ORTHOGONAL POLYNOMIALS

I: Legendre

Name: Legendre *Symbol:* $P_n(x)$ *Interval:* $[-1, 1]$
Differential Equation: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Explicit Expression: $P_n(x) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} (-1)^m \binom{n}{m} \binom{2n-2m}{n} x^{n-2m}$
Recurrence Relation: $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
Weight: 1
Standardization: $P_n(1) = 1$
Norm: $\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$
Rodrigues' Formula: $P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\{(1-x^2)^n\}$
Generating Function: $R^{-1} = \sum_{n=0}^{\infty} P_n(x) z^n; -1 < x < 1, \quad |z| \quad < 1,$
$R = \sqrt{1 - 2xz + z^2}$
Inequality: $|P_n(x)| \leq 1, -1 \leq x \leq 1.$

II: Tschebysheff, First Kind

Name: Tschebysheff, First Kind *Symbol:* $T_n(x)$ *Interval:*$[-1, 1]$
Differential Equation: $(1-x^2)y - xy' + n^2 y = 0$
Explicit Expression: $\frac{n}{2} \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m} = \cos(n \arccos x) = T_n(x)$
Recurrence Relation: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$
Weight: $(1-x^2)^{-1/2}$
Standardization: $T_n(1) = 1$
Norm: $\int_{-1}^{+1} (1-x^2)^{-1/2} [T_n(x)]^2 dx = \begin{cases} \pi/2, & n \neq 0 \\ \pi, & n = 0 \end{cases}$
Rodrigues' Formula: $\frac{(-1)^n (1-x^2)^{1/2} \sqrt{\pi}}{2^{n+1} \Gamma(n+\frac{1}{2})} \frac{d^n}{dx^n} \{(1-x^2)^{n-(1/2)}\} = T_n(x)$
Generating Function: $\frac{1-xz}{1-2xz-z^2} = \sum_{n=0}^{\infty} T_n(x) z^n, \ -1 < x < 1, \quad |z| \ < 1$
Inequality: $|T_n(x)| \leq 1, -1 \leq x \leq 1.$

III: Tschebysheff, Second Kind

Name: Tschebysheff, Second Kind *Symbol* $U_n(x)$ *Interval:* [-1, 1]
Differential Equation: $(1-x^2)y'' - 3xy' + n(n+2)y = 0$
Explicit Expression: $U_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(m-n)!}{m!(n-2m)!} (2x)^{n-2m}$
$U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta}$

Recurrence Relation: $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$
Weight: $(1-x^2)^{1/2}$ *Standardization:* $U_n(1) = n+1$
Norm: $\int_{-1}^{+1} (1-x^2)^{1/2} [U_n(x)]^2 dx = \frac{\pi}{2}$
Rodrigues' Formula: $U_n(x) = \frac{(-1)^n (n+1) \sqrt{\pi}}{(1-x^2)^{1/2} 2^{n+1} \Gamma(n+\frac{3}{2})} \frac{d^n}{dx^n} \{(1-x^2)^{n+(1/2)}\}$
Generating Function: $\frac{1}{1-2xz+z^2} = \sum_{n=0}^{\infty} U_n(x) z^n, -1 < x < 1, \quad |z| \ < 1$
Inequality: $|U_n(x)| \leq n+1, -1 \leq x \leq 1.$

IV: Jacobi

Name: Jacobi *Symbol:* $P_n^{(\alpha,\beta)}(x)$ *Interval:* $[-1, 1]$
Differential Equation: $(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0$

Explicit Expression: $P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m$

Recurrence Relation:

$$\begin{aligned}&2(n+1)\,(n+\alpha+\beta+1)\,(2n+\alpha+\beta)\,P_{n+1}^{(\alpha,\beta)}(x)\\&\quad=(2n+\alpha+\beta+1)[(\alpha^2-\beta^2)+(2n+\alpha+\beta+2)(2n+\alpha+\beta)x]\,P_n^{(\alpha,\beta)}(x)\\&\quad\quad-2(n+\alpha)\,(n+\beta)\,(2n+\alpha+\beta+2)\,P_{n-1}^{(\alpha,\beta)}(x)\end{aligned}$$

Weight: $(1-x)^\alpha(1+x)^\beta;\ \alpha,\beta > 1$
Standardization: $P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}$
Norm: $\int_{-1}^{+1}(1-x)^\alpha(1+x)^\beta[P_n^{(\alpha,\beta)}(x)]^2dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}$
Rodrigues' Formula: $P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!(1-x)^\alpha(1+x)^\beta}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}$

Generating Function: $R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta} = \sum_{n=0}^{\infty}2^{-\alpha-\beta}P_n^{(\alpha,\beta)}(x)z^n,$

$$R=\sqrt{1-2xz+z^2},\quad |z|<1$$

Inequality: $\max_{-1\le x\le 1}|P_n^{(\alpha,\beta)}(x)| = \begin{cases}\binom{n+q}{n}\sim n^q \text{if } q=\max(\alpha,\beta)\ge -\frac{1}{2}\\ |P_n^{(\alpha,\beta)}(x')|\sim n^{-1/2}\text{if } q<-\frac{1}{2}\\ x' \text{ is one of the two maximum points nearest}\\ \frac{\beta-\alpha}{\alpha+\beta+1}\end{cases}$

V: Generalized Laguerre

Name: Generalized Laguerre *Symbol:* $L_n^{(\alpha)}(x)$ *Interval:* $[0,\infty]$
Differential Equation: $xy''+(\alpha+1-x)y'+ny=0$
Explicit Expression: $L_n^{(\alpha)}(x) = \sum_{m=0}^{n}(-1)^m\binom{n+\alpha}{n-m}\frac{1}{m!}x^m$
Recurrence Relation: $(n+1)L_n^{(\alpha)}+1(x) = [(2n+\alpha+1)-x]L_n^{(\alpha)}(x)-(n+\alpha)L_n^{(\alpha)}-1(x)$
Weight: $x^\alpha e^{-x}, \alpha > -1$ *Standardization:* $L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}x^n+\cdots$
Norm: $\int_0^\infty x^\alpha e^{-x}[L_n^{(\alpha)}(x)]^2dx = \frac{\Gamma(n+\alpha+1)}{n!}$
Rodrigues' Formula: $L_n^{(\alpha)}(x) = \frac{1}{n!x^\alpha e^{-x}}\frac{d^n}{dx^n}\{x^{n+\alpha}e^{-x}\}$
Generating Function: $(1-z)^{-\alpha-1}\exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty}L_n^{(\alpha)}(x)z^n$
Inequality: $|L_n^{(\alpha)}(x) \le \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}e^{x/2};\quad \begin{matrix}x\ge 0\\ \alpha>0\end{matrix}$

$$|L_n^{(a)}(x)| \le \left[2-\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}\right]e^{x/2};\quad \begin{matrix}x\ge 0\\ -1<\alpha<0\end{matrix}$$

VI: Hermite

Name: Hermite *Symbol:* $H_n(x)$ *Interval:* $[-\infty,\infty]$
Differential Equation: $y''-2xy'+2ny=0$
Explicit Expression: $H_n(x) = \sum_{m=0}^{[n/2]}\frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$
Recurrence Relation: $H_{n+1}(x) = 2xH_n(x)-2nH_{n-1}(x)$
Weight: e^{-x^2} *Standardization:* $H_n(1) = 2^n x^n+\cdots$
Norm: $\int_{-\infty}^{\infty}e^{-x^2}[H_n(x)]^2\,dx = 2^n n!\sqrt{\pi}$
Rodrigues' Formula: $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$
Generating Function: $e^{-x^2+2zx} = \sum_{n=0}^{\infty}H_n(x)\frac{z^n}{n!}$
Inequality: $|H_n(x)|e^{x^2/2}k2^{n/2}\sqrt{n!}\ k\approx 1.086435$