CLEBSCH–GORDAN COEFFICIENTS

$$\left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\,\middle|\, \begin{matrix} j \\ m \end{matrix}\right) = \delta_{m,m_1+m_2}\sqrt{\frac{(j_1+j_2-j)!(j+j_1-j_2)!(j+j_2-j_1)!(2j+1)}{(j+j_1+j_2+1)!}}$$
$$\times \sum_k \frac{(-1)^k\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!}}{k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j-j_2+m_1+k)!(j-j_1-m_2+k)!}.$$

1. Conditions:

   (a) Each of $\{j_1, j_2, j, m_1, m_2, m\}$ may be an integer, or half an integer. Additionally: $j > 0$, $j_1 > 0$, $j_2 > 0$ and $j + j_1 + j_2$ is an integer.

   (b) $j_1 + j_2 - j \geq 0$.

   (c) $j_1 - j_2 + j \geq 0$.

   (d) $-j_1 + j_2 + j \geq 0$.

   (e) $|m_1| \leq j_1$, $|m_2| \leq j_2$, $|m| \leq j$.

2. Special values:

   (a) $\left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\,\middle|\, \begin{matrix} j \\ m \end{matrix}\right) = 0$ if $m_1 + m_2 \neq m$.

   (b) $\left(\begin{matrix} j_1 & 0 \\ m_1 & 0 \end{matrix}\,\middle|\, \begin{matrix} j \\ m \end{matrix}\right) = \delta_{j_1,j}\delta_{m_1,m}$.

   (c) $\left(\begin{matrix} j_1 & j_2 \\ 0 & 0 \end{matrix}\,\middle|\, \begin{matrix} j \\ 0 \end{matrix}\right) = 0$ when $j_1 + j_2 + j$ is an odd integer.

   (d) $\left(\begin{matrix} j_1 & j_1 \\ m_1 & m_1 \end{matrix}\,\middle|\, \begin{matrix} j \\ m \end{matrix}\right) = 0$ when $2j_1 + j$ is an odd integer.

3. Symmetry relations: all of the following are equal to $\left(\begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix}\,\middle|\, \begin{matrix} j \\ m \end{matrix}\right)$:

   (a) $\left(\begin{matrix} j_2 & j_1 \\ -m_2 & -m_1 \end{matrix}\,\middle|\, \begin{matrix} j \\ -m \end{matrix}\right)$,

   (b) $(-1)^{j_1+j_2-j}\left(\begin{matrix} j_2 & j_1 \\ m_1 & m_2 \end{matrix}\,\middle|\, \begin{matrix} j \\ m \end{matrix}\right)$,

   (c) $(-1)^{j_1+j_2-j}\left(\begin{matrix} j_1 & j_2 \\ -m_1 & -m_2 \end{matrix}\,\middle|\, \begin{matrix} j \\ -m \end{matrix}\right)$,

   (d) $\sqrt{\frac{2j+1}{2j_1+1}}(-1)^{j_2+m_2}\left(\begin{matrix} j & j_2 \\ -m & m_2 \end{matrix}\,\middle|\, \begin{matrix} j_1 \\ -m_1 \end{matrix}\right)$,

   (e) $\sqrt{\frac{2j+1}{2j_1+1}}(-1)^{j_1-m_1+j-m}\left(\begin{matrix} j & j_2 \\ m & -m_2 \end{matrix}\,\middle|\, \begin{matrix} j_1 \\ m_1 \end{matrix}\right)$,

   (f) $\sqrt{\frac{2j+1}{2j_1+1}}(-1)^{j-m+j_1-m_1}\left(\begin{matrix} j_2 & j \\ m_2 & -m \end{matrix}\,\middle|\, \begin{matrix} j_1 \\ -m_1 \end{matrix}\right)$,

   (g) $\sqrt{\frac{2j+1}{2j_2+1}}(-1)^{j_1-m_1}\left(\begin{matrix} j_1 & j \\ m_1 & -m \end{matrix}\,\middle|\, \begin{matrix} j_2 \\ -m_2 \end{matrix}\right)$,

   (h) $\sqrt{\frac{2j+1}{2j_2+1}}(-1)^{j_1-m_1}\left(\begin{matrix} j & j_1 \\ m & -m_1 \end{matrix}\,\middle|\, \begin{matrix} j_2 \\ m_2 \end{matrix}\right)$.

By use of the symmetry relations, Clebsch–Gordan coefficients may be put in the standard form $j_1 \leq j_2 \leq j$ and $m \geq 0$.

m_2	m	j_1	j	$\begin{pmatrix} j_1 & \frac{1}{2} & \| & j \\ m_1 & m_2 & \| & m \end{pmatrix}$	
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{\sqrt{2}}{2}$	≈ 0.707107
0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	≈ 0.866025
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{\sqrt{2}}{2}$	≈ 0.707107
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	≈ 0.866025
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	≈ 1.000000

m_2	m	j_1	j	$\begin{pmatrix} j_1 & \mathbf{1} & \| & j \\ m_1 & m_2 & \| & m \end{pmatrix}$	
-1	0	1	1	$\frac{\sqrt{2}}{2}$	≈ 0.707107
-1	0	1	2	$\frac{\sqrt{6}}{6}$	≈ 0.408248
$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\sqrt{2}}{2}$	≈ 0.707107
$-\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{3}{4}$	≈ 0.750000
$-\frac{1}{2}$	$\frac{1}{2}$	1	2	$\frac{\sqrt{5}}{4}$	≈ 0.559017
0	0	1	2	$\frac{\sqrt{6}}{3}$	≈ 0.816496
0	0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\sqrt{3}}{2}$	≈ 0.866025
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\sqrt{6}}{3}$	≈ 0.8164967
0	$\frac{1}{2}$	1	1	$\frac{\sqrt{2}}{4}$	≈ 0.353553
0	$\frac{1}{2}$	1	2	$\frac{\sqrt{10}}{4}$	≈ 0.790569
0	1	1	1	$\frac{\sqrt{2}}{2}$	≈ 0.707107

m_2	m	j_1	j	$\begin{pmatrix} j_1 & \mathbf{1} & \| & j \\ m_1 & m_2 & \| & m \end{pmatrix}$	
0	1	1	2	$\frac{\sqrt{2}}{2}$	≈ 0.707107
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\sqrt{2}}{2}$	≈ 0.707107
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$-\frac{\sqrt{2}}{4}$	≈ -0.353553
$\frac{1}{2}$	$\frac{1}{2}$	1	2	$\frac{\sqrt{10}}{4}$	≈ 0.790569
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\sqrt{30}}{6}$	≈ 0.912871
$\frac{1}{2}$	$\frac{3}{2}$	1	2	$\frac{\sqrt{105}}{12}$	≈ 0.853913
1	0	1	1	$-\frac{\sqrt{2}}{2}$	≈ -0.707107
1	0	1	2	$\frac{\sqrt{6}}{6}$	≈ 0.408248
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\sqrt{3}}{3}$	≈ 0.577350
1	$\frac{1}{2}$	1	1	$-\frac{3}{4}$	≈ -0.750000
1	$\frac{1}{2}$	1	2	$\frac{\sqrt{5}}{4}$	≈ 0.559017
1	1	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{\sqrt{10}}{4}$	≈ 0.790569
1	1	1	1	$-\frac{\sqrt{2}}{2}$	≈ -0.707107
1	1	1	2	$\frac{\sqrt{2}}{2}$	≈ 0.707107
1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	1	≈ 1.000000
1	$\frac{3}{2}$	1	2	$\frac{\sqrt{105}}{12}$	≈ 0.853913
1	2	1	2	1	≈ 1.000000