6. The zeros of $J_0(x)$ and $J_1(x)$.

If j_{0s} and j_{1s} are the sth zeros of $J_0(x)$ and $J_1(x)$ respectively, and if $a = 4_s - 1$, $b = 4_s + 1$

$$j_{0,s} \sim \frac{1}{4}\pi a\left\{1 + \frac{2}{\pi^2 a^2} - \frac{62}{3\pi^4 a^4} + \frac{15,116}{15\pi^6 a^6} - \frac{12,554,474}{105\pi^8 a^8} + \frac{8,368,654,292}{315\pi^{10}a^{10}} - + \cdots\right\}$$
$$j_{1,s} \sim \frac{1}{4}\pi b\left\{1 - \frac{6}{\pi^2 b^2} + \frac{6}{\pi^4 b^4} - \frac{4716}{5\pi^6 b^6} + \frac{3,902,418}{35\pi^8 b^8} - \frac{895,167,324}{35\pi^{10}b^{10}} + \cdots\right\}$$
$$J_1(j_{0,s}) \sim \frac{(-1)^{s+1}2^{\frac{3}{2}}}{\pi a^{\frac{1}{2}}}\left\{1 - \frac{56}{3\pi^4 a^4} + \frac{9664}{5\pi^6 a^6} - \frac{7,381,280}{21\pi^8 a^8} + \cdots\right\}$$
$$J_0(j_{1,s}) \sim \frac{(-1)^{s}2^{\frac{3}{2}}}{\pi b^{\frac{1}{2}}}\left\{1 + \frac{24}{\pi^4 b^4} - \frac{19,584}{10\pi^6 b^6} + \frac{2,466,720}{7\pi^8 b^8} - \cdots\right\}$$

7. Table of zeros for $J_0(x)$ and $J_1(x)$

Define $\{\alpha_n, \beta_n\}$ by $J_1(\alpha_n) = 0$ and $J_0(\beta_n) = 0$.

Roots α_n	$J_1(\alpha_n)$	Roots β_n	$J_0(\beta_n)$
2.4048	0.5191	0.0000	1.0000
5.5201	-0.3403	3.8317	-0.4028
8.6537	0.2715	7.0156	0.3001
11.7915	-0.2325	10.1735	-0.2497
14.9309	0.2065	13.3237	0.2184
18.0711	-0.1877	16.4706	-0.1965
21.2116	0.1733	19.6159	0.1801

8. Recurrence formulas

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \qquad nJ_n(x) + xJ_n'(x) = xJ_{n-1}(x)$$
$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \qquad nJ_n(x) - xJ_n'(x) = xJ_{n+1}(x)$$

9. If J_n is written for $J_n(x)$ and $J_n^{(k)}$ is written for $\frac{d^k}{dx^k}\{J_n(x)\}$, then the following derivative relationships are important

$$J_0^{(r)} = -J_1^{(r-1)}$$
$$J_0^{(2)} = -J_0 + \tfrac{1}{x}J_1 = \tfrac{1}{2}(J_2 - J_0)$$
$$J_0^{(3)} = \tfrac{1}{x}J_0 + \left(1 - \tfrac{2}{x^2}\right)J_1 = \tfrac{1}{4}(-J_3 + 3J_1)$$
$$J_0^{(4)} = \left(1 - \tfrac{3}{x^2}\right)J_0 - \left(\tfrac{2}{x} - \tfrac{6}{x^3}\right)J_1 = \tfrac{1}{8}(J_4 - 4J_2 + 3J_0), \text{ etc.}$$

10. Half order Bessel functions

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$
$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$
$$J_{n+\frac{3}{2}}(x) = -x^{n+\frac{1}{2}} \frac{d}{dx}\{x^{-(n+\frac{1}{2})} J_{n+\frac{1}{2}}(x)\}$$
$$J_{n-\frac{1}{2}}(x) = x^{-(n+\frac{1}{2})} \frac{d}{dx}\{x^{n+\frac{1}{2}} J_{n+\frac{1}{2}}(x)\}$$

n	$\left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x)$	$\left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{-(n+\frac{1}{2})}(x)$
0	$\sin x$	$\cos x$
1	$\frac{\sin x}{x} - \cos x$	$-\frac{\cos x}{x} - \sin x$
2	$\left(\frac{3}{x^2} - 1\right) \sin x - \frac{3}{x} \cos x$	$\left(\frac{3}{x^2} - 1\right) \cos x + \frac{3}{x} \sin x$
3	$\left(\frac{15}{x^3} - \frac{6}{x}\right) \sin x - \left(\frac{15}{x^2} - 1\right) \cos x$ etc.	$-\left(\frac{15}{x^3} - \frac{6}{x}\right) \cos x - \left(\frac{15}{x^2} - 1\right) \sin x$

11. Additional solutions to Bessel's equation are

$Y_n(x)$ (also called Weber's function, and sometimes denoted by $N_n(x)$)
$H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ (also called Hankel functions)

These solutions are defined as follows

$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} & n \text{ not an integer} \\ \lim\limits_{\nu \to n} \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} & n \text{ an integer} \end{cases} \qquad \begin{aligned} H_n^{(1)}(x) &= J_n(x) + iY_n(x) \\ H_n^{(2)}(x) &= J_n(x) - iY_n(x) \end{aligned}$$

The additional properties of these functions may all be derived from the above relations and the known properties of $J_n(x)$.

12. Complete solutions to Bessel's equation may be written as

$$c_1 J_n(x) + c_2 J_{-n}(x) \qquad \text{if } n \text{ is not an integer}$$

or, for any value of n,

$$c_1 J_n(x) + c_2 Y_n(x) \qquad \text{or} \qquad c_1 H_n^{(1)} x + c_2 H_n^{(2)}(x)$$

13. The modified (or hyperbolic) Bessel's differential equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2) y = 0$$

14. When n is not an integer, two independent solutions of the equation are $I_n(x)$ and $I_{-n}(x)$, where

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

15. If n is an integer,

$$I_n(x) = I_{-n}(x) = \frac{x^n}{2^n n!} \left\{1 + \frac{x^2}{2^2 \cdot 1!(n+1)} + \frac{x^4}{2^4 \cdot 2!(n+1)(n+2)} + \frac{x^6}{2^6 \cdot 3!(n+1)(n+2)(n+3)} + \cdots \right\}$$

16. For $n = 0$ and $n = 1$, this formula becomes

$$I_0(x) = 1 + \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \frac{x^6}{2^6(3!)^2} + \frac{x^8}{2^8(4!)^2} + \cdots$$
$$I_1(x) = \frac{x}{2} + \frac{x^3}{2^3 \cdot 1!2!} + \frac{x^5}{2^5 \cdot 2!3!} + \frac{x^7}{2^7 \cdot 3!4!} + \frac{x^9}{2^9 \cdot 4!5!} + \cdots$$

17. Another solution to the modified Bessel's equation is

$$K_n(x) = \begin{cases} \frac{1}{2}\pi \frac{I_{-n}(x) - I_n(x)}{\sin(n\pi)} & n \text{ not an integer} \\ \lim_{\nu \to n} \frac{1}{2}\pi \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)} & n \text{ an integer} \end{cases}$$

This function is linearly independent of $I_n(x)$ for all values of n. Thus the complete solution to the modified Bessel's equation may be written as

$$c_1 I_n(x) + c_2 I_{-n}(x) \quad n \text{ not an integer}$$

or

$$c_1 I_n(x) + c_2 K_n(x) \quad \text{any } n$$

18. The following relations hold among the various Bessel functions:

$$I_n(z) = i^{-m} J_m(iz)$$
$$Y_n(iz) = (i)^{n+1} I_n(z) - \tfrac{2}{\pi} i^{-n} K_n(z)$$

Most of the properties of the modified Bessel function may be deduced from the known properties of $J_n(x)$ by use of these relations and those previously given.

19. Recurrence formulas

$$I_{n-1}(x) - I_{n+1}(x) = \tfrac{2n}{x} I_n(x) \qquad I_{n-1}(x) + I_{n+1}(x) = 2I_n'(x)$$
$$I_{n-1}(x) - \tfrac{n}{x} I_n(x) = I_n'(x) \qquad I_n'(x) = I_{n+1}(x) + \tfrac{n}{x} I_n(z)$$