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STRUCTURAL THEORY

Structural Theory Application to Model Structure to Predict Its Behavior

Structure design is the application of structural theory to ensure that buildings and other structures are built to support all loads and resist all constraining forces that may be reasonably expected to be imposed on them during their expected service life, without hazard to occupants or users and preferably without dangerous deformations, excessive sidesway (drift), or annoying vibrations. In addition, good design requires that this objective be achieved economically.

Applying structural theory to mathematic models is an essential and important tool in structural engineering. Over the past 200 years, many of the most significant contributions to the understanding of the structures have been made by scientist engineers while working on mathematical models, which were used for real structures.

Application of mathematical models of any sort to any real structural system must be idealized in some fashion; that is, an analytical model must be developed. There has never been an analytical model which is a precise representation of the physical system. While the performance of the structure is the result of natural effects, the development and thus the performance of the model is entirely under the control of the analyst. The validity of the results obtained from applying mathematical theory to the study of the model therefore rests on the accuracy of the model. While this is true, it does not mean that all analytical models must be elaborate, conceptually sophisticated devices. In some cases very simple models give surprisingly accurate results. While in some other cases they may yield answers, which deviate markedly from the true physical behavior of the model, yet be completely satisfactory for the problem at hand.

Provision should be made in the application of structural theory to design for abnormal as well as normal service conditions. Abnormal conditions may arise as a result of accidents, fire, explosions, tornadoes, severer-than-anticipated earthquakes, floods, and inadvertent or even deliberate overloading of building components. Under such conditions, parts of a building may be damaged. The structural system, however, should be so designed that the damage will be limited in extent and undamaged portions of the building will remain stable. For the purpose, structural elements should be proportioned and arranged to form a stable system under normal service conditions. In addition, the system should have sufficient continuity and ductility, or energy-absorption capacity, so that if any small portion of it should sustain damage, other parts will transfer loads (at least until repairs can be made) to remaining structural components capable of transmitting the loads to the ground.

("Steel Design Handbook, LRFD Method", Akbar R. Tamboli Ed., McGraw-Hill 1997. "Design Methods for Reducing the Risk of Progressive Collapse in Buildings", NBS Buildings Science Series 98, National Institute of Standards and

Technology, 1977. "Handbook of Structural Steel Connection Design and Details," Akbar R. Tamboli Ed., McGraw-Hill 1999.)

6.1 Structural Integrity

Provision should be made in application of structural theory to design for abnormal as well as normal service conditions. Abnormal conditions may arise as a result of accidents, fire, explosions, tornadoes, severer-than-anticipated earthquakes, floods, and inadvertent or even deliberate overloading of building components. Under such conditions, parts of a building may be damaged. The structural system, however, should be so designed that the damage will be limited in extent and undamaged portions of the building will remain stable. For the purpose, structural elements should be proportioned and arranged to form a stable system under normal service conditions. In addition, the system should have sufficient continuity, redundancy and ductility, or energy-absorption capacity, so that if any small portion of it should sustain damage, other parts will transfer loads (at least until repairs can be made) to remaining structural components capable of transmitting the loads to the ground.

If a structure does not possess this capability, failure of a single component can lead, through progressive collapse of adjoining components, to collapse of a major part or all of the structure. For example, if the corner column of a multistory building should be removed in a mishap and the floor it supports should drop to the floor below, the lower floor and the column supporting it may collapse, throwing the debris to the next lower floor. This action may progress all the way to the ground. One way of avoiding this catastrophe is to design the structure so that when a column fails all components that had been supported by it will cantilever from other parts of the building, although perhaps with deformations normally considered unacceptable.

This example indicates that resistance to progressive collapse may be provided by inclusion in design of alternate load paths capable of absorbing the load from damaged or failed components. An alternative is to provide, in design, reserve strength against mishaps. In both methods, connections of components should provide continuity, redundancy and ductility.

(D. M. Schultz, F. F. P. Burnett, and M. Fintel, "A Design Approach to General Structural Integrity," in "Design and Construction of Large-Panel Concrete Structures," U.S. Department of Housing and Urban Development, 1977; E. V. Leyendecker and B. R. Ellingwood, "Design Methods for Reducing the Risk of Progressive Collapse in Buildings," NBS Buildings Science Series 98, National Institute of Standards and Technology, 1977.)

Equilibrium

6.2 Types of Load

Loads are the external forces acting on a structure. Stresses are the internal forces that resist the loads.

Tensile forces tend to stretch a component, **compressive forces** tend to shorten it, and **shearing forces** tend to slide parts of it past each other.

Loads also may be classified as static or dynamic. **Static loads** are forces that are applied slowly and then remain nearly constant, such as the weight, or dead load, of a floor system. **Dynamic loads** vary with time. They include repeated loads, such as alternating forces from oscillating machinery; moving loads, such as trucks or trains on bridges; impact loads, such as that from a falling weight striking a floor or the shock wave from an explosion impinging on a wall; and seismic loads or other forces created in a structure by rapid movements of supports.

Loads may be considered distributed or concentrated. **Uniformly distributed loads** are forces that are, or for practical purposes may be considered, constant over a surface of the supporting member; dead weight of a rolled-steel beam is a good example. **Concentrated loads** are forces that have such a small contact area as to be negligible compared with the entire surface area of the supporting member. For example, a beam supported on a girder, may, for all practical purposes, be considered a concentrated load on the girder.

In addition, loads may be axial, eccentric, or torsional. An **axial load** is a force whose resultant passes through the centroid of a section under consideration and is perpendicular to the plane of the section. An **eccentric load** is a force perpendicular to the plane of the section under consideration but not passing through the centroid of the section, thus bending the supporting member. **Torsional loads** are forces that are offset from the shear center of the section under consideration and are inclined to or in the plane of the section, thus twisting the supporting member.

Also, loads are classified according to the nature of the source. For example: **Dead loads** include materials, equipment, constructions, or other elements of weight supported in, on, or by a structural element, including its own weight, that are intended to remain permanently in place. **Live loads** include all occupants, materials, equipment, constructions, or other elements of weight supported in, on, or by a structural element that will or are likely to be moved or relocated during the expected life of the structure. **Impact loads** are a fraction of the live loads used to account for additional stresses and deflections resulting from movement of the live loads. **Wind loads** are maximum forces that may be applied to a structural element by wind in a mean recurrence interval, or a set of forces that will produce equivalent stresses. Mean recurrence intervals generally used are 25 years for structures with no occupants or offering negligible risk to life, 50 years for ordinary permanent structures, and 100 years for permanent structures with a high degree of sensitivity to wind and an unusually high degree of hazard to life and property in case of failure. **Snow loads** are maximum forces that may be applied by snow accumulation in a mean recurrence interval. **Seismic loads** are forces that produce maximum stresses or deformations in a structural element during an earthquake, or equivalent forces.

Probable maximum loads should be used in design. For buildings, minimum design load should be that specified for expected conditions in the local building code or, in the absence of an applicable local code, in "Minimum Design Loads for Buildings and Other Structures," ASCE 7-93, American Society of Civil Engineers, Reston, VA, (www.asce.org). For highways and highway bridges, minimum design loads should be those given in "Standard Specifications for Highway Bridges," American Association of State Highway and Transportation Officials, Washington, D.C. (www.transportation.org). For railways and railroad bridges, minimum design loads should be those given in "Manual for Railway Engineering," American Railway Engineering and Maintenance-of-Way Association, Chicago (www.arema.org).

6.3 Static Equilibrium

If a structure and its components are so supported that after a small deformation occurs no further motion is possible, they are said to be in equilibrium. Under such circumstances, external forces are in balance and internal forces, or stresses, exactly counteract the loads.

Since there is no translatory motion, the vector sum of the external forces must be zero. Since there is no rotation, the sum of the moments of the external forces about any point must be zero. For the same reason, if we consider any portion of the structure and the loads on it, the sum of the external and internal forces on the boundaries of that section must be zero. Also, the sum of the moments of these forces must be zero.

In Fig. 6.1, for example, the sum of the forces R_L and R_R needed to support the truss is equal to the 20-kip load on the truss (1 **kip** = 1 kilopound = 1000 lb = 0.5 ton). Also, the sum of the moments of the external forces is zero about any point; about the right end, for instance, it is $40 \times 15 - 30 \times 20 = 600 - 600$.

Figure 6.2 shows the portion of the truss to the left of section AA. The internal forces at the cut members balance the external load and hold this piece of the truss in equilibrium.

When the forces act in several directions, it generally is convenient to resolve them into components parallel to a set of perpendicular axes that will simplify computations. For example, for forces in a single plane, the most useful technique is to resolve them into horizontal and vertical components. Then, for a structure in equilibrium, if H represents the horizontal components, V the

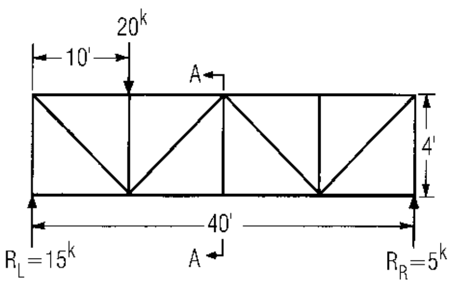


Fig. 6.1 Truss in equilibrium under load. Upward-acting forces, or reactions, R_L and R_R, equal the 20-kip downward-acting force.

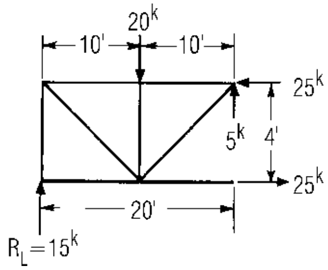


Fig. 6.2 Section of the truss shown in Fig. 6.1 is kept in equilibrium by stresses in the components.

vertical components, and M the moments of the components about any point in the plane,

$$\Sigma H = 0 \quad \Sigma V = 0 \quad \text{and} \quad \Sigma M = 0 \qquad (6.1)$$

These three equations may be used to determine three unknowns in any nonconcurrent coplanar force system, such as the truss in Figs. 6.1 and 6.2. They may determine the magnitude of three forces for which the direction and point of application already are known, or the magnitude, direction, and point of application of a single force. Suppose, for the truss in Fig. 6.1, the reactions at the supports are to be computed. Take the sum of the moments about the right support and equate them to zero to find the left reaction: $40R_L - 30 \times 20 = 0$, from which $R_L = 600/40 = 15$ kips. To find the right reaction, take moments about the left support and equate the sum to zero: $10 \times 20 - 40R_R = 0$, from which $R_R = 5$ kips. As an alternative, equate the sum of the vertical forces to zero to obtain R_R after finding R_L: $20 - 15 - R_R = 0$, from which $R_R = 5$ kips.

Stress and Strain

6.4 Unit Stress and Strain

It is customary to give the strength of a material in terms of unit stress, or internal force per unit of area. Also, the point at which yielding starts generally is expressed as a *unit stress.* Then, in some design methods, a safety factor is applied to either of these stresses to determine a unit stress that should not be exceeded when the member carries design loads. That unit stress is known as the *allowable stress*, or *working stress.*

In working-stress design, to determine whether a structural member has adequate load-carrying capacity, the designer generally has to compute the maximum unit stress produced by design loads in the member for each type of internal force—tensile, compressive, or shearing—and compare it with the corresponding allowable unit stress.

When the loading is such that the unit stress is constant over a section under consideration, the stress may be computed by dividing the force by the area of the section. But, generally, the unit stress varies from point to point. In those cases, the unit stress at any point in the section is the limiting value of the ratio of the internal force on any small area to that area, as the area is taken smaller and smaller.

Unit Strain ▪ Sometimes in the design of a structure, the designer may be more concerned with limiting deformation or strain than with strength. Deformation in any direction is the total change in the dimension of a member in that direction. *Unit strain* in any direction is the deformation per unit of length in that direction.

When the loading is such that the unit strain is constant over the length of a member, it may be computed by dividing the deformation by the original length of the member. In general, however, unit strain varies from point to point in a member. Like a varying unit stress, it represents the limiting value of a ratio.

6.5 Stress-Strain Relations

When a material is subjected to external forces, it develops one or more of the following types of strain: linear elastic, nonlinear elastic, viscoelastic, plastic, and anelastic. Many structural materials exhibit linear elastic strains under design loads. For these materials, unit strain is proportional to unit stress until a certain stress, the proportional limit, is exceeded (point A in Fig. 6.3*a* to *c*). This relationship is known as **Hooke's law**.

For axial tensile or compressive loading, this relationship may be written

$$f = E\varepsilon \quad \text{or} \quad \varepsilon = \frac{f}{E} \qquad (6.2)$$

where f = unit stress

e = unit strain

E = Young's modulus of elasticity

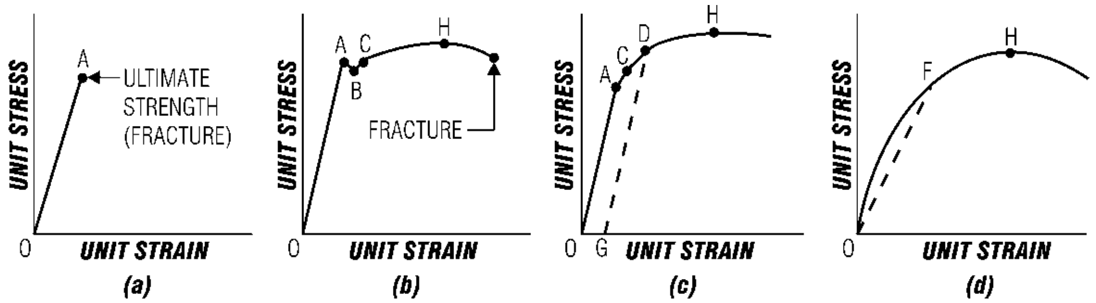


Fig. 6.3 Relationship of unit stress and unit strain for various materials. (*a*) Brittle. (*b*) Linear elastic with distinct proportional limit. (*c*) Linear elastic with an indistinct proportional limit. (*d*) Nonlinear.

Within the elastic limit, there is no permanent residual deformation when the load is removed. Structural steels have this property.

In nonlinear elastic behavior, stress is not proportional to strain, but there is no permanent residual deformation when the load is removed. The relation between stress and strain may take the form

$$\varepsilon = \left(\frac{f}{K}\right)^n \tag{6.3}$$

where K = pseudoelastic modulus determined by test

n = constant determined by test

Viscoelastic behavior resembles linear elasticity. The major difference is that in linear elastic behavior, the strain stops increasing if the load does; but in viscoelastic behavior, the strain continues to increase although the load becomes constant and a residual strain remains when the load is removed. This is characteristic of many plastics.

Anelastic deformation is time-dependent and completely recoverable. Strain at any time is proportional to change in stress. Behavior at any given instant depends on all prior stress changes. The combined effect of several stress changes is the sum of the effects of the several stress changes taken individually.

Plastic strain is not proportional to stress, and a permanent deformation remains on removal of the load. In contrast with anelastic behavior, plastic deformation depends primarily on the stress and is largely independent of prior stress changes.

When materials are tested in axial tension and corresponding stresses and strains are plotted, stress-strain curves similar to those in Fig. 6.3 result. Figure 6.3*a* is typical of a brittle material, which deforms in accordance with Hooke's law up to fracture. The other curves in Fig. 6.3 are characteristic of ductile materials; because strains increase rapidly near fracture with little increase in stress, they warn of imminent failure, whereas brittle materials fail suddenly.

Figure 6.3*b* is typical of materials with a marked proportional limit *A*. When this is exceeded, there is a sudden drop in stress, then gradual stress increase with large increases in strain to a maximum before fracture. Figure 6.3*c* is characteristic of materials that are linearly elastic over a substantial range but have no definite proportional limit. And Fig. 6.3*d* is a representative curve for materials that do not behave linearly at all.

Modulus of Elasticity ▪ *E* is given by the slope of the straight-line portion of the curves in Fig. 6.3*a* to *c*. It is a measure of the inherent rigidity or stiffness of a material. For a given geometric configuration, a material with a larger *E* deforms less under the same stress.

At the termination of the linear portion of the stress-strain curve, some materials, such as low-carbon steel, develop an upper and lower **yield point** (*A* and *B* in Fig. 6.3*b*). These points mark a range in which there appears to be an increase in strain with no increase or a small decrease in stress. This behavior may be a consequence of inertia effects in the testing machine and the deformation characteristics of the test specimen. Because of the location of the yield points, the yield stress sometimes is used erroneously as a synonym for proportional limit and elastic limit.

The **proportional limit** is the maximum unit stress for which Hooke's law is valid. The **elastic limit** is the largest unit stress that can be developed without a permanent set remaining after removal of the load (*C* in Fig. 6.3). Since the elastic limit is always difficult to determine and many materials do not have a well-defined proportional limit, or even have one at all, the offset yield strength is used as a measure of the beginning of plastic deformation.

The **offset yield strength** is defined as the stress corresponding to a permanent deformation, usually 0.01% (0.0001 in/in) or 0.20% (0.002 in/in). In Fig. 6.3*c* the yield strength is the stress at *D*, the intersection of the stress-strain curve and a line *GD* parallel to the straight-line portion and starting at the given unit strain. This stress sometimes is called the **proof stress**.

For materials with a stress-strain curve similar to that in Fig. 6.3*d*, with no linear portion, a **secant modulus**, represented by the slope of a line, such as *OF*, from the origin to a specified point on the curve, may be used as a measure of stiffness. An alternative measure is the **tangent modulus**, the slope of the stress-strain curve at a specified point.

Ultimate tensile strength is the maximum axial load observed in a tension test divided by the original cross-sectional area. Characterized by the beginning of necking down, a decrease in cross-sectional area of the specimen, or local instability, this stress is indicated by *H* in Fig. 6.3.

Ductility is the ability of a material to undergo large deformations without fracture. It is measured by elongation and reduction of area in a tension test and expressed as a percentage. Ductility depends on temperature and internal stresses as well as the characteristics of the material; a material that may be ductile under one set of conditions may have a brittle failure at lower temperatures or under tensile stresses in two or three perpendicular directions.

Modulus of rigidity, or shearing modulus of elasticity, is defined by

$$G = \frac{\nu}{\gamma} \tag{6.4}$$

where G = modulus of rigidity

ν = unit shearing stress

γ = unit shearing strain

It is related to the modulus of elasticity in tension and compression E by the equation

$$G = \frac{E}{2(1 + \mu)} \tag{6.5}$$

where μ is a constant known as Poisson's ratio (Art. 6.7).

Toughness is the ability of a material to absorb large amounts of energy. Related to the area under the stress-strain curve, it depends on both strength and ductility. Because of the difficulty of determining toughness analytically, often toughness is measured by the energy required to fracture a specimen, usually notched and sometimes at low temperatures, in impact tests. Charpy and Izod, both applying a dynamic load by pendulum, are the tests most commonly used.

Hardness is a measure of the resistance a material offers to scratching and indention. A relative numerical value usually is determined for this property in such tests as Brinell, Rockwell, and Vickers. The numbers depend on the size of an indentation made under a standard load. Scratch resistance is measured on the Mohs scale by comparison with the scratch resistance of 10 minerals arranged in order of increasing hardness from talc to diamond.

Creep is a property of certain materials like concrete that deforms with time under constant load. Shrinkage for concrete is the volume reduction with time. It is unrelated to load application. **Relaxation** is a decrease in load or stress under a sustained constant deformation.

If stresses and strains are plotted in an axial tension test as a specimen enters the inelastic range and then is unloaded, the curve during unloading, if the material was elastic, descends parallel to the straight portion of the curve (for example, *DG* in Fig. 6.3*c*). Completely unloaded, the specimen has a permanent set (*OG*). This also will occur in compression tests.

If the specimen now is reloaded, strains are proportional to stresses (the curve will practically follow *DG*) until the curve rejoins the original curve at *D*. Under increasing load, the reloading curve coincides with that for a single loading. Thus, loading the specimen into the inelastic range, but not to ultimate strength, increases the apparent elastic range. The phenomenon, called **strain**

hardening, or work hardening, appears to increase the yield strength. Usually, when the yield strength of a material is increased through strain hardening, the ductility of the material is reduced.

But if the reloading is reversed in compression, the compressive yield strength is decreased, which is called the **Bauschinger effect**.

6.6 Constant Unit Stress

The simplest cases of stress and strain are those in which the unit stress and strain are constant. Stresses caused by an axial tension or compression load, a centrally applied shear, or a bearing load are examples. These conditions are illustrated in Figs. 6.4 to 6.7.

For constant unit stress, the equation of equilibrium may be written

$$P = Af \tag{6.6}$$

where P = load, lb

A = cross-sectional area (normal to load) for tensile or compressive forces, or area on which sliding may occur for shearing forces, or contact area for bearing loads, in^2

f = tensile, compressive, shearing, or bearing unit stress, psi

For torsional stresses, see Art. 6.18.

Unit strain for the axial tensile and compressive loads is given by

$$\varepsilon = \frac{e}{L} \tag{6.7}$$

where ε = unit strain, in/in

e = total lengthening or shortening of member, in

L = original length of the member, in

Application of Hooke's law and Eq. (6.6) to Eq. (6.7) yields a convenient formula for the deformation:

$$e = \frac{PL}{AE} \tag{6.8}$$

where P = load on member, lb

A = its cross-sectional area, in^2

E = modulus of elasticity of material, psi

[Since long compression members tend to buckle, Eqs. (6.6) to (6.8) are applicable only to short members. See Arts. 6.39 to 6.41.]

Although tension and compression strains represent a simple stretching or shortening of a member, shearing strain is a distortion due to a small rotation. The load on the small rectangular portion of the member in Fig. 6.6 tends to distort it into a parallelogram. The unit shearing strain is the change in the right angle, measured in radians. (See also Art. 6.5.)

6.7 Poisson's Ratio

When a material is subjected to axial tensile or compressive loads, it deforms not only in the

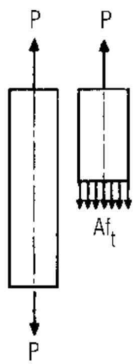


Fig. 6.4
Tension member axially loaded.

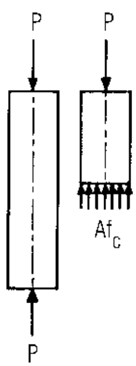


Fig. 6.5
Compression member axially loaded.

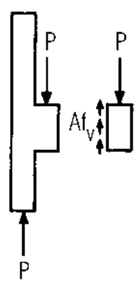


Fig. 6.6
Bracket in shear.

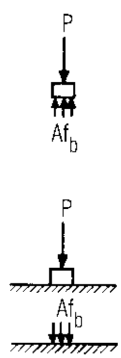


Fig. 6.7
Bearing load.

direction of the loads but normal to them. Under tension, the cross section of a member decreases, and under compression, it increases. The ratio of the unit lateral strain to the unit longitudinal strain is called *Poisson's ratio.*

Within the elastic range, Poisson's ratio is a constant for a material. For materials such as concrete, glass, and ceramics, it may be taken as 0.25; for structural steel, 0.3. It gradually increases beyond the proportional limit and tends to approach a value of 0.5.

Assume, for example, that a steel hanger with an area of 2 in^2 carries a 40-kip (40,000-lb) load. The unit stress is 40/2, or 20 ksi. The unit tensile strain, with modulus of elasticity of steel $E = 30{,}000$ ksi, is 20/30,000, or 0.00067 in/in. With Poisson's ratio as 0.3, the unit lateral strain is -0.3×0.00067, or a shortening of 0.00020 in/in.

6.8 Thermal Stresses

When the temperature of a body changes, its dimensions also change. Forces are required to prevent such dimensional changes, and stresses are set up in the body by these forces.

If α is the coefficient of expansion of the material and T the change in temperature, the unit strain in a bar restrained by external forces from expanding or contracting is

$$\varepsilon = \alpha T \tag{6.9}$$

According to Hooke's law, the stress f in the bar is

$$f = E\alpha T \tag{6.10}$$

where $E =$ modulus of elasticity.

When a circular ring, or hoop, is heated and then slipped over a cylinder of slightly larger diameter d than d_r, the original hoop diameter, the hoop will develop a tensile stress on cooling. If the diameter is very large compared with the hoop thickness, so that radial stresses can be neglected, the unit tensile stresses may be assumed constant. The unit strain will be

$$\varepsilon = \frac{\pi d - \pi d_1}{\pi d_1} = \frac{d - d_1}{d_1}$$

and the hoop stress will be

$$f = \frac{(d - d_1)E}{d_1} \tag{6.11}$$

6.9 Axial Stresses in Composite Members

In a homogeneous material, the centroid of a cross section lies at the intersection of two perpendicular axes so located that the moments of the areas on opposite sides of an axis about that axis are zero. To find the centroid of a cross section containing two or more materials, the moments of the products of the area A of each material and its modulus of elasticity E should be used, in the elastic range.

Consider now a prism composed of two materials, with modulus of elasticity E_1 and E_2, extending the length of the prism. If the prism is subjected to a load acting along the centroidal axis, then the unit strain ε in each material will be the same. From the equation of equilibrium and Eq. (6.8), noting that the length L is the same for both materials,

$$\varepsilon = \frac{P}{A_1E_1 + A_2E_2} = \frac{P}{\Sigma AE} \tag{6.12}$$

where A_1 and A_2 are the cross-sectional areas of each material and P the axial load. The unit stresses in each material are the products of the unit strain and its modulus of elasticity:

$$f_1 = \frac{PE_1}{\Sigma AE} \qquad f_2 = \frac{PE_2}{\Sigma AE} \tag{6.13}$$

6.10 Stresses in Pipes and Pressure Vessels

In a cylindrical pipe under internal radial pressure, the circumferential unit stresses may be assumed constant over the pipe thickness t, in, if the diameter is relatively large compared with the thickness (at least 15 times as large). Then, the circumferential unit stress, in pounds per square inch, is given by

$$f = \frac{pR}{t} \tag{6.14}$$

where $p =$ internal pressure, psi

$R =$ average radius of pipe, in (see also Art. 21.14)

In a closed cylinder, the pressure against the ends will be resisted by longitudinal stresses in the cylinder. If the cylinder is thin, these stresses, psi, are given by

$$f_2 = \frac{pR}{2t} \tag{6.15}$$

Equation (6.15) also holds for the stress in a thin spherical tank under internal pressure p with R the average radius.

In a thick-walled cylinder, the effect of radial stresses f_r becomes important. Both radial and circumferential stresses may be computed from Lamé's formulas:

$$f_r = p\frac{r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right) \tag{6.16}$$

$$f = p\frac{r_i^2}{r_o^2 - r_o^2}\left(1 + \frac{r_o^2}{r^2}\right) \tag{6.17}$$

where r_i = internal radius of cylinder, in

r_o = outside radius of cylinder, in

r = radius to point where stress is to be determined, in

The equations show that if the pressure p acts outward, the circumferential stress f will be tensile (positive) and the radial stress compressive (negative). The greatest stresses occur at the inner surface of the cylinder ($r = r_i$):

$$\text{Max } f_r = -p \tag{6.18}$$

$$\text{Max } f = \frac{k^2 + 1}{k^2 - 1}p \tag{6.19}$$

where $k = r_o/r_i$. Maximum shear stress is given by

$$\text{Max } f_v = \frac{k^2}{k^2 - 1}p \tag{6.20}$$

For a closed cylinder with thick walls, the longitudinal stress is approximately

$$f_z = \frac{p}{r_i(k^2 - 1)} \tag{6.21}$$

But because of end restraints, this stress will not be correct near the ends.

(S. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill Book Company, New York.)

6.11 Strain Energy

Stressing a bar stores energy in it. For an axial load P and a deformation e, the energy stored called strain energy is

$$U = \frac{1}{2}Pe \tag{6.22a}$$

assuming the load is applied gradually and the bar is not stressed beyond the proportional limit. The equation represents the area under the load-deformation curve up to the load P. Application of Eqs. (6.2) and (6.6) to Eq. (6.22*a*) yields another useful equation for energy, in-lb:

$$U = \frac{f^2}{2E}AL \tag{6.22b}$$

where f = unit stress, psi

E = modulus of elasticity of material, psi

A = cross-sectional area, in^2

L = length of bar, in

Since AL is the volume of the bar, the term $f^2/2E$ gives the energy stored per unit of volume. It represents the area under the stress-strain curve up to the stress f.

Modulus of resilience is the energy stored per unit of volume in a bar stressed by a gradually applied axial load up to the proportional limit. This modulus is a measure of the capacity of the material to absorb energy without danger of being permanently deformed. It is important in designing members to resist energy loads.

Equation (6.22*a*) is a general equation that holds true when the **principle of superposition** applies (the total deformation produced at a point by a system of forces is equal to the sum of the deformations produced by each force). In the general sense, P in Eq. (6.22*a*) represents any group of statically interdependent forces that can be completely defined by one symbol, and e is the corresponding deformation.

The strain-energy equation can be written as a function of either the load or the deformation. For axial tension or compression, strain energy, in inch-pounds, is given by

$$U = \frac{P^2L}{2AE} \qquad U = \frac{AEe^2}{2L} \tag{6.23a}$$

where P = axial load, lb

e = total elongation or shortening, in

L = length of member, in

A = cross-sectional area, in^2

E = modulus of elasticity, psi

For pure shear:

$$U = \frac{V^2 L}{2AG} \qquad U = \frac{AGe^2}{2L} \tag{6.23b}$$

where V = shearing load, lb

e = shearing deformation, in

L = length over which deformation takes place, in

A = shearing area, in^2

G = shearing modulus, psi

For torsion:

$$U = \frac{T^2 L}{2JG} \qquad U = \frac{JG\phi^2}{2L} \tag{6.23c}$$

where T = torque, in-lb

ϕ = angle of twist, rad

L = length of shaft, in

J = polar moment of inertia of cross section, in^4

G = shearing modulus, psi

For pure bending (constant moment):

$$U = \frac{M^2 L}{2EI} \qquad U = \frac{EI\theta^2}{2L} \tag{6.23d}$$

where M = bending moment, in-lb

θ = angle of rotation of one end of beam with respect to other, rad

L = length of beam, in

I = moment of inertia of cross section, in^4

E = modulus of elasticity, psi

For beams carrying transverse loads, the total strain energy is the sum of the energy for bending and that for shear. (See also Art. 6.54.)

Stresses at a Point

Tensile and compressive stresses sometimes are referred to as *normal stresses* because they act normal to the cross section. Under this concept, tensile stresses are considered positive normal stresses and compressive stresses negative.

6.12 Stress Notation

Consider a small cube extracted from a stressed member and placed with three edges along a set of x, y, z coordinate axes. The notations used for the components of stress acting on the sides of this element and the direction assumed as positive are shown in Fig. 6.8.

For example, for the sides of the element perpendicular to the z axis, the normal component of stress is denoted by f_z. The shearing stress v is resolved into two components and requires two subscript letters for a complete description. The first letter indicates the direction of the normal to the plane under consideration; the second letter gives the direction of the component of stress. Thus, for the sides perpendicular to the z axis, the shear component in the x direction is labeled v_{zx} and that in the y direction v_{zy}.

6.13 Stress Components

If, for the small cube in Fig. 6.8, moments of the forces acting on it are taken about the x axis, and assuming the lengths of the edges as dx, dy, and dz, the equation of equilibrium requires that

$$(v_{zy}\, dx\, dy)\, dz = (v_{yz}\, dx\, dz)\, dy$$

(Forces are taken equal to the product of the area of the face and the stress at the center.) Two similar equations can be written for moments taken about the y and z axes. These equations show that

$$v_{xy} = v_{yx} \qquad v_{zx} = v_{xz} \qquad v_{zy} = v_{yz} \tag{6.24}$$

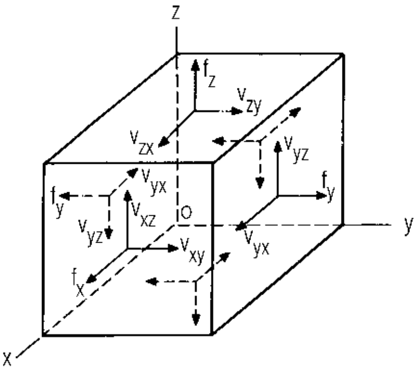


Fig. 6.8 Stresses at a point in a rectangular coordinate system.

Thus, components of shearing stress on two perpendicular planes and acting normal to the intersection of the planes are equal. Consequently, to describe the stresses acting on the coordinate planes through a point, only six quantities need be known: the three normal stresses f_x, f_y, f_z and three shearing components $\nu_{xy} = \nu_{yx}$, $\nu_{zx} = \nu_{xz}$, and $\nu_{zy} = \nu_{yz}$.

If only the normal stresses are acting, the unit strains in the x, y, and z directions are

$$\varepsilon_x = \frac{1}{E}[f_x - \mu(f_y + f_z)]$$

$$\varepsilon_y = \frac{1}{E}[f_y - \mu(f_x + f_z)] \qquad (6.25)$$

$$\varepsilon_z = \frac{1}{E}[f_z - \mu(f_x + f_y)]$$

where μ = Poisson's ratio. If only shearing stresses are acting, the distortion of the angle between edges parallel to any two coordinate axes depends only on shearing-stress components parallel to those axes. Thus, the unit shearing strains are (see Art. 6.5)

$$\gamma_{xy} = \frac{1}{G}\nu_{xy} \quad \gamma_{yz} = \frac{1}{G}\nu_{yz} \quad \gamma_{zx} = \frac{1}{G}\nu_{zx} \qquad (6.26)$$

6.14 Two-Dimensional Stress

When the six components of stress necessary to describe the stresses at a point are known (Art. 6.13), the stresses on any inclined plane through the same point can be determined. For two-dimensional stress, only three stress components need be known.

Assume, for example, that at a point O in a stressed plate, the components f_x, f_y, and ν_{xy} are known (Fig. 6.9). To find the stresses on any other plane through the z axis, take a plane parallel to it close to O, so that this plane and the coordinate planes form a tiny triangular prism. Then, if α is the angle the normal to the plane makes with the x axis, the normal and shearing stresses on the inclined plane, to maintain equilibrium, are

$$f = f_x \cos^2\alpha + f_y \sin^2\alpha + 2\nu_{xy}\sin\alpha\cos\alpha \qquad (6.27)$$

$$\nu = \nu_{xy}(\cos^2\alpha - \sin^2\alpha) + (f_y - f_x)\sin\alpha\cos\alpha \qquad (6.28)$$

(See also Art. 6.17.)

Note: All structural members are three-dimensional. While two-dimensional stress calculations may be sufficiently accurate for most practical purposes, this is not always the case. For

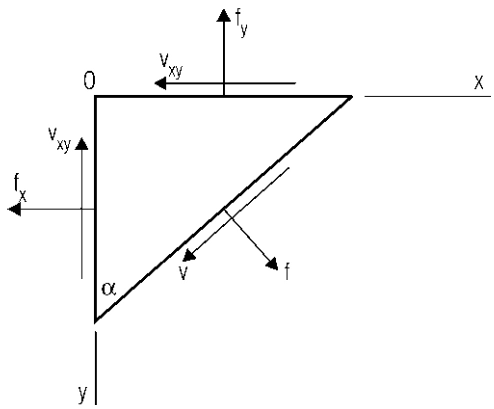


Fig. 6.9 Stresses at a point on a plane inclined to the axes.

example, although loads may create normal stresses on two perpendicular planes, a third normal stress also exists, as computed with Poisson's ratio. [See Eq. (6.25).]

6.15 Principal Stresses

If a plane at a point O in a stressed plate is rotated, it reaches a position for which the normal stress on it is a maximum or a minimum. The directions of maximum and minimum normal stress are perpendicular to each other, and on the planes in those directions, there are no shearing stresses.

The directions in which the normal stresses become maximum or minimum are called *principal directions*, and the corresponding normal stresses are called *principal stresses.* To find the principal directions, set the value of ν given by Eq. (6.28) equal to zero. Then, the normals to the principal planes make an angle with the x axis given by

$$\tan 2\alpha = \frac{2\nu_{xy}}{f_x - f_y} \qquad (6.29)$$

If the x and y axes are taken in the principal directions, $\nu_{xy} = 0$. In that case, Eqs. (6.27) and (6.28) simplify to

$$f = f_x \cos^2\alpha + f_y \sin^2\alpha \qquad (6.30)$$

$$\nu = \frac{1}{2}(f_y - f_x)\sin 2\alpha \qquad (6.31)$$

where f_x and f_y are the principal stresses at the point, and f and ν are, respectively, the normal and shearing stress on a plane whose normal makes an angle α with the x axis.

If only shearing stresses act on any two perpendicular planes, the state of stress at the point is said to be one of pure shear or simple shear. Under such conditions, the principal directions bisect the angles between the planes on which these shearing stresses act. The principal stresses are equal in magnitude to the pure shears.

6.16 Maximum Shearing Stress at a Point

The maximum unit shearing stress occurs on each of two planes that bisect the angles between the planes on which the principal stresses at a point act. The maximum shear equals half the algebraic difference of the principal stresses:

$$\text{Max}\, v = \frac{f_1 - f_2}{2} \tag{6.32}$$

where f_1 is the maximum principal stress and f_2 the minimum.

6.17 Mohr's Circle

As explained in Art. 6.14, if the stresses on any plane through a point in a stressed plate are known, the stresses on any other plane through the point can be computed. This relationship between the stresses may be represented conveniently on Mohr's circle (Fig. 6.10). In this diagram, normal stress f and shear stress v are taken as rectangular coordinates. Then, for each plane through the point there will correspond a point on the circle, the coordinates of which are the values of f and v for the plane.

Given the principal stresses f_1 and f_2 (Art. 6.15), to find the stresses on a plane making an angle α with the plane on which f_1 acts: Mark off the principal stresses on the f axis (points A and B in Fig. 6.10). Measure tensile stresses to the right of the v axis and compressive stresses to the left. Construct a circle passing through A and B and having its center on the f axis. This is the Mohr's circle for the given stresses at the point under consideration. Draw a radius making an angle 2α with the f axis, as indicated in Fig. 6.10. The coordinates of the intersection with the circle represent the normal and shearing stresses f and v acting on the plane.

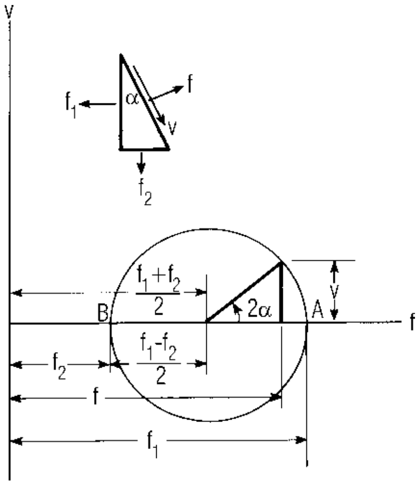


Fig. 6.10 Mohr's circle for stresses at a point—constructed from known principal stresses f_1 and f_2 in a plane.

Given the stresses on any two perpendicular planes f_x, f_y, and v_{xy}, but not the principal stresses f_1 and f_2, to draw the Mohr's circle: Plot the two points representing the known stresses with respect to the f and v axes (points C and D in Fig. 6.11). The line joining these points is a

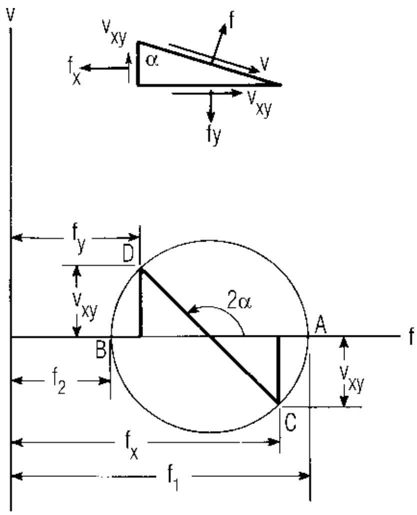


Fig. 6.11 Stress circle constructed from two known normal positive stresses f_x and f_y and a known shear v_{xy}.

diameter of the circle, so bisect CD to find the center of the circle and draw the circle. Its intersections with the f axis determine f_1 and f_2.

(S. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill Book Company, New York, books.mcgraw-hill.com.)

6.18 Torsion

Forces that cause a member to twist about a longitudinal axis are called torsional loads. Simple torsion is produced only by a couple, or moment, in a plane perpendicular to the axis.

If a couple lies in a nonperpendicular plane, it can be resolved into a torsional moment, in a plane perpendicular to the axis, and bending moments, in planes through the axis.

Shear Center ▪ The point in each normal section of a member through which the axis passes and about which the section twists is called the shear center. If the loads on a beam, for example, do not pass through the shear center, they cause the beam to twist. See also Art. 6.36.

If a beam has an axis of symmetry, the shear center lies on it. In doubly symmetrical beams, the shear center lies at the intersection of two axes of symmetry and hence coincides with the centroid.

For any section composed of two narrow rectangles, such as a T beam or an angle, the shear center may be taken as the intersection of the longitudinal center lines of the rectangles.

For a channel section with one axis of symmetry, the shear center is outside the section at a distance from the centroid equal to $e(1 + h^2A/4I)$, where e is the distance from the centroid to the center of the web, h is the depth of the channel, A the cross-sectional area, and I the moment of inertia about the axis of symmetry. (The web lies between the centroid and the shear center.)

Locations of shear centers for several other sections are given in Freidrich Bleich, "Buckling Strength of Metal Structures," chap. 3, McGraw-Hill Publishing Company, New York, 1952, books.mcgraw-hill.com.

Stresses Due to Torsion ▪ Simple torsion is resisted by internal shearing stresses. These can be resolved into radial and tangential shearing stresses, which being normal to each other also are equal (see Art. 6.13). Furthermore, on planes that bisect the angles between the planes on which the shearing stresses act, there also occur compressive and tensile stresses. The magnitude of these normal stresses is equal to that of the shear. Therefore, when torsional loading is combined with other types of loading, the maximum stresses occur on inclined planes and can be computed by the methods of Arts. 6.14 and 6.17.

Circular Sections ▪ If a circular shaft (hollow or solid) is twisted, a section that is plane before twisting remains plane after twisting. Within the proportional limit, the shearing stress at any point in a transverse section varies with the distance from the center of the section. The maximum shear, psi, occurs at the circumference and is given by

$$v = \frac{Tr}{J} \tag{6.33}$$

where T = torsional moment, in-lb

r = radius of section, in

J = polar moment of inertia, in^4

Polar moment of inertia of a cross section is defined by

$$J = \int \rho^2 dA \tag{6.34}$$

where ρ = radius from shear center to any point in section

dA = differential area at point

In general, J equals the sum of the moments of inertia about any two perpendicular axes through the shear center. For a solid circular section, $J = \pi r^4/2$. For a hollow circular section with diameters D and d, $J = \pi(D^4 - d^4)/32$.

Within the proportional limit, the angular twist between two points L inches apart along the axis of a circular bar is, in radians (1 rad = 57.3°):

$$\theta = \frac{TL}{GJ} \tag{6.35}$$

where G is the shearing modulus of elasticity (see Art. 6.5).

Noncircular Sections ▪ If a shaft is not circular, a plane transverse section before twisting does not remain plane after twisting. The resulting warping increases the shearing stresses in some parts of the section and decreases them in others, compared with the shearing stresses that would occur if the section remained plane. Consequently, shearing stresses in a noncircular section are not proportional to distances from the shear center. In elliptical and rectangular sections, for example, maximum shear occurs on the circumference at a point nearest the shear center.

For a solid rectangular section, this maximum shear stress may be expressed in the following form:

$$v = \frac{T}{kb^2 d} \tag{6.36}$$

where b = short side of rectangle, in

d = long side, in

k = constant depending on ratio of these sides:

d/b =	1.0	1.5	2.0	2.5	3	4	5	10	∞
k =	0.208	0.231	0.246	0.258	0.267	0.282	0.291	0.312	0.333

(S. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill Publishing Company, New York, books.mcgraw-hill.com.)

Hollow Tubes ▪ If a thin-shell hollow tube is twisted, the shearing force per unit of length on a cross section (**shear flow**) is given approximately by

$$H = \frac{T}{2A} \tag{6.37}$$

where A is the area enclosed by the mean perimeter of the tube, in^2. And the unit shearing stress is given approximately by

$$v = \frac{H}{t} = \frac{T}{2At} \tag{6.38}$$

where t is the thickness of the tube, in. For a rectangular tube with sides of unequal thickness, the total shear flow can be computed from Eq. (6.37) and the shearing stress along each side from Eq. (6.38), except at the corners, where there may be appreciable stress concentration.

Channels and I Beams ▪ For a narrow rectangular section, the maximum shear is very nearly equal to

$$v = \frac{T}{1/3b^2 d} \tag{6.39}$$

This formula also can be used to find the maximum shearing stress due to torsion in members, such as I beams and channels, made up of thin rectangular components. Let $J = 1/3\Sigma b^3 d$, where b is the thickness of each rectangular component and d the corresponding length. Then, the maximum shear is given approximately by

$$v = \frac{Tb'}{J} \tag{6.40}$$

where b' is the thickness of the web or the flange of the member. Maximum shear will occur at the center of one of the long sides of the rectangular part that has the greatest thickness.

(A. P. Boresi, O. Sidebottom, F. B. Seely, and J. O. Smith, "Advanced Mechanics of Materials," John Wiley & Sons, Inc., New York, www.wiley.com.)

Straight Beams

6.19 Types of Beams

Bridge decks and building floors and roofs frequently are supported on a rectangular grid of flexural members. Different names often are given to the components of the grid, depending on the type of structure and the part of the structure supported on the grid. In general, though, the members spanning between main supports are called **girders** and those they support are called **beams** (Fig. 6.12). Hence, this type of framing is known as beam-and-girder framing.

In bridges, the smaller structural members parallel to the direction in which traffic moves may be called **stringers** and the transverse members **floor beams**. In building roofs, the grid components may be referred to as **purlins** and **rafters;** and in floors, they may be called **joists** and **girders**.

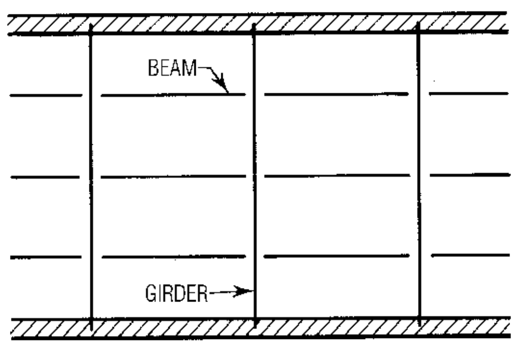


Fig. 6.12 Beam-and-girder framing.

Fig. 6.13 Simple beam, both ends free to rotate.

Fig. 6.14 Cantilever beam.

Fig. 6.15 Beam with one end fixed.

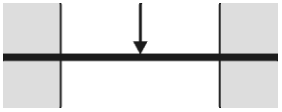


Fig. 6.16 Fixed-end beam.

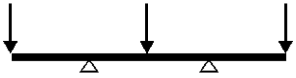


Fig. 6.17 Beam with overhangs.

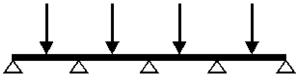


Fig. 6.18 Continuous beam.

Beam-and-girder framing usually is used for relatively short spans and where shallow members are desired to provide ample headroom underneath.

Beams and trusses are similar in behavior as flexural members. The term beam, however, usually is applied to members with top continuously connected to bottom throughout their length, while those with top and bottom connected at intervals are called trusses.

There are many ways in which beams may be supported. Some of the most common methods are shown in Figs. 6.13 to 6.19. The beam in Fig. 6.13 is called a simply supported beam, or **simple beam**. It has supports near its ends that restrain it only against vertical movement. The ends of the beam are free to rotate. When the loads have a horizontal component, or when change in length of the beam due to temperature may be important, the supports may also have to prevent horizontal motion, in which case horizontal restraint at one support generally is sufficient. The distance between the supports is called the **span**. The load carried by each support is called a **reaction**.

The beam in Fig. 6.14 is a **cantilever**. It has a support only at one end. The support provides restraint against rotation and horizontal and vertical movement. Such support is called a **fixed end**. Placing a support under the free end of the cantilever produces the beam in Fig. 6.15. Fixing the free end yields a **fixed-end beam** (Fig. 6.16); no rotation or vertical movement can occur at either end. In actual practice, however, a fully fixed end can seldom be obtained. Most support conditions are intermediate between those for a simple beam and those for a fixed-end beam.

Fig. 6.19 Hung-span (suspended-span) construction.

Figure 6.17 shows a beam that overhangs both its simple supports. The overhangs have a free end, like a cantilever, but the supports permit rotation.

Two types of beams that extend over several supports are illustrated in Figs. 6.18 and 6.19. Figure 6.18 shows a **continuous beam**. The one in Fig. 6.19 has one or two hinges in certain spans; it is called **hung-span**, or suspended-span, construction. In effect, it is a combination of simple beams and beams with overhangs.

Reactions for the beams in Figs. 6.13, 6.14, and 6.17 and the type of beam in Fig. 6.19 with hinges suitably located may be found from the equations of equilibrium, which is why they are classified as **statically determinate beams**.

The equations of equilibrium, however, are not sufficient to determine the reactions of the beams in Figs. 6.15, 6.16, and 6.18. For those beams, there are more unknowns than equations. Additional equations must be obtained based on a knowledge of the deformations, for example, that a fixed end permits no rotation. Such beams are classified as **statically indeterminate**. Methods for finding the stresses in that type of beam are given in Arts. 6.51 to 6.63.

6.20 Reactions

As pointed out in Art. 6.19, the loads imposed by a simple beam on its supports can be found by application of the equations of equilibrium [Eq. (6.1)]. Consider, for example, the 60-ft-long beam with overhangs in Fig. 6.20. This beam carries a uniform

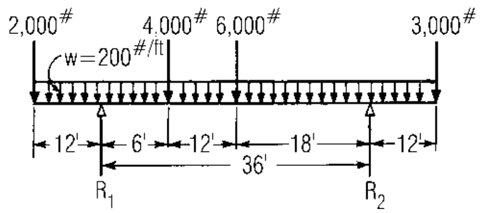


Fig. 6.20 Beam with overhangs loaded with both uniform and concentrated loads.

load of 200 lb/lin ft over its entire length and several concentrated loads. The span is 36 ft.

To find reaction R_1, take moments about R_2 and equate the sum of the moments to zero (assume clockwise rotation to be positive, counterclockwise, negative):

$$-2000 \times 48 + 36R_1 - 4000 \times 30 - 6000 \times 18 + 3000 \times 12 - 200 \times 60 \times 18 = 0$$

$$R_1 = 14{,}000 \text{ lb}$$

In this calculation, the moment of the uniform load was found by taking the moment of its resultant, 200×60, which acts at the center of the beam.

To find R_2, proceed in a similar manner by taking moments about R_1 and equating the sum to zero, or equate the sum of the vertical forces to zero. Generally it is preferable to use the moment equation and apply the other equation as a check.

As an alternative procedure, find the reactions caused by uniform and concentrated loads separately and sum the results. Use the fact that the reactions due to symmetrical loading are equal, to simplify the calculation. To find R_2 by this procedure, take half the total uniform load

$$0.5 \times 200 \times 60 = 6000 \text{ lb}$$

and add it to the reaction caused by the concentrated loads, found by taking moments about R_1, dividing by the span, and summing:

$$-2000 \times \frac{12}{36} + 4000 \times \frac{6}{36} + 6000 \times \frac{18}{36} + 3000 \times \frac{48}{36} = 7000 \text{ lb}$$

$$R_2 = 6000 + 7000 = 13{,}000 \text{ lb}$$

Check to see that the sum of the reactions equals the total applied load:

$$14{,}000 + 13{,}000 = 2000 + 4000 + 6000 + 3000 + 200 \times 60$$

$$27{,}000 = 27{,}000$$

Reactions for simple beams with various loads are given in Figs. 6.33 to 6.38.

To find the reactions of a continuous beam, first determine the end moments and shears (Arts. 6.58 to 6.63); then if the continuous beam is considered as a series of simple beams with these applied as external loads, the beam will be statically determinate and the reactions can be determined from the equations of equilibrium. (For an alternative method, see Art. 6.57.)

6.21 Internal Forces

At every section of a beam in equilibrium, internal forces act to prevent motion. For example, assume the beam in Fig. 6.20 cut vertically just to the right of its center. Adding the external forces, including the reaction, to the left of this cut (see Fig. 6.21*a*) yields an unbalanced downward load of 4000 lb. Evidently, at the cut section, an upward-acting internal force of 4000 lb must be present to maintain equilibrium. Also, taking moments of the external forces about the section yields an unbalanced moment of 54,000 ft-lb. To maintain

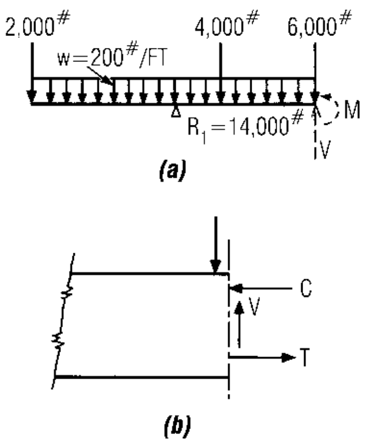


Fig. 6.21 Sections of beam kept in equilibrium by internal stresses.

equilibrium, there must be an internal moment of 54,000 ft-lb resisting it.

This internal, or resisting, moment is produced by a couple consisting of a force *C* acting on the top part of the beam and an equal but opposite force *T* acting on the bottom part (Fig. 6.21*b*). For this type of beam and loading, the top force is the resultant of compressive stresses acting over the upper portion of the beam, and the bottom force is the resultant of tensile stresses acting over the bottom part. The surface at which the stresses change from compression to tension—where the stress is zero—is called the **neutral surface**.

6.22 Shear Diagrams

As explained in Art. 6.21, at a vertical section through a beam in equilibrium, external forces on one side of the section are balanced by internal forces. The unbalanced external vertical force at the section is called the shear. It equals the algebraic sum of the forces that lie on either side of the section. For forces on the left of the section, those acting upward are considered positive and those acting downward negative. For forces on the right of the section, signs are reversed.

A shear diagram represents graphically the shear at every point along the length of a beam. The shear diagram for the beam in Fig. 6.20 is shown in Fig. 6.22*b*. The beam is drawn to scale and the loads and reactions are located at the points at which they act. Then, a convenient zero axis is drawn horizontally from which to plot the shears to scale. Start at the left end of the beam, and directly under the 2000-lb load there, scale off -2000 from the zero axis. Next, determine the shear just to the left of the next concentrated load, the left support: $-2000 - 200 \times 12 = -4400$ lb. Plot this downward under R_1. Note that in passing from just to the left of the support to just to the right, the shear changes by the magnitude of the reaction, from -4400 to $-4400 + 14{,}000$ or 9600 lb, so plot this value also under R_1. Under the 4000-lb load, plot the shear just to the left of it, $9600 - 200 \times 6$, or 8400 lb, and the shear just to the right, $8400 - 4000$, or 4400 lb. Proceed in this manner to the right end, where the shear is 3000 lb, equal to the load on the free end.

To complete the diagram, the points must be connected. Straight lines can be used because shear varies uniformly for a uniform load (see Fig. 6.24*b*)

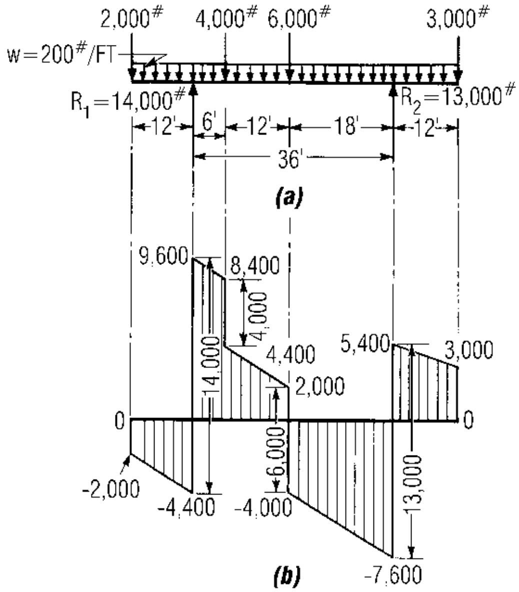


Fig. 6.22 Shear diagram for beam in Fig. 6.20.

6.23 Bending-Moment Diagrams

About a vertical section through a beam in equilibrium, there is an unbalanced moment due to external forces, called *bending moment*. For forces on the left of the section, clockwise moments are considered positive and counterclockwise moments negative. For forces on the right of the section, the signs are reversed. Thus, when the bending moment is positive, the bottom of a simple beam is in tension and the top is in compression.

A bending-moment diagram represents graphically the bending moment of every point along the length of the beam. Figure 6.23*c* is the bending-moment diagram for the beam with concentrated loads in Fig. 6.23*a*. The beam is drawn to scale, and the loads and reactions are located at the points at which they act. Then, a horizontal line is drawn to represent the zero axis from which to plot the bending moments to scale. Note that the bending moment at both supports for this simple beam is zero. Between the supports and the first load, the bending moment is proportional to the distance from the support since the bending moment in that region equals the reaction times the distance from the support. Hence, the bending-moment diagram for this portion of the beam is a sloping straight line.

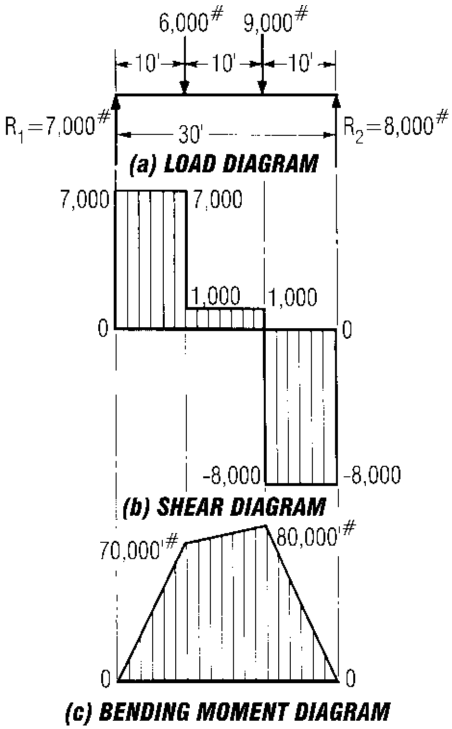


Fig. 6.23 Shear and moment diagrams for beam with concentrated loads.

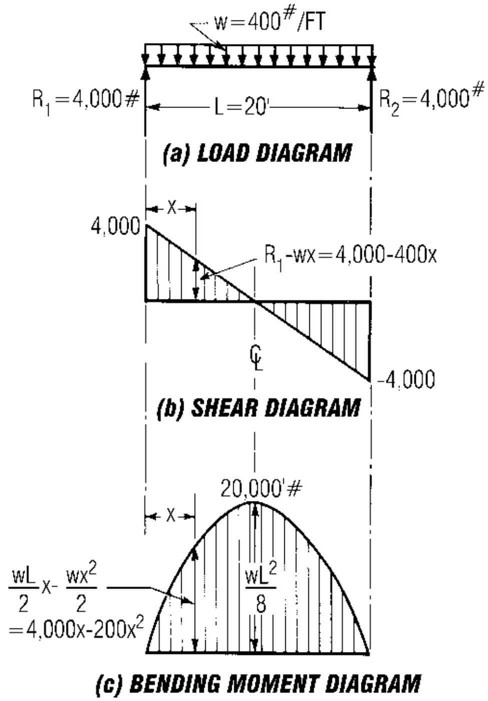


Fig. 6.24 Shear and moment diagrams for uniformly loaded beam.

To find the bending moment under the 6000-lb load, consider only the forces to the left of it, in this case only the reaction R_1. Its moment about the 6000-lb load is 7000 × 10, or 70,000 ft-lb. The bending-moment diagram, then, between the left support and the first concentrated load is a straight line rising from zero at the left end of the beam to 70,000, plotted, to a convenient scale, under the 6000-lb load.

To find the bending moment under the 9000-lb load, add algebraically the moments of the forces to its left: 7000 × 20 − 6000 × 10 = 80,000 ft-lb. (This result could have been obtained more easily by considering only the portion of the beam on the right, where the only force present is R_2, and reversing the sign convention: 8000 × 10 = 80,000 ft-lb.) Since there are no other loads between the 6000- and 9000-lb loads, the bending-moment diagram between them is a straight line.

If the bending moment and shear are known at any section, the bending moment at any other section can be computed if there are no un-known forces between the sections. The rule is:

The bending moment at any section of a beam equals the bending moment at any section to the left, plus the shear at that section times the distance between sections, minus the moments of intervening loads. If the section with known moment and shear is on the right, the sign convention must be reversed.

For example, the bending moment under the 9000-lb load in Fig. 6.23*a* also could have been determined from the moment under the 6000-lb load and the shear just to the right of that load. As indicated in the shear diagram (Fig. 6.23*b*), that shear is 1000 lb. Thus, the moment is given by 70,000 + 1000 × 10 = 80,000 ft-lb.

Bending-moment diagrams for simple beams with various loadings are shown in Figs. 6.33 to 6.38. To obtain bending-moment diagrams for loading conditions that can be represented as a sum of the loadings shown, sum the bending moments at corresponding locations on the beam as given on the diagram for the component loads.

For a simple beam carrying a uniform load, the bending-moment diagram is a parabola (Fig. 6.24*c*). The maximum moment occurs at the center and equals $wL^2/8$ or $WL/8$, where w is the load per linear foot and $W = wL$ is the total load on the beam.

The bending moment at any section of a simply supported, uniformly loaded beam equals one-half the product of the load per linear foot and the distances to the section from both supports:

$$M = \frac{w}{2}x(L - x) \tag{6.41}$$

6.24 Shear-Moment Relationship

The slope of the bending-moment curve at any point on a beam equals the shear at that point. If V is the shear, M the moment, and x the distance along the beam,

$$V = \frac{dM}{dx} \tag{6.42}$$

Since maximum bending moment occurs when the slope changes sign, or passes through zero, maximum moment (positive or negative) occurs at the point of zero shear.

Integration of Eq. (6.42) yields

$$M_1 - M_2 = \int_{x2}^{x1} V dx \tag{6.43}$$

Thus, the change in bending moment between any two sections of a beam equals the area of the shear diagram between ordinates at the two sections.

6.25 Moving Loads and Influence Lines

Influence lines are a useful device for solving problems involving moving loads. An influence line indicates the effect at a given section of a unit load placed at any point on the structure.

For example, to plot the influence line for bending moment at a point on a beam, compute the moments produced at that point as a unit load moves along the beam and plot these moments under the corresponding positions of the unit load. Actually, the unit load need not be placed at every point along the beam. The equation of the influence line can be determined in many cases by placing the load at an arbitrary point and computing the bending moment in general terms. (See also Art. 6.55.)

To draw the influence line for reaction at A for a simple beam AB (Fig. 6.25*a*), place a unit load at an arbitrary distance xL from B. The reaction at A due to this load is $1\ xL/L = x$. Then, $R_A = x$ is the equation of the influence line. It represents a straight line sloping downward from unity at A, when the unit load is at that end of the beam, to zero at B, when the load is at B (Fig. 6.25*a*).

Figure 6.25*b* shows the influence line for bending moment at the center of a beam. It resembles in appearance the bending-moment diagram for a load at the center of the beam, but its significance is entirely different. Each ordinate gives the moment at midspan for a load at the location of the ordinate. The diagram indicates that if a unit load is placed at a distance xL from one end, it produces a bending moment of $xL/2$ at the center of the span.

Figure 6.25*c* shows the influence line for shear at the quarter point of a beam. When the load is to the right of the quarter point, the shear is positive and equal to the left reaction. When the load is to the left, the shear is negative and equals the right reaction. Thus, to produce maximum shear at the quarter point, loads should be placed only to the right of the quarter point, with the largest load at the quarter point, if possible. For a uniform load, maximum shear results when the load extends from the right end of the beam to the quarter point.

Suppose, for example, that a 60-ft crane girder is to carry wheel loads of 20 and 10 kips, 5 ft apart. For maximum shear at the quarter point, place the 20-kip wheel there and the 10-kip wheel 5 ft to the right. The corresponding ordinates of the influence line (Fig. 6.25*c*) are $\frac{3}{4}$ and $40/45 \times \frac{3}{4}$. Hence, the maximum shear is $20 \times \frac{3}{4} + 10 \times 40/45 \times \frac{3}{4} = 21.7$ kips.

Figure 6.25*d* shows influence lines for bending moment at several points on a beam. The apexes of the triangular diagrams fall on a parabola, as indicated by the dashed line. From the diagram, it can be concluded that the maximum moment produced at any section by a single concentrated load moving along a beam occurs when the load is at that section. And the magnitude of the maximum moment increases when the section is moved toward midspan, in accordance with the equation for the parabola given in Fig. 6.25*d*.

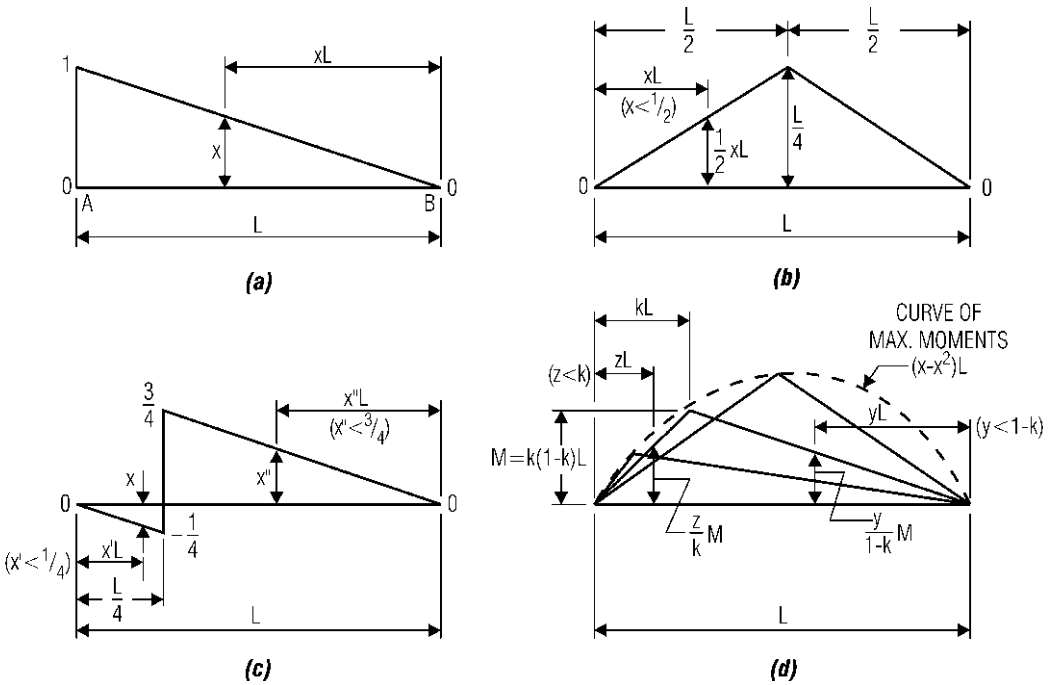


Fig. 6.25 Influence lines for (*a*) reaction at *A*, (*b*) midspan bending moment, (*c*) quarter-point shear, and (*d*) bending moments at several points in a beam.

6.26 Maximum Bending Moment

When a span is to carry several moving concentrated loads, an influence line is useful when determining the position of the loads for which bending moment is a maximum at a given section (see Art. 6.25). For a simple beam, maximum bending moment will occur at a section *C* as loads move across the beam when one of the loads is at *C*. The load to place at *C* is the one for which the expression $W_a/a - W_b/b$ (Fig. 6.26) changes sign as that load passes from one side of *C* to the other. (W_a is the sum of the loads on one side of *C* and W_b the sum of the loads on the other side of *C*.)

When several concentrated loads move along a simple beam, the maximum moment they produce in the beam may be near but not necessarily at midspan. To find the maximum moment, first determine the position of the loads for maximum moment at midspan. Then, shift the loads until the load P_2 (Fig. 6.27) that was at the center of the beam is as far from midspan as

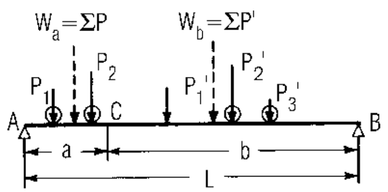


Fig. 6.26 Moving loads on simple beam *AB* placed for maximum moment at *C*.

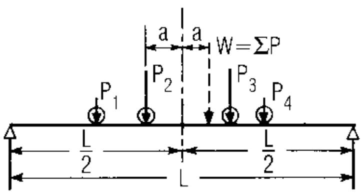


Fig. 6.27 Moving loads placed for maximum moment in a simple beam.

the resultant of all the loads on the span is on the other side of midspan. Maximum moment will occur under P_2. When other loads move on or off the span during the shift of P_2 away from midspan, it may be necessary to investigate the moment under one of the other loads when it and the new resultant are equidistant from midspan.

6.27 Bending Stresses in a Beam

The commonly used flexure formula for computing bending stresses in a beam is based on the following assumptions:

1. The unit stress parallel to the bending axis at any point of a beam is proportional to the unit strain in the same direction at the point. Hence, the formula holds only within the proportional limit.
2. The modulus of elasticity in tension is the same as that in compression.
3. The total and unit axial strain at any point are both proportional to the distance of that point from the neutral surface. (Cross sections that are plane before bending remain plane after bending. This requires that all fibers have the same length before bending, thus that the beam be straight.)
4. The loads act in a plane containing the centroidal axis of the beam and are perpendicular to that axis. Furthermore, the neutral surface is perpendicular to the plane of the loads. Thus, the plane of the loads must contain an axis of symmetry of each cross section of the beam. (The flexure formula does not apply to a beam with cross sections loaded unsymmetrically.)
5. The beam is proportioned to preclude prior failure or serious deformation by torsion, local buckling, shear, or any cause other than bending.

Equating the bending moment to the resisting moment due to the internal stresses at any section of a beam yields the **flexure formula**:

$$M = \frac{fI}{c} \tag{6.44}$$

where M = bending moment at section, in-lb

f = normal unit stress at distance c, in, from the neutral axis (Fig. 6.28), psi

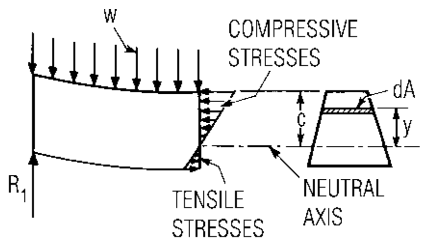


Fig. 6.28 Unit stresses on a beam section produced by bending.

I = moment of inertia of cross section with respect to neutral axis, in^4

Generally, c is taken as the distance to the outermost fiber to determine maximum f.

6.28 Moment of Inertia

The neutral axis in a symmetrical beam coincides with the centroidal axis; that is, at any section the neutral axis is so located that

$$\int y \, dA = 0 \tag{6.45}$$

where dA is a differential area parallel to the axis (Fig. 6.28), y is its distance from the axis, and the summation is taken over the entire cross section.

Moment of inertia with respect to the neutral axis is given by

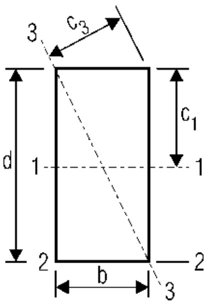
$$I = \int y^2 \, dA \tag{6.46}$$

Values for I for several common cross sections are given in Fig. 6.29. Values for standard structural-steel sections are listed in manuals of the American Institute of Steel Construction. When the moments of inertia of other types of sections are needed, they can be computed directly by applying Eq. (6.46) or by breaking the section up into components for which the moment of inertia is known.

With the following formula, the moment of inertia of a section can be determined from that of its components:

$$I' = I + Ad^2 \tag{6.47}$$

where I = moment of inertia of component about its centroidal axis, in^4



RECTANGLE

$A = bd$

$c_1 = d/2$

$c_2 = d$

$c_3 = \frac{bd}{\sqrt{b^2+d^2}}$

$S_1 = \frac{bd^2}{6}$

$r_1 = \frac{d}{\sqrt{12}}$

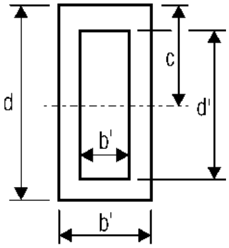
$I_1 = \frac{bd^3}{12}$

$I_2 = \frac{bd^3}{3}$

$I_3 = \frac{b^3d^3}{6(b^2+d^2)}$

$S_3 = \frac{b^2d^2}{6\sqrt{b^2+d^2}}$

$r_3 = \frac{bd}{\sqrt{6(b^2+d^2)}}$



HOLLOW RECTANGLE

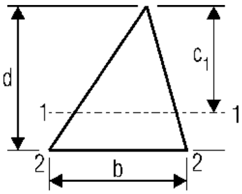
$A = bd - bd'$

$c = d/2$

$I = \frac{bd^3 - bd'^3}{12}$

$S = \frac{bd^3 - bd'^3}{6d}$

$r = \sqrt{\frac{bd^3 - bd'^3}{12(bd - bd')}}$



TRIANGLE

$A = \frac{bd}{2}$

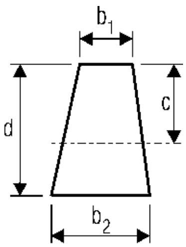
$I_1 = \frac{bd^3}{36}$

$S_1 = \frac{bd^2}{24}$

$c_1 = \frac{2d}{3}$

$I_2 = \frac{bd^3}{12}$

$r_1 = \frac{d}{\sqrt{18}}$



TRAPEZOID

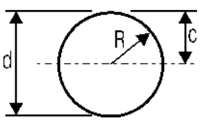
$A = \frac{(b_1+b_2)d}{2}$

$c = \frac{(b_1+2b_2)d}{3(b_1+b_2)}$

$I = \frac{(b_1^2+4b_1b_2+b_2^2)}{36(b_1+b_2)}d^3$

$S = \frac{(b_1^2+4b_1b_2+b_2^2)}{12(b_1+2b_2)}d^2$

$r = \frac{d}{6(b_1+b_2)}\sqrt{2(b_1^2+4b_1b_2)+b_2^2}$



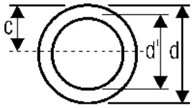
CIRCLE

$A = \pi R^2 = \frac{\pi d^2}{4}$ $c = \frac{d}{2}$

$I = \frac{\pi R^4}{4} = \frac{\pi d^4}{64}$

$S = \frac{\pi R^3}{4} = \frac{\pi d^3}{32}$

$r = \frac{R}{2} = \frac{d}{4}$

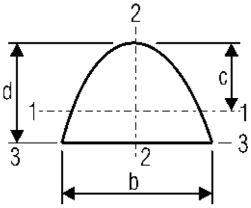


HOLLOW CIRCLE

$A = \frac{\pi(d^2 - d'^2)}{4}$ $c = \frac{d}{2}$

$I = \frac{\pi(d^4 - d'^4)}{64}$ $S = \frac{\pi(d^4 - d'^4)}{32d}$

$r = \frac{\sqrt{d^2+d'^2}}{4}$



PARABOLA

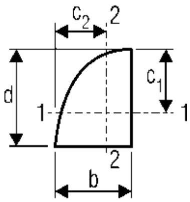
$A = \frac{2}{3}bd$

$I_1 = \frac{8}{175}bd^3$

$I_3 = \frac{16}{105}bd$

$c = \frac{3}{5}d$

$I_2 = \frac{b^3d}{30}$



HALF PARABOLA

$A = \frac{2}{3}bd$

$c_1 = \frac{3}{5}d$ $c_2 = \frac{5}{8}b$

$I_1 = \frac{8}{175}bd^3$ $I_2 = \frac{19}{480}b^3d$

Fig. 6.29 Geometric properties of sections.

I' = moment of inertia of component about parallel axis, in^4

A = cross-sectional area of component, in^2

d = distance between centroidal and parallel axes, in

The formula enables computation of the moment of inertia of a component about the centroidal axis of a section from the moment of inertia about the component's centroidal axis, usually obtainable from Fig. 6.29 or the AISC manual. By summing up the transferred moments of inertia for all the components, the moment of inertia of the section is obtained.

When the moments of inertia of an area with respect to any two perpendicular axes are known, the moment of inertia with respect to any other axis passing through the point of intersection of the two axes may be obtained by using Mohr's circle as for stresses (Fig. 6.11). In this analog, I_x corresponds with f_x, I_y with f_y, and the **product of inertia** I_{xy} with v_{xy} (Art. 6.17)

$$I_{xy} = \int xy\, dA \tag{6.48}$$

The two perpendicular axes through a point about which the moments of inertia are a maximum or a minimum are called the principal axes. The product of inertia is zero for the principal axes.

6.29 Section Modulus

The ratio $S = I/c$, relating bending moment and maximum bending stresses within the elastic range in a beam [Eq. (6.44)], is called the *section modulus*. I is the moment of inertia of the cross section about the neutral axis and c the distance from the neutral axis to the outermost fiber. Values of S for common types of sections are given in Fig. 6.29. Values for standard structural-steel sections are listed in manuals of the American Institute of Steel Construction.

6.30 Shearing Stresses in a Beam

Vertical shear at any section in a beam is resisted by nonuniformly distributed, vertical unit stresses (Fig. 6.30). At every point in the section, there also is a horizontal unit stress, which is equal in magnitude to the vertical unit shearing stress there [see Eq. (6.24)].

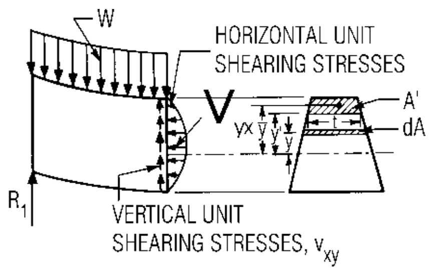


Fig. 6.30 Unit shearing stresses on a beam section.

At any distance y' from the neutral axis, both the horizontal and vertical shearing unit stresses are equal to

$$v = \frac{V}{It} A'\bar{y} \tag{6.49}$$

where V = vertical shear at cross section, lb

t = thickness of beam at distance y' from neutral axis, in

I = moment of inertia of section about neutral axis, in^4

A' = area between outermost surface and surface for which shearing stress is being computed, in^2

$\bar{y}$ = distance of center of gravity of this area from neutral axis, in

For a rectangular beam, with width $t = b$ and depth d, the maximum shearing stress occurs at middepth. Its magnitude is

$$v = \frac{V}{(bd^3/12)b} \frac{bd}{2} \frac{d}{4} = \frac{3}{2} \frac{V}{bd}$$

That is, the maximum shear stress is 50% greater than the average shear stress on the section. Similarly, for a circular beam, the maximum is one-third greater than the average. For an I or wide-flange beam, however, the maximum shear stress in the web is not appreciably greater than the average for the web section alone, assuming that the flanges take no shear.

6.31 Combined Shear and Bending Stress

For deep beams on short spans and beams with low tensile strength, it sometimes is necessary to determine the maximum normal stress f' due to a combination of shear stress v and bending stress f. This maximum or principal stress (Art. 6.15) occurs on a plane inclined to that of v and of f. From Mohr's circle (Fig. 6.11) with $f = f_x, f_y = 0$, and $v = v_{xy}$,

$$f' = \frac{f}{2} + \sqrt{v^2 + \left(\frac{f}{2}\right)^2} \qquad (6.50)$$

6.32 Beam Deflections

The **elastic curve** is the position taken by the longitudinal centroidal axis of a beam when it deflects under load. The radius of curvature at any point of this curve is

$$R = \frac{EI}{M} \qquad (6.51)$$

where M = bending moment at point

E = modulus of elasticity

I = moment of inertia of cross section about neutral axis

Since the slope of the elastic curve is very small, $1/R$ is approximately d^2y/dx^2, where y is the deflection of the beam at a distance x from the origin of coordinates. Hence, Eq. (6.51) may be rewritten

$$M = EI\frac{d^2y}{dx^2} \qquad (6.52)$$

To obtain the slope and deflection of a beam, this equation may be integrated, with M expressed as a function of x. Constants introduced during the integration must be evaluated in terms of known points and slopes of the elastic curve.

After integration, Eq. (6.52) yields

$$\theta_B - \theta_A = \int_A^B \frac{M}{EI}\,dx \qquad (6.53)$$

in which θ_A and θ_B are the slopes of the elastic curve at any two points A and B. If the slope is zero at one of the points, the integral in Eq. (6.53) gives the slope of the elastic curve at the other. The integral represents the area of the bending-moment diagram between A and B with each ordinate divided by EI.

The **tangential deviation** t of a point on the elastic curve is the distance of this point, measured in a direction perpendicular to the original position of the beam, from a tangent drawn at some other point on the curve.

$$t_B - t_A = \int_A^B \frac{Mx}{EI}\,dx \qquad (6.54)$$

Equation (6.54) indicates that the tangential deviation of any point with respect to a second point on the elastic curve equals the moment about the first point of the area of the M/EI diagram between the two points. The moment-area method for determining beam deflections is a technique employing Eqs. (6.53) and (6.54).

Moment-Area Method ▪ Suppose, for example, the deflection at midspan is to be computed for a beam of uniform cross section with a concentrated load at the center (Fig. 6.31). Since the deflection at midspan for this loading is the maximum for the span, the slope of the elastic curve at midspan is zero; that is, the tangent is parallel to the undeflected position of the beam. Hence, the deviation of either support from the midspan tangent equals the deflection at the center of the beam. Then, by the moment-area theorem [Eq. (6.54)], the deflection y_c is given by the moment about either support of the area of the M/EI diagram included between an ordinate at the center of the beam and that support

$$y_c = \left(\frac{1}{2}\frac{PL}{4EI}\frac{L}{2}\right)\frac{L}{3} = \frac{PL^3}{48EI}$$

Suppose now that the deflection y at any point D at a distance xL from the left support (Fig. 6.31) is to be determined. Note that from similar triangles, $xL/L = DE/t_{AB}$, where DE is the distance from the undeflected position of D to the tangent to the elastic curve at support A, and t_{AB} is the tangential deviation of B from that tangent. But DE also equals $y + t_{AD}$, where t_{AD} is the tangential deviation of D from the tangent at A. Hence,

$$y + t_{AD} = xt_{AB}$$

This equation is perfectly general for the deflection of any point of a simple beam, no matter how loaded. It may be rewritten to give the deflection directly:

$$y = xt_{AB} - t_{AD} \qquad (6.55)$$

But t_{AB} is the moment of the area of the M/EI diagram for the whole beam about support B, and

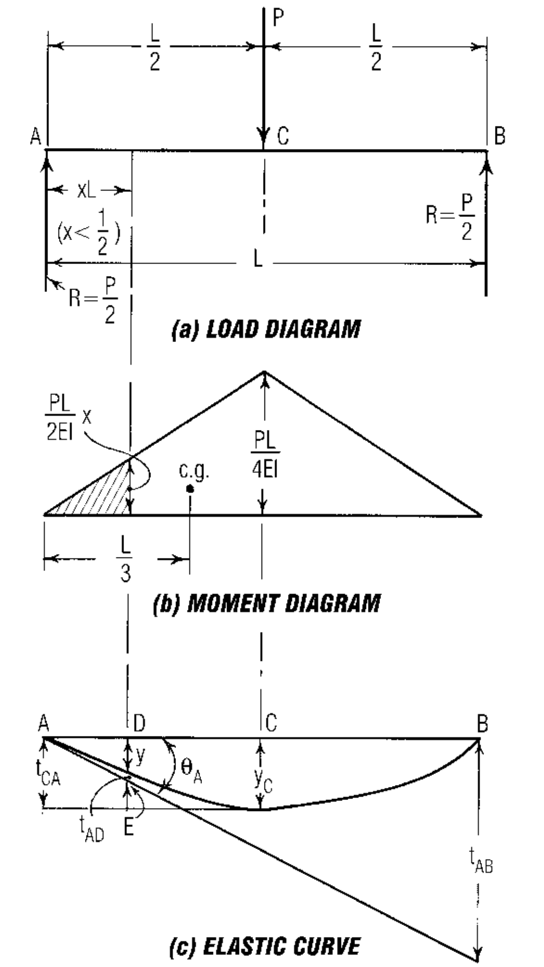


Fig. 6.31 Elastic curve for a simple beam and tangential deviations at ends.

t_{AD} is the moment about D of the area of the M/EI diagram included between ordinates at A and D. So at any point x of the beam in Fig. 6.31, the deflection is

$$y = x\left[\frac{1}{2}\frac{PL}{4EI}\frac{L}{2}\left(\frac{L}{3}+\frac{2L}{3}\right)\right] - \frac{1}{2}\frac{PLx}{2EI}(xL)\frac{xL}{3} = \frac{PL^3}{48EI}x(3-4x^2)$$

It also is noteworthy that, since the tangential deviations are very small distances, the slope of the elastic curve at A is given by

$$\theta_A = \frac{t_{AB}}{L} \qquad (6.56)$$

This holds, in general, for all simple beams regardless of the type of loading.

Conjugate-Beam Method ▪ The procedure followed in applying Eq. (6.55) to the deflection of the loaded beam in Fig. 6.31 is equivalent to finding the bending moment at D with the M/EI diagram serving as the load diagram. The technique of applying the M/EI diagram as a load and determining the deflection as a bending moment is known as the conjugate-beam method.

The conjugate beam must have the same length as the given beam; it must be in equilibrium with the M/EI load and the reactions produced by the load; and the bending moment at any section must be equal to the deflection of the given beam at the corresponding section. The last requirement is equivalent to specifying that the shear at any section of the conjugate beam with the M/EI load be equal to the slope of the elastic curve at the corresponding section of the given beam. Figure 6.32 shows the conjugates for various types of beams.

Deflection Computations ▪ Deflections for several types of loading on simple beams are given in Figs. 6.33 and 6.35 to 6.38 and for cantilevers and beams with overhangs in Figs. 6.39 to 6.44.

When a beam carries several different types of loading, the most convenient method of computing its deflection usually is to find the deflections separately for the uniform and concentrated loads and add them.

For several concentrated loads, the easiest method of obtaining the deflection at a point on a beam is to apply the reciprocal theorem (Art. 6.55). According to this theorem, if a concentrated load is applied to a beam at a point A, the deflection the load produces at point B equals the deflection at A for the same load applied at B ($d_{AB} = d_{BA}$). So place the loads one at a time at the point for which the deflection is to be found, and from the equation of the elastic curve determine the deflections at the actual location of the loads. Then, sum these deflections.

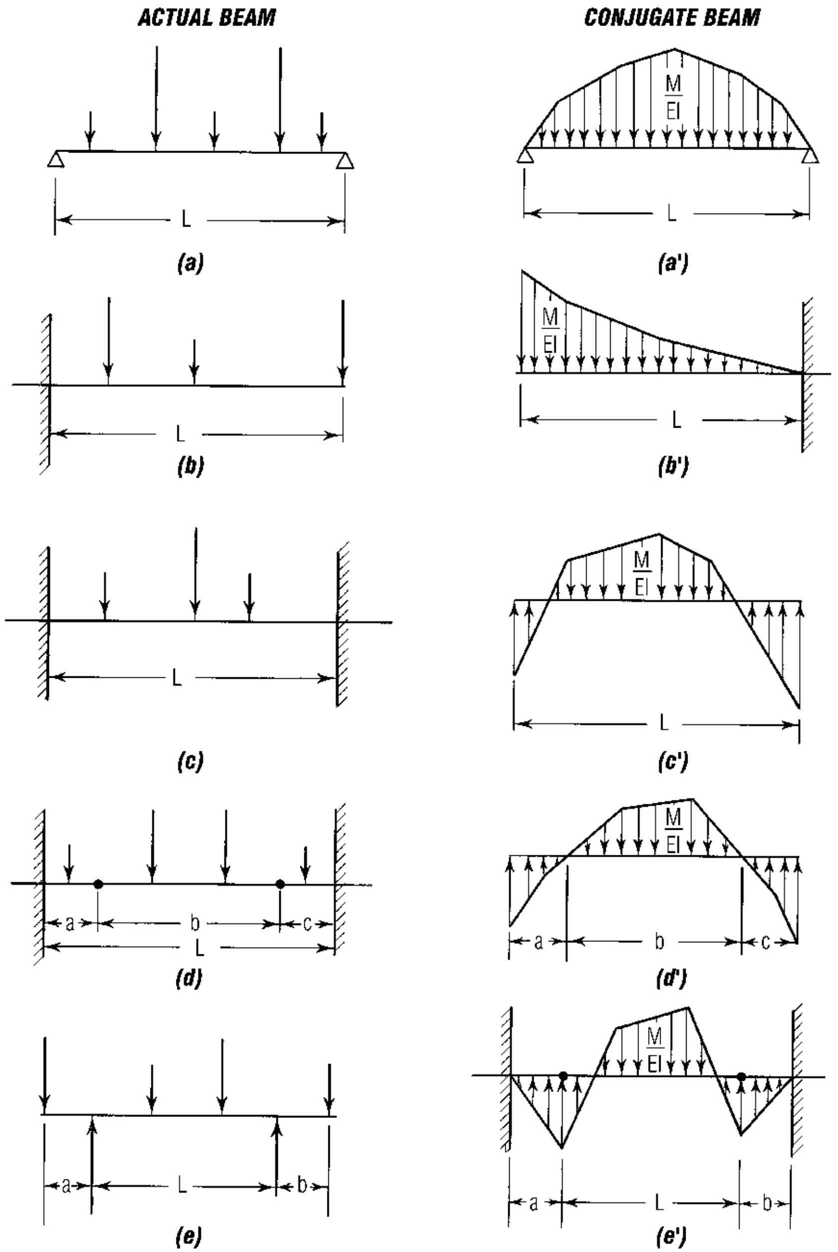


Fig. 6.32 Conjugate beams.

Suppose, for example, the midspan deflection is to be computed. Assume each load in turn applied at the center of the beam and compute the deflection at the point where it originally was applied from the equation of the elastic curve given in Fig. 6.36. The sum of these deflections is the total midspan deflection.

Another method for computing deflections is presented in Art. 6.54. This method also may be used to determine the deflection of a beam due to shear.

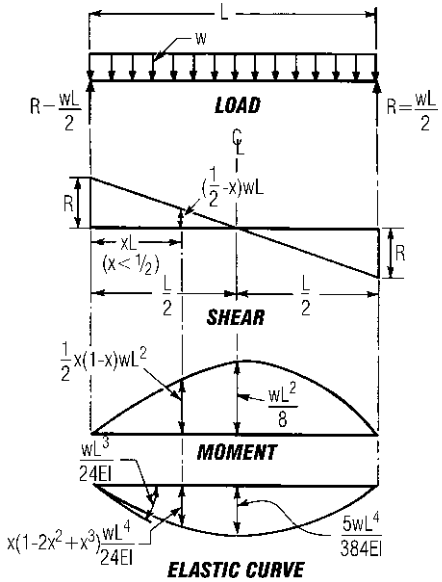


Fig. 6.33 Shears, moments, and deflections for full uniform load on a simply supported, prismatic beam.

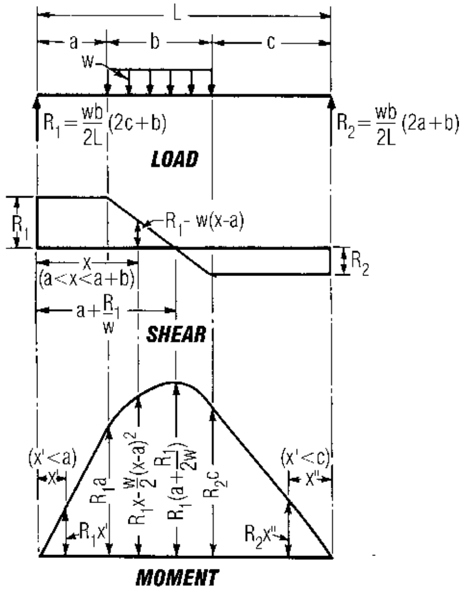


Fig. 6.34 Shears and moments for a uniformly distributed load over part of a simply supported beam.

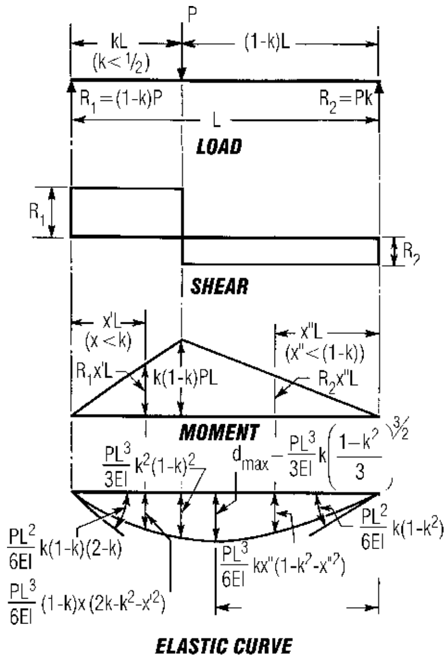


Fig. 6.35 Shears, moments, and deflections for a concentrated load at any point of a simply supported, prismatic beam.

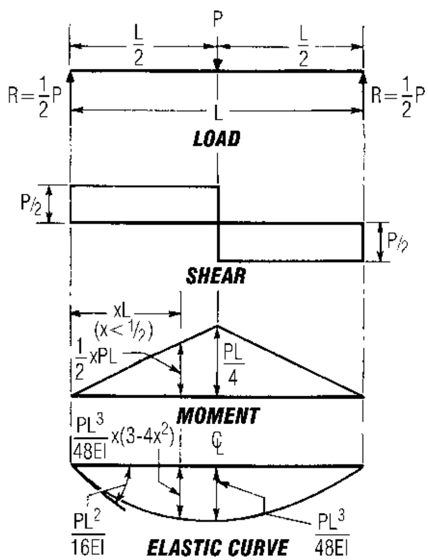


Fig. 6.36 Shears, moments, and deflections for a concentrated load at midspan of a simply supported, prismatic beam.

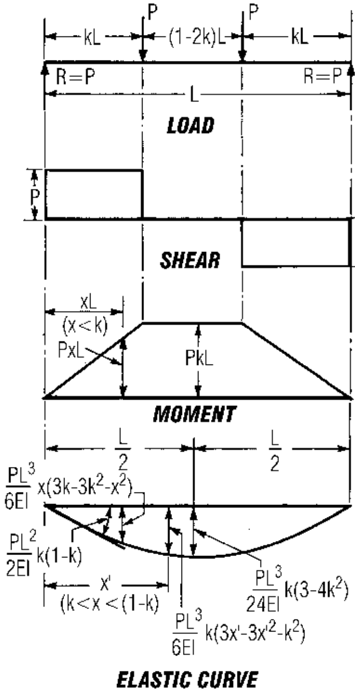


Fig. 6.37 Shears, moments, and deflections for two equal concentrated loads on a simply supported, prismatic beam.

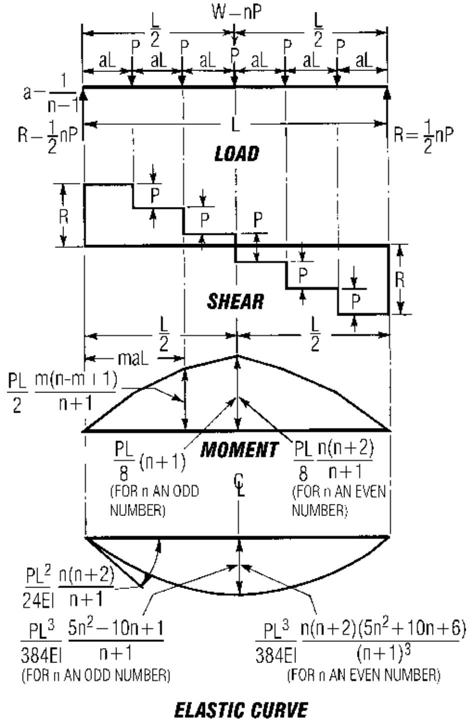


Fig. 6.38 Shears, moments, and deflections for several equal loads equally spaced on a simply supported, prismatic beam.

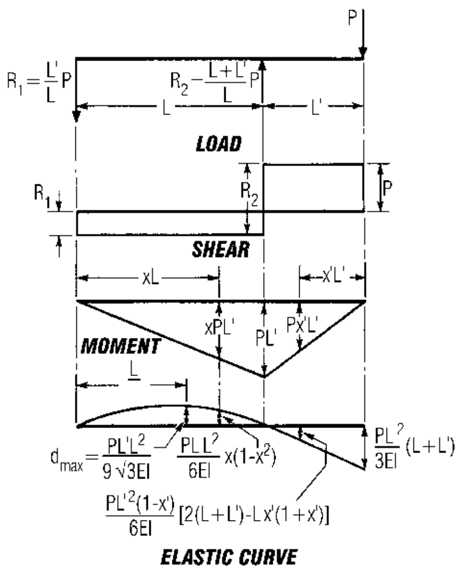


Fig. 6.39 Shears, moments, and deflections for a concentrated load on a beam overhang.

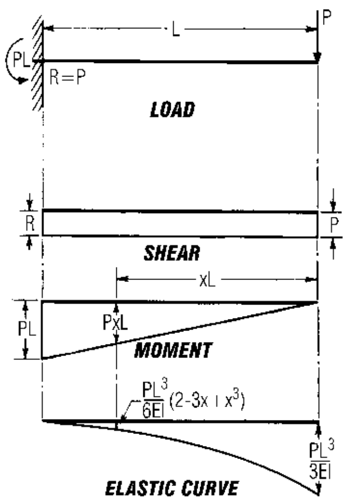


Fig. 6.40 Shears, moments, and deflections for a concentrated load on the end of a prismatic cantilever.

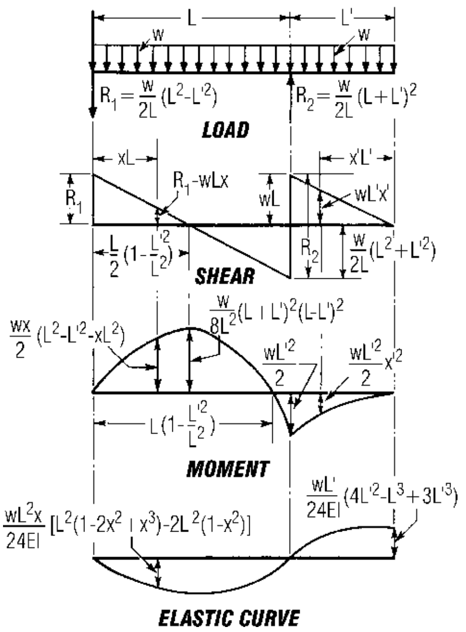


Fig. 6.41 Shears, moments, and deflections for a uniform load over a beam with overhang.

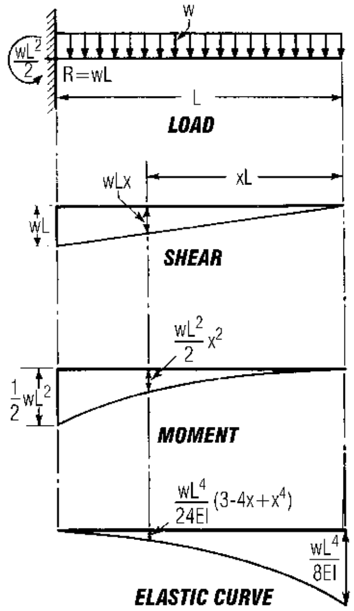


Fig. 6.42 Shears, moments, and deflections for a uniform load over the length of a cantilever.

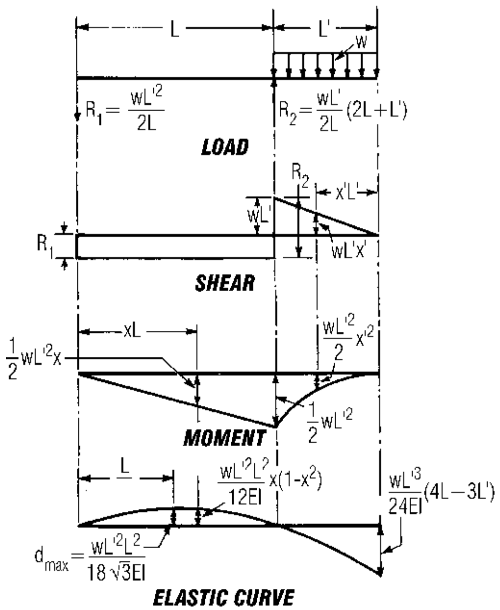


Fig. 6.43 Shears, moments, and deflections for a uniform load on a beam overhang.

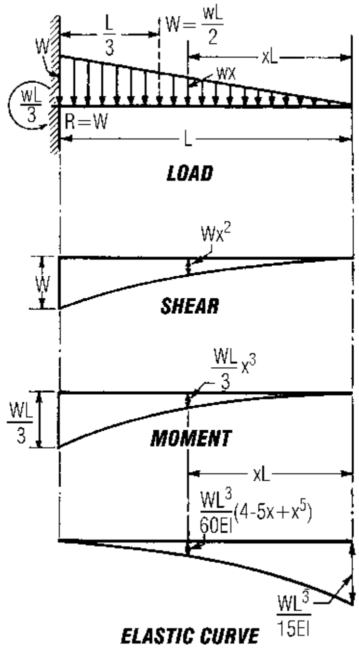


Fig. 6.44 Shears, moments, and deflections for a triangular loading on a prismatic cantilever.

6.33 Unsymmetrical Bending

When a beam is subjected to loads that do not lie in a plane containing a principal axis of each cross section, unsymmetrical bending occurs. Assuming that the bending axis of the beam lies in the plane of the loads, to preclude torsion (see Art. 6.36), and that the loads are perpendicular to the bending axis, to preclude axial components, the stress, psi, at any point in a cross section is

$$f = \frac{M_x y}{I_x} \pm \frac{M_y x}{I_y} \tag{6.57}$$

where M_x = bending moment about principal axis XX, in-lb

M_y = bending moment about principal axis YY, in-lb

x = distance from point where stress is to be computed to YY axis, in

y = distance from point to XX, in

I_x = moment of inertia of cross section about XX, in^4

I_y = moment of inertia about YY, in^4

If the plane of the loads makes an angle θ with a principal plane, the neutral surface will form an angle α with the other principal plane such that

$$\tan\alpha = \frac{I_x}{I_y}\tan\theta$$

6.34 Combined Axial and Bending Loads

For short beams, subjected to both transverse and axial loads, the stresses are given by the principle of superposition if the deflection due to bending may be neglected without serious error. That is, the total stress is given with sufficient accuracy at any section by the sum of the axial stress and the bending stresses. The maximum stress, psi, equals

$$f = \frac{P}{A} + \frac{Mc}{I} \tag{6.58a}$$

where P = axial load, lb

A = cross-sectional area, in^2

M = maximum bending moment, in-lb

c = distance from neutral axis to outermost fiber at section where maximum moment occurs, in

I = moment of inertia about neutral axis at that section, in^4

When the deflection due to bending is large and the axial load produces bending stresses that cannot be neglected, the maximum stress is given by

$$f = \frac{P}{A} + (M + Pd)\frac{c}{I} \tag{6.58b}$$

where d is the deflection of the beam. For axial compression, the moment Pd should be given the same sign as M, and for tension, the opposite sign, but the minimum value of $M + Pd$ is zero. The deflection d for axial compression and bending can be obtained by applying Eq. (6.52).

(S. Timoshenko and J. M. Gere, "Theory of Elastic Stability," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; Friedrich Bleich, "Buckling Strength of Metal Structures," McGraw-Hill Book Company, New York, books.mcgraw-hill.com.) But it may be closely approximated by

$$d = \frac{d_o}{1 - (P/P_c)} \tag{6.59}$$

where d_o = deflection for transverse loading alone, in

P_c = critical buckling load, $\pi^2 EI/L^2$ (see Art. 6.39), lb

6.35 Eccentric Loading

If an eccentric longitudinal load is applied to a bar in the plane of symmetry, it produces a bending moment Pe, where e is the distance, in, of the load P from the centroidal axis. The total unit stress is the sum of the stress due to this moment and the stress due to P applied as an axial load:

$$f = \frac{P}{A} \pm \frac{Pec}{I} = \frac{P}{A}\left(1 \pm \frac{ec}{r^2}\right) \tag{6.60}$$

where A = cross-sectional area, in^2

c = distance from neutral axis to outermost fiber, in

I = moment of inertia of cross section about neutral axis, in^4

r = **radius of gyration** = $\sqrt{I/A}$, in

Figure 6.29 gives values of the radius of gyration for several cross sections.

If there is to be no tension on the cross section under a compressive load, e should not exceed r^2/c. For a rectangular section with width b and depth d, the eccentricity, therefore, should be less than $b/6$ and $d/6$; i.e., the load should not be applied outside the middle third. For a circular cross section with diameter D, the eccentricity should not exceed $D/8$.

When the eccentric longitudinal load produces a deflection too large to be neglected in computing the bending stress, account must be taken of the additional bending moment Pd, where d is the deflection, in. This deflection may be computed by using Eq. (6.52) or closely approximated by

$$d = \frac{4eP/P_c}{\pi(1 - P/P_c)} \tag{6.61}$$

P_c is the critical buckling load $\pi^2 EI/L^2$ (see Art. 6.39), lb.

If the load P does not lie in a plane containing an axis of symmetry, it produces bending about the two principal axes through the centroid of the section. The stresses, psi, are given by

$$f = \frac{P}{A} \pm \frac{Pe_x c_x}{I_y} \pm \frac{Pe_y c_y}{I_x} \tag{6.62}$$

where A = cross-sectional area in^2

e_x = eccentricity with respect to principal axis YY, in

e_y = eccentricity with respect to principal axis XX, in

c_x = distance from YY to outermost fiber, in

c_y = distance from XX to outermost fiber, in

I_x = moment of inertia about XX, in^4

I_y = moment of inertia about YY, in^4

The principal axes are the two perpendicular axes through the centroid for which the moments of inertia are a maximum or a minimum and for which the products of inertia are zero.

6.36 Beams with Unsymmetrical Sections

The derivation of the flexure formula $f = Mc/I$ (Art. 6.27) assumes that a beam bends, without twisting, in the plane of the loads and that the neutral surface is perpendicular to the plane of the loads. These assumptions are correct for beams with cross sections symmetrical about two axes when the plane of the loads contains one of these axes. They are not necessarily true for beams that are not doubly symmetrical because in beams that are doubly symmetrical, the bending axis coincides with the centroidal axis, whereas in unsymmetrical sections the two axes may be separate. In the latter case, if the plane of the loads contains the centroidal axis but not the bending axis, the beam will be subjected to both bending and torsion.

The **bending axis** is the longitudinal line in a beam through which transverse loads must pass to preclude the beam's twisting as it bends. The point in each section through which the bending axis passes is called the **shear center**, or center of twist. The shear center also is the center of rotation of the section in pure torsion (Art. 6.18). Its location depends on the dimensions of the section.

Computation of stresses and strains in members subjected to both bending and torsion is complicated, because warping of the cross section and buckling may occur and should be taken into account. Such computations may not be necessary if twisting is prevented by use of bracing or avoided by selecting appropriate shapes for the members and by locating and directing loads to pass through the bending axis.

(F. Bleich, "Buckling Strength of Metal Structures," McGraw-Hill Book Company, New York, books.mcgraw-hill.com.)

Curved Beams

Structural members, such as arches, crane hooks, chain links, and frames of some machines, that have considerable initial curvature in the plane of loading are called curved beams. The flexure formula of Art. 6.27, $f = Mc/I$, cannot be applied to them with any reasonable degree of accuracy unless the depth of the beam is small compared with the radius of curvature.

Unlike the condition in straight beams, unit strains in curved beams are not proportional to the distance from the neutral surface, and the centroidal axis does not coincide with the neutral axis. Hence the stress distribution on a section is not linear but more like the distribution shown in Fig. 6.45*c*.

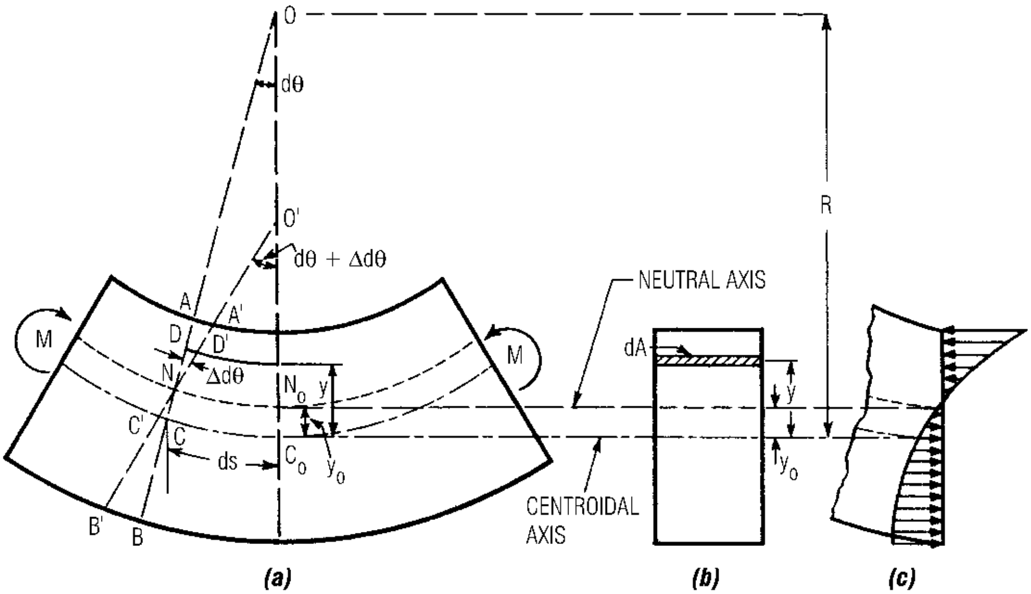


Fig. 6.45 Bending stresses in a curved beam.

6.37 Stresses in Curved Beams

Just as for straight beams, the assumption that plane sections before bending remain plane after bending generally holds for curved beams. So the total strains are proportional to the distance from the neutral axis. But since the fibers are initially of unequal length, the unit strains are a more complex function of this distance. In Fig. 6.45*a*, for example, the bending couples have rotated section *AB* of the curved beam into section *A′B′* through an angle $\Delta d\theta$. If ε_o is the unit strain at the centroidal axis and ω is the angular unit strain $\Delta d\theta / d\theta$, then if *M* is the bending moment:

$$\varepsilon_o = \frac{M}{ARE} \quad \text{and} \quad \omega = \frac{M}{ARE}\left(1 + \frac{AR^2}{I'}\right) \qquad (6.63)$$

where *A* is the cross-sectional area, *E* the modulus of elasticity, and

$$I' = \int \frac{y^2\,dA}{1 - y/R} = \int y^2\left(1 + \frac{y}{R} + \frac{y^2}{R^2} + \cdots\right)dA \qquad (6.64)$$

It should be noted that I' is very nearly equal to the moment of inertia *I* about the centroidal axis when the depth of the section is small compared with *R*, so that the maximum ratio of *y* to *R* is small compared with unity. *M* is positive when it decreases the radius of curvature.

The stresses in the curved beam can be obtained from Fig. 6.45*a* with the use of ε_o and ω from Eq. (6.63):

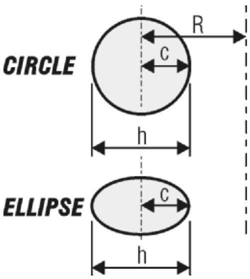
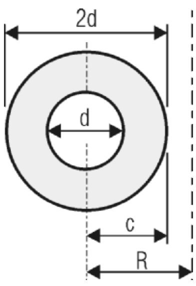
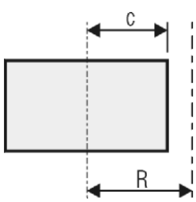
$$f = \frac{M}{A\,R} - \frac{My}{I'}\frac{1}{1 - y/R} \qquad (6.65)$$

Equation (6.65) for bending stresses in curved beams subjected to end moments in the plane of curvature can be expressed for the inside and outside beam faces in the form:

$$f = \frac{Mc}{I}K \qquad (6.66)$$

where c = distance from the centroidal axis to the inner or outer surface. Table 6.1 gives values of *K* calculated from Eq. (6.66) for circular, elliptical, and rectangular cross sections.

Table 6.1 Values of K for Curved Beams

Section	R/c	K		y_o/R
		Inside face	Outside face	
	1.2	3.41	0.54	0.224
	1.4	2.40	0.60	0.151
	1.6	1.96	0.65	0.108
	1.8	1.75	0.68	0.084
	2.0	1.62	0.71	0.069
	3.0	1.33	0.79	0.030
	4.0	1.23	0.84	0.016
	6.0	1.14	0.89	0.0070
	8.0	1.10	0.91	0.0039
	10.0	1.08	0.93	0.0025
	1.2	3.28	0.58	0.269
	1.4	2.31	0.64	0.182
	1.6	1.89	0.68	0.134
	1.8	1.70	0.71	0.104
	2.0	1.57	0.73	0.083
	3.0	1.31	0.81	0.038
	4.0	1.21	0.85	0.020
	6.0	1.13	0.90	0.0087
	8.0	1.10	0.92	0.0049
	10.0	1.07	0.93	0.0031
	1.2	2.89	0.57	0.305
	1.4	2.13	0.63	0.204
	1.6	1.79	0.67	0.149
	1.8	1.63	0.70	0.112
	2.0	1.52	0.73	0.090
	3.0	1.30	0.81	0.041
	4.0	1.20	0.85	0.021
	6.0	1.12	0.90	0.0093
	8.0	1.09	0.92	0.0052
	10.0	1.07	0.94	0.0033

If Eq. (6.65) is applied to I or T beams or tubular members, it may indicate circumferential flange stresses that are much lower than will actually occur. The error is due to the fact that the outer edges of the flanges deflect radially. The effect is equivalent to having only part of the flanges active in resisting bending stresses. Also, accompanying the flange deflections, there are transverse bending stresses in the flanges. At the junction with the web, these reach a maximum, which may be greater than the maximum circumferential stress. Furthermore, there are radial stresses (normal stresses acting in the direction of the radius of curvature) in the web that also may have maximum values greater than the maximum circumferential stress.

If a curved beam carries an axial load P as well as bending loads, the maximum unit stress is

$$f = \frac{P}{A} \pm \frac{Mc}{I}K \qquad (6.67)$$

M is taken positive in this equation when it increases the curvature, and P is positive when it is a tensile force, negative when compressive.

6.38 Slope and Deflection of Curved Beams

If we consider two sections of a curved beam separated by a differential distance ds (Fig. 6.45*a*), the change in angle $\Delta d\theta$ between the sections caused by a bending moment M and an axial load P may be obtained from Eq. (6.63), noting that $d\theta = ds/R$.

$$\Delta d\theta = \frac{M\,ds}{EI'}\left(1 + \frac{I'}{A\,R^2}\right) + \frac{P\,ds}{A\,RE} \qquad (6.68)$$

where E is the modulus of elasticity, A the cross-sectional area, R the radius of curvature of the centroidal axis, and I' is defined by Eq. (6.64).

If P is a tensile force, the length of the centroidal axis increases by

$$\Delta ds = \frac{P\,ds}{A\,E} + \frac{M\,ds}{A\,RE} \qquad (6.69)$$

The effect of curvature on shearing deformations for most practical applications is negligible.

For shallow sections (depth of section less than about one-tenth the span), the effect of axial forces on deformations may be neglected. Also, unless the radius of curvature is very small compared with the depth, the effect of curvature may be ignored. Hence, for most practical applications, Eq. (6.68) may be used in the simplified form:

$$\Delta d\theta = \frac{M\,ds}{EI} \qquad (6.70)$$

For deeper beams, the action of axial forces, as well as bending moments, should be taken into account; but unless the curvature is sharp, its effect on deformations may be neglected. So only Eq. (6.70) and the first term in Eq. (6.69) need be used.

(S. Timoshenko and D. H. Young, "Theory of Structures," McGraw-Hill Publishing Company, New York, books.mcgraw-hill.com.) See also Arts. 6.69 and 6.70.

Buckling of Columns

Columns are compression members whose cross-sectional dimensions are small compared with their length in the direction of the compressive force. Failure of such members occurs because of instability when a certain load (called the critical or **Euler load**) is equaled or exceeded. The member may bend, or buckle, suddenly and collapse.

Hence, the strength of a column is determined not by the unit stress in Eq. (6.6) ($P = Af$) but by the maximum load it can carry without becoming unstable. The condition of instability is characterized by disproportionately large increases in lateral deformation with slight increase in load. It may occur in slender columns before the unit stress reaches the elastic limit.

6.39 Equilibrium of Columns

Figure 6.46 represents an axially loaded column with ends unrestrained against rotation. If the member is initially perfectly straight, it will remain straight as long as the load P is less than the critical load P_c (also called Euler load). If a small transverse force is applied, it will deflect, but it will return to

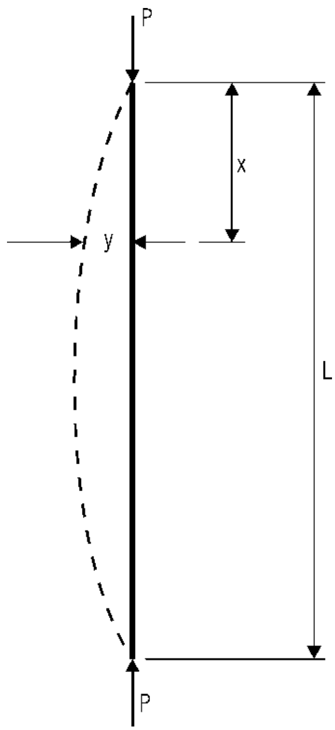


Fig. 6.46 Buckling of a column.

the straight position when this force is removed. Thus, when P is less than P_c, internal and external forces are in stable equilibrium.

If $P = P_c$ and a small transverse force is applied, the column again will deflect, but this time, when the force is removed, the column will remain in the bent position (dashed line in Fig. 6.46).

The equation of this elastic curve can be obtained from Eq. (6.52):

$$EI\frac{d^2y}{dx^2} = -P_c y \tag{6.71}$$

in which E = modulus of elasticity, psi

I = least moment of inertia of cross section, in^4

y = deflection of bent member from straight position at distance x from one end, in

This assumes that the stresses are within the elastic limit.

Solution of Eq. (6.71) gives the smallest value of the Euler load as

$$P_c = \frac{\pi^2 EI}{L^2} \tag{6.72}$$

Equation (6.72) indicates that there is a definite magnitude of an axial load that will hold a column in equilibrium in the bent position when the stresses are below the elastic limit. Repeated application and removal of small transverse forces or small increases in axial load above this critical load will cause the member to fail by buckling. Internal and external forces are in a state of unstable equilibrium.

It is noteworthy that the Euler load, which determines the load-carrying capacity of a column, depends on the stiffness of the member, as expressed by the modulus of elasticity, rather than on the strength of the material of which it is made.

By dividing both sides of Eq. (6.72) by the cross-sectional area A, in^2, and substituting r^2 for I/A (r is the radius of gyration of the section), we can write the solution of Eq. (6.71) in terms of the average unit stress on the cross section:

$$\frac{P_c}{A} = \frac{\pi^2 E}{(L/r)^2} \tag{6.73}$$

This holds only for the elastic range of buckling, that is, for values of the **slenderness ratio** L/r above a certain limiting value that depends on the properties of the material.

Effects of End Conditions ▪ Equation (6.73) was derived on the assumption that the ends of the columns are free to rotate. It can be generalized, however, to take into account the effect of end conditions:

$$\frac{P_c}{A} = \frac{\pi^2 E}{(kL/r)^2} \tag{6.74}$$

where k is a factor that depends on the end conditions. For a pin-ended column, $k = 1$; for a column with both ends fixed, $k = \frac{1}{2}$; for a column with one end fixed and one end pinned, k is about 0.7; and for a column with one end fixed and one end free from all restraint, $k = 2$. When a column has different restraints or different radii of gyration about its principal axes, the largest value of kL/r for a principal axis should be used in Eq. (6.74).

Inelastic Buckling ▪ Equations (6.72) to (6.74), having been derived from Eq. (6.71), the differential equation for the elastic curve, are based on the assumption that the critical average stress is below the elastic limit when the state of unstable equilibrium is reached. In members with slenderness ratio L/r below a certain limiting value, however, the elastic limit is exceeded before the column buckles. As the axial load approaches the critical load, the modulus of elasticity varies with the stress. Hence, Eqs. (6.72) to (6.74), based on the assumption that E is a constant, do not hold for these short columns.

After extensive testing and analysis, prevalent engineering opinion favors the Engesser equation for metals in the inelastic range:

$$\frac{P_t}{A} = \frac{\pi^2 E_t}{(kL/r)^2} \tag{6.75}$$

This differs from Eq. (6.74) only in that the tangent modulus E_t (the actual slope of the stress-strain curve for the stress P_t/A) replaces E, the modulus of elasticity in the elastic range. P_t is the smallest axial load for which two equilibrium positions are possible, the straight position and a deflected position.

Another solution to the inelastic-buckling problem is called the double modulus method, in which the bending stiffness of the cross section is expressed in terms of E_t and E, representing the loading and unloading portions of materials on the cross section respectively. The critical stress obtained is higher than that of the Engesser equation.

Eccentric Loading ▪ Under eccentric loading, the maximum unit stress in short compression members is given by Eqs. (6.60) and (6.62), with the eccentricity *e* increased by the deflection given by Eq. (6.61). For columns, the stress within the elastic range is given by the **secant formula**:

$$f = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\frac{kL}{2r}\sqrt{\frac{P}{AE}}\right) \tag{6.76}$$

When the slenderness ratio L/r is small, the formula approximates Eq. (6.60).

6.40 Column Curves

The result of plotting the critical stress in columns for various values of slenderness ratios (Art. 6.39) is called a column curve. For axially loaded, initially straight columns, it consists of two parts: the Euler critical values [Eq. (6.73)] and the Engresser, or tangent-modulus, critical values [Eq. (6.75)], with $k = 1$.

The second part of the curve is greatly affected by the shape of the stress-strain curve for the material of which the column is made, as indicated in Fig. 6.47. The stress-strain curve for a material, such as an aluminum alloy or high-strength steel, which does not have a sharply defined yield point, is shown in Fig. 6.47*a.* The corresponding column curve is plotted in Fig. 6.47*b.* In contrast, Fig. 6.47*c* presents the stress-strain curve for structural steel,

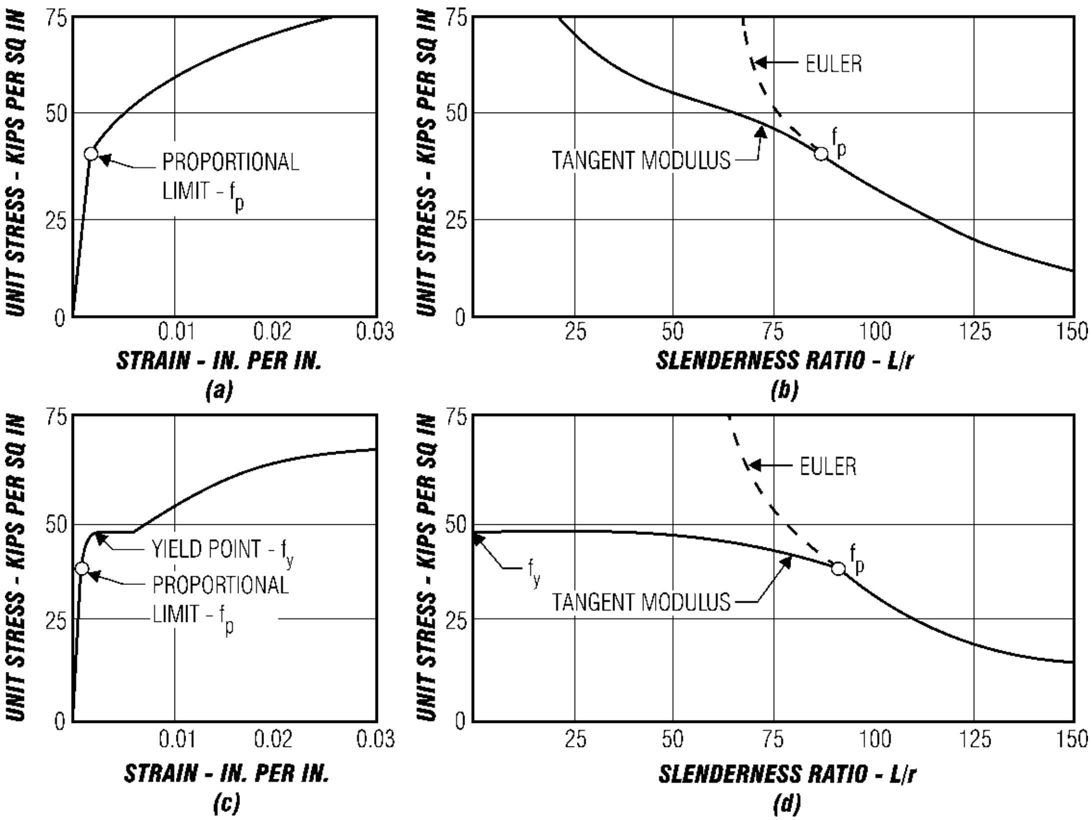


Fig. 6.47 Column curves: (*a*) Stress-strain curve for a material without a sharply defined yield point; (*b*) column curve for the material in (*a*); (*c*) stress-strain curve for a material with a sharply defined yield point; (*d*) column curve for the material in (*c*).

with a sharply defined yield point, and Fig. 6.47*d* the related column curve. This curve becomes horizontal as the critical stress approaches the yield strength of the material and the tangent modulus becomes zero, whereas the column curve in Fig. 6.47*b* continues to rise with decreasing values of the slenderness ratio.

Examination of Fig. 6.47*d* also indicates that slender columns, which fall in the elastic range, where the column curve has a large slope, are very sensitive to variations in the factor *k*, which represents the effect of end conditions. On the other hand, in the inelastic range, where the column curve is relatively flat, the critical stress is relatively insensitive to changes in *k*. Hence, the effect of end conditions is of much greater significance for long columns than for short columns.

6.41 Behavior of Actual Columns

For many reasons, columns in structures behave differently from the ideal column assumed in deriving Eqs. (6.72) to (6.76). A major consideration is the effect of accidental imperfections, such as nonhomogeneity of materials, initial crookedness, and unintentional eccentricities of the axial load. These effects can be taken into account by a proper choice of safety factor.

There are, however, other significant conditions that must be considered in any design procedure: continuity in framed structures and eccentricity of the load. Continuity affects column action two ways: The restraint and sidesway at column ends determine the value of *k*, and bending moments are transmitted to the columns by adjoining structural members.

Because of the deviation of the behavior of actual columns from the ideal, columns generally are designed by empirical formulas. Separate equations usually are given for short columns, intermediate columns, and long columns, and still other equations for combinations of axial load and bending moment.

Furthermore, a column may fail not by buckling of the member as a whole but, as an alternative, by buckling of one of its components. Hence, when members like I beams, channels, and angles are used as columns, or when sections are built up of plates, the possibility that the critical load on a component (leg, half flange, web, lattice bar) will be less than the critical load on the column as a whole should be investigated.

Similarly, the possibility of buckling of the compression flange or the web of a beam should be investigated.

Local buckling, however, does not always result in a reduction in the load-carrying capacity of a column; sometimes it results in a redistribution of the stresses, which enables the member to carry additional load.

For more details on column action, see S. Timoshenko and J. M. Gere, "Theory of Elastic Stability," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; B. G. Johnston, "Guide to Stability Design Criteria for Metal Structures," John Wiley & Sons, Inc., New Jersey, www.wiley.com; F. Bleich, "Buckling Strength of Metal Structures," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; and T. V. Galambos, "Guide to Stability Design Criteria for Metal Structures," John Wiley & Sons, Inc., Hoboken, N.J., 1988, www.wiley.com.

Graphic-Statics Fundamentals

Since a force is completely determined when it is known in magnitude, direction, and point of application, any force may be represented by the length, direction, and position of a straight line. The length of line to a given scale represents the magnitude of the force. The position of the line parallels the line of action of the force, and an arrowhead on the line indicates the direction in which the force acts.

6.42 Force Polygons

Graphically represented, a force may be designated by a letter, sometimes followed by a subscript, such as P_1 and P_2 in Fig. 6.48. Or each extremity of the line may be indicated by a letter and the force referred to by means of these letters (Fig. 6.48*a*). The order of the letters indicates the direction of the force; in Fig. 6.48*a*, referring to P_1 as *OA* indicates it acts from *O* toward *A*.

Forces are concurrent when their lines of action meet. If they lie in the same plane, they are coplanar.

Parallelogram of Forces ▪ The **resultant** of several forces is a single force that would produce

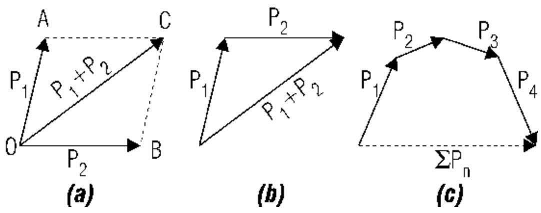


Fig. 6.48 Addition of forces by (*a*) parallelogram law, (*b*) triangle construction, and (*c*) polygon construction.

the same effect on a rigid body. The resultant of two concurrent forces is determined by the **parallelogram law**:

If a parallelogram is constructed with two forces as sides, the diagonal represents the resultant of the forces (Fig. 6.48*a*).

The resultant is said to be equal to the sum of the forces, sum here meaning vectorial sum, or addition by the parallelogram law. Subtraction is carried out in the same manner as addition, but the direction of the force to be subtracted is reversed.

If the direction of the resultant is reversed, it becomes the **equilibrant**, a single force that will hold the two given forces in equilibrium.

Resolution of Forces ▪ Any force may be resolved into two components acting in any given direction. To resolve a force into two components, draw a parallelogram with the force as a diagonal and sides parallel to the given directions. The sides then represent the components.

The procedure is: (1) Draw the given force. (2) From both ends of the force draw lines parallel to the directions in which the components act. (3) Draw the components along the parallels through the origin of the given force to the intersections with the parallels through the other end. Thus, in Fig. 6.48*a*, P_1 and P_2 are the components in directions *OA* and *OB* of the force represented by *OC*.

Force Triangles and Polygons ▪ Examination of Fig. 6.48*a* indicates that a step can be saved in adding forces P_1 and P_2. The same resultant could be obtained by drawing only the upper half of the parallelogram. Hence, to add two forces, draw the first force; then draw the second force at the end of the first one. The resultant is the force drawn from the origin of the first force to the end of the second force, as shown in Fig. 6.48*b*.

This diagram is called a force triangle. Again, the equilibrant is the resultant with direction reversed. If it is drawn instead of the resultant, the arrows representing the direction of the forces will all point in the same direction around the triangle. From the force triangle, an important conclusion can be drawn:

If three forces meeting at a point are in equilibrium, they form a closed force triangle.

To add several forces $P_1, P_2, P_3, \ldots, P_n$, draw P_2 from the end of P_1, P_3 from the end of P_2, and so on. The force required to complete the force polygon is the resultant (Fig. 6.48*c*).

If a group of concurrent forces is in equilibrium, they form a closed force polygon.

6.43 Equilibrium Polygons

When forces are coplanar but not concurrent, the force polygon will yield the magnitude and direction of the resultant but not its point of application. To complete the solution, the easiest method generally is to employ an auxiliary force polygon, called an equilibrium, or funicular (string), polygon. Sides of this polygon represent the lines of action of certain components of the given forces; more specifically, they take the configuration of a weightless string holding the forces in equilibrium.

In Fig. 6.49*a*, the forces P_1, P_2, P_3, and P_4 acting on the given body are not in equilibrium. The magnitude and direction of their resultant *R* are obtained from the force polygon *abcde* (Fig. 6.49*b*). The line of action may be obtained as follows:

From any point *O* in the force polygon, draw a line to each vertex of the polygon. Since the lines *Oa* and *Ob* form a closed triangle with force P_1, they represent two forces S_5 and S_1 that hold P_1 in equilibrium—two forces that may replace P_1 in a force diagram. So, as in Fig. 6.49*a*, at any point *m* on the line of action of P_1, draw lines *mn* and *mv* parallel to S_1 and S_5, respectively, to represent the lines of action of these forces. Similarly, S_1 and S_2 represent two forces that may replace P_2. The line of action of S_1 already is indicated by the line *mn*, and it intersects P_2 at *n*. So through *n* draw a line parallel to S_2, intersecting P_3 at *r*. Through *r*, draw *rs* parallel to S_3, and through *s*, draw *st* parallel to S_4. Lines *mv* and *st*, parallel to S_5 and S_4, respectively, represent the lines of action of S_5 and S_4. But these two forces form a closed force triangle with the resultant *ae* (Fig. 6.49*b*), and therefore the three

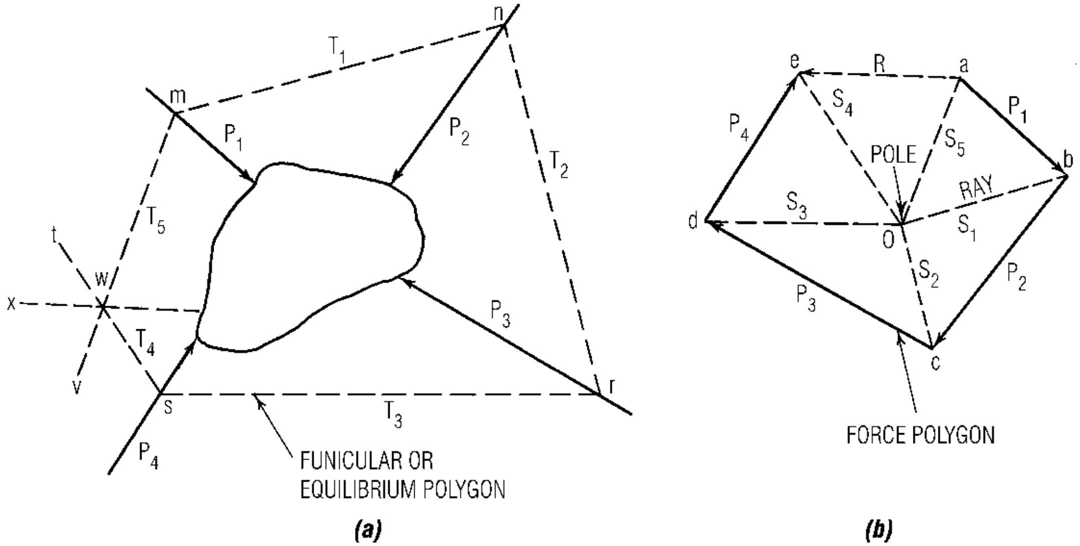


Fig. 6.49 Force and equilibrium polygons for a system of forces in equilibrium.

forces must be concurrent. Hence, the line of action of the resultant must pass through the intersection *w* of the lines *mv* and *st.* The resultant of the four given forces is thus fully determined. A force of equal magnitude but acting in the opposite direction, from *e* to *a*, will hold P_1, P_2, P_3, and P_4 in equilibrium.

The polygon *mnrsw* is called an *equilibrium polygon.* Point *O* is called the *pole*, and $S_1 \ldots S_5$ are called the *rays of the force polygon.*

Stresses in Trusses

A truss is a coplanar system of structural members joined at their ends to form a stable framework. Usually, analysis of a truss is based on the assumption that the joints are hinged. Neglecting small changes in the lengths of the members due to loads, the relative positions of the joints cannot change. Stresses due to joint rigidity or deformations of the members are called **secondary stresses**.

6.44 Truss Characteristics

Three bars pinned together to form a triangle represent the simplest type of truss. Some of the more common types of trusses are shown in Fig. 6.50.

The top members are called the **upper chord**, the bottom members the **lower chord**, and the verticals and diagonals **web members**.

Trusses act like long, deep girders with cutout webs. Roof trusses have to carry not only their own weight and the weight of roof framing but wind loads, snow loads, suspended ceilings and equipment, and a live load to take care of construction, maintenance, and repair loading. Bridge trusses have to support their own weight and that of deck framing and deck, live loads imposed by traffic (automobiles, trucks, railroad trains, pedestrians, and so on) and impact caused by live load, plus wind on structural members and vehicles. **Deck trusses** carry the live load on the upper chord and **through trusses** on the lower chord.

Loads generally are applied at the intersection of members, or panel points, so that the members will be subjected principally to direct stresses—tension or compression. To simplify stress analysis, the weight of the truss members is apportioned to upper- and lower-chord panel points. The members are assumed to be pinned at their ends, even though this may actually not be the case. However, if the joints are of such nature as to restrict relative rotation substantially, then

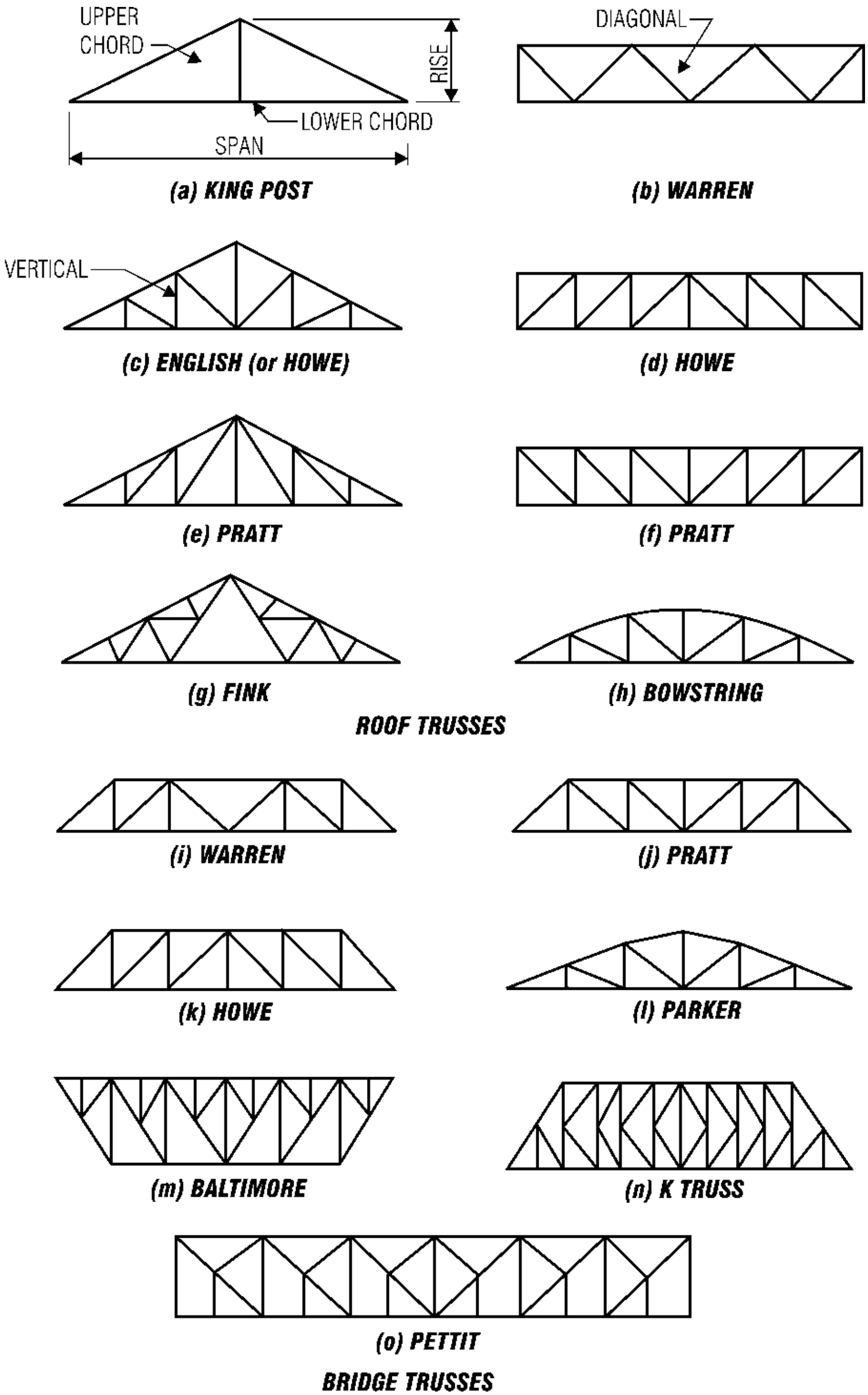


Fig. 6.50 Common types of trusses.

the "secondary" stresses set up as a result should be computed and superimposed on the stresses obtained with the assumption of pin ends.

6.45 Bow's Notation

In analysis of trusses, especially in graphical analysis, Bow's notation is useful for identifying truss members, loads, and stresses. Capital letters are placed in the spaces between truss members and between forces; each member and load is then designated by the letters on opposite sides of it. For example, in Fig. 6.51*a*, the upper-chord members are *AF*, *BH*, *CJ*, and *DL*. The loads are *AB*, *BC*, and *CD*, and the reactions are *EA* and *DE*. Stresses in the members generally are designated by the same letters but in lowercase.

6.46 Method of Sections for Truss Stresses

A convenient method of computing the stresses in truss members is to isolate a portion of the truss by a section so chosen as to cut only as many members with unknown stresses as can be evaluated by the laws of equilibrium applied to that portion of the truss. The stresses in the members cut by the section are treated as external forces and must hold the loads on that portion of the truss in equilibrium. Compressive forces act toward each joint or panel point, and tensile forces away from the joint.

Joint Isolation ▪ A choice of section that often is convenient is one that isolates a joint with only two unknown stresses. Since the stresses and load at the joint must be in equilibrium, the sum of the

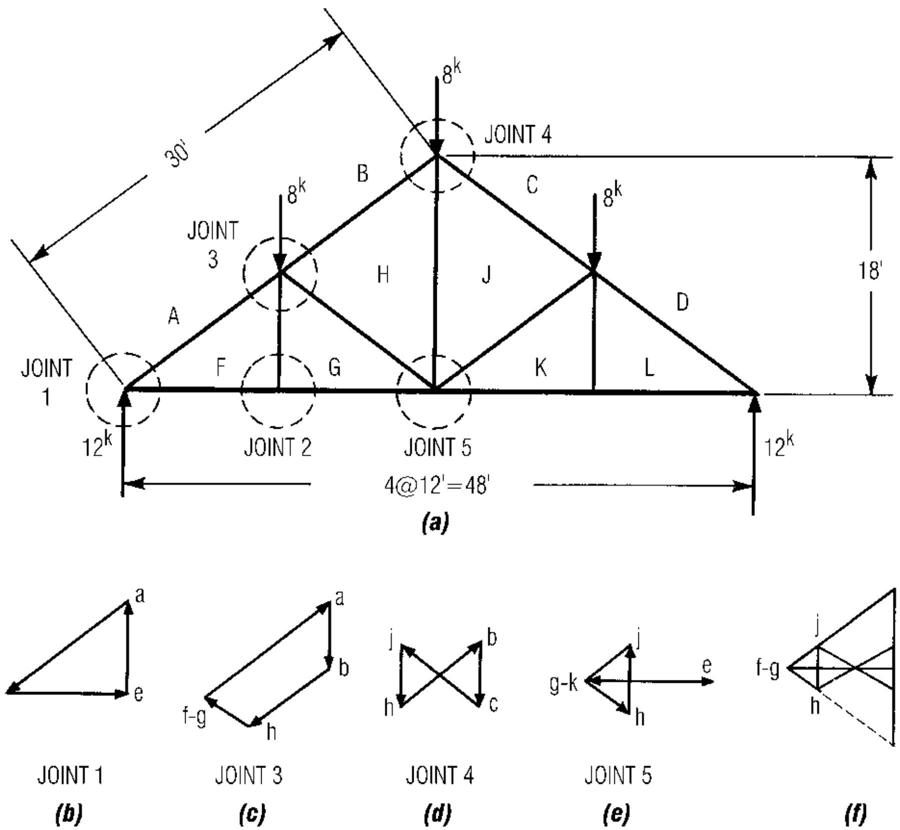


Fig. 6.51 Graphical determination of stresses at each joint of the truss in (*a*) may be expedited by constructing the single Maxwell diagram in (*f*).

horizontal components of the forces must be zero, and so must be the sum of the vertical components. Since the lines of action of all the forces are known (the stresses act along the longitudinal axes of the truss members), we can therefore compute two unknown magnitudes of stresses at each joint by this method.

To apply it to joint 1 of the truss in Fig. 6.51*a*, first equate the sum of the vertical components to zero. This equation shows that the vertical component *af* of the top chord must be equal and opposite to the reaction, 12 kips (see Fig. 6.51*b* and Bow's notation, Art. 6.45). The stress in the top chord *ea* at this joint, then, must be a compression equal to $12 \times 30/18 = 20$ kips. Next, equate the sum of the horizontal components to zero. This equation indicates that the stress in the bottom chord *fe* at the joint must be equal and opposite to the horizontal component of the top chord. Hence, the stress in the bottom chord must be a tension equal to $20 \times 24/30 = 16$ kips.

Taking a section around joint 2 in Fig. 6.51*a* reveals that the stress in the vertical *fg* is zero since there are no loads at the joint and the bottom chord is perpendicular to the vertical. Also, the stress must be the same in both bottom-chord members at the joint since the sum of the horizontal components must be zero.

After joints 1 and 2 have been solved, a section around joint 3 cuts only two unknown stresses: S_{BH} in top chord *BH* and S_{HG} in diagonal *HG*. Application of the laws of equilibrium to this joint yields the following two equations, one for the vertical components and the second for the horizontal components:

$$\Sigma V = 0.6S_{FA} - 8 - 0.6S_{BH} + 0.6S_{HG} = 0 \tag{6.77}$$

$$\Sigma H = 0.8S_{FA} - 0.8S_{BH} - 0.8S_{HG} = 0 \tag{6.78}$$

Both unknown stresses are assumed to be compressive, i.e., acting toward the joint. The stress in the vertical does not appear in these equations because it already was determined to be zero. The stress in *FA*, S_{FA}, was found from analysis of joint 1 to be 20 kips. Simultaneous solution of the two equations yields $S_{HG} = 6.7$ kips and $S_{BH} = 13.3$ kips. (If these stresses had come out with a negative sign, it would have indicated that the original assumption of their directions was incorrect; they would, in that case, be tensile forces instead of compressive forces.)

Examination of the force polygons in Fig. 6.51 indicates that each stress occurs in two force polygons. Hence, the graphical solution can be shortened by combining the polygons. The combination of the various polygons for all the joints into one stress diagram is known as a **Maxwell diagram** (Fig. 6.51*f*).

Wind loads on a roof truss with a sloping top chord are assumed to act normal to the roof, in which case the load polygon will be an inclined line or a true polygon. The reactions are computed generally on the assumption either that both are parallel to the resultant of the wind loads or that one end of the truss is free to move horizontally and therefore will not resist the horizontal components of the loads. The stress diagram is plotted in the same manner as for vertical loads after the reactions have been found.

Some trusses are complex and require special methods of analysis. (C. H. Norris et al., "Elementary Structural Analysis," McGraw-Hill Book Company, New York, 1976, books.mcgraw-hill.com.)

Parallel-Chord Trusses ▪ A convenient section for determining the stresses in diagonals of parallel-chord trusses is a vertical one, such as *N-N* in Fig. 6.52*a*. The sum of the forces acting on that portion of the truss to the left of *N-N* equals the vertical component of the stress in diagonal *cD* (see Fig. 6.52*b*). Thus, if θ is the acute angle between *cD* and the vertical,

$$R_1 - P_1 - P_2 + S\cos\theta = 0 \tag{6.79}$$

But $R_1 - P_1 - P_2$ is the algebraic sum of all the external vertical forces on the left of the section and is the vertical shear in the section. It may be designated as V. Therefore,

$$V + S\cos\theta = 0 \quad \text{or} \quad S = -V\sec\theta \tag{6.80}$$

From this it follows that for trusses with horizontal chords and single-web systems, the stress in any web member, other than the subverticals, equals the vertical shear in the member multiplied by the secant of the angle that the member makes with the vertical.

Nonparallel Chords ▪ A vertical section also can be used to determine the stress in diagonals

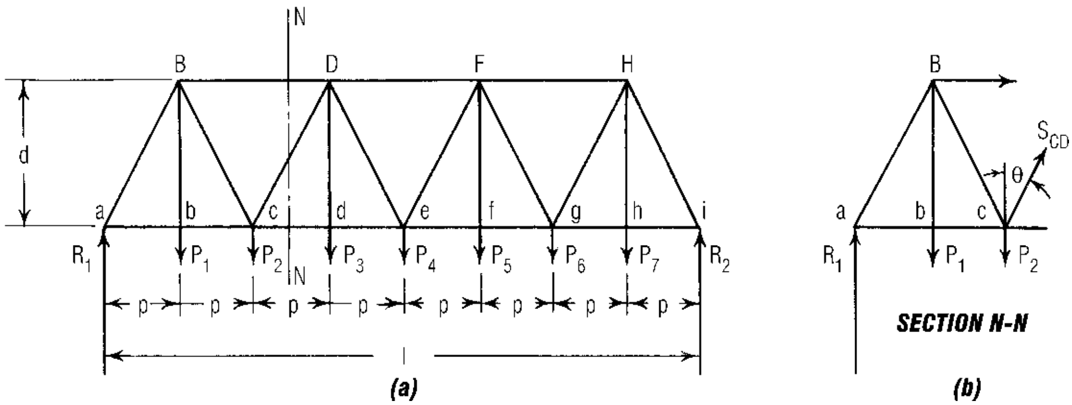


Fig. 6.52 Vertical section through the truss in (*a*) enables determination of stress in the diagonal (*b*).

when the chords are not parallel, but the previously described procedure must be modified. Suppose, for example, that the stress in the diagonal *Bc* of the Parker truss in Fig. 6.53 is to be found. Take a vertical section to the left of joint *c*. This section cuts *BC*, the top chord, and *Bc*, both of which have vertical components, as well as the horizontal bottom chord *bc*. Now, extend *BC* and *bc* until they intersect, at *O*. If *O* is used as the center for taking moments of all the forces, the moments of the stresses in *BC* and *bc* will be zero since the lines of action pass through *O*. Since *Bc* remains the only stress with a moment about *O*, *Bc* can be computed from the fact that the sum of the moments about *O* must equal zero, for equilibrium.

Generally, the calculation can be simplified by determining first the vertical component of the diagonal and from it the stress. So resolve *Bc* into its horizontal and vertical components Bc_H and Bc_V, at *c*, so that the line of action of the horizontal component passes through *O*. Taking moments about *O* yields

$$(Bc_v \times Oc) - (R \times Oa) + (P_1 \times Ob) = 0 \qquad (6.81)$$

from which Bc_V may be determined. The actual stress in *Bc* is Bc_v multiplied by the secant of the angle that *Bc* makes with the vertical.

The stress in verticals, such as *Cc*, can be found in a similar manner. But take the section on a slope so as not to cut the diagonal but only the vertical and the chords. The moment equation about the intersection of the chords yields the stress in the vertical directly since it has no horizontal component.

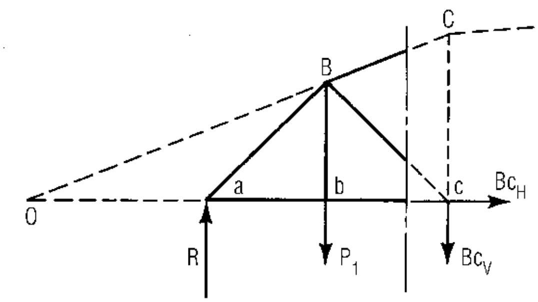


Fig. 6.53 Stress in a truss diagonal is determined by taking a vertical section and computing moments about the intersection of top and bottom chords.

Subdivided Panels ▪ In a truss with parallel chords and subdivided panels, such as the one in Fig. 6.54*a*, the subdiagonals may be either tension or compression. In Fig. 6.54*a*, the subdiagonal *Bc* is in compression and *d′E* is in tension. The vertical component of the stress in any subdiagonal, such as *d′E*, equals half the stress in the vertical *d′d* at the intersection of the subdiagonal and main diagonal. See Fig. 6.54*b*.

For a truss with inclined chords and subdivided panels, this is not the case. For example, the stress in *d′E* for a truss with nonparallel chords is $d'd \times l/h$, where *l* is the length of *d′E* and *h* is the length of *Ee*.

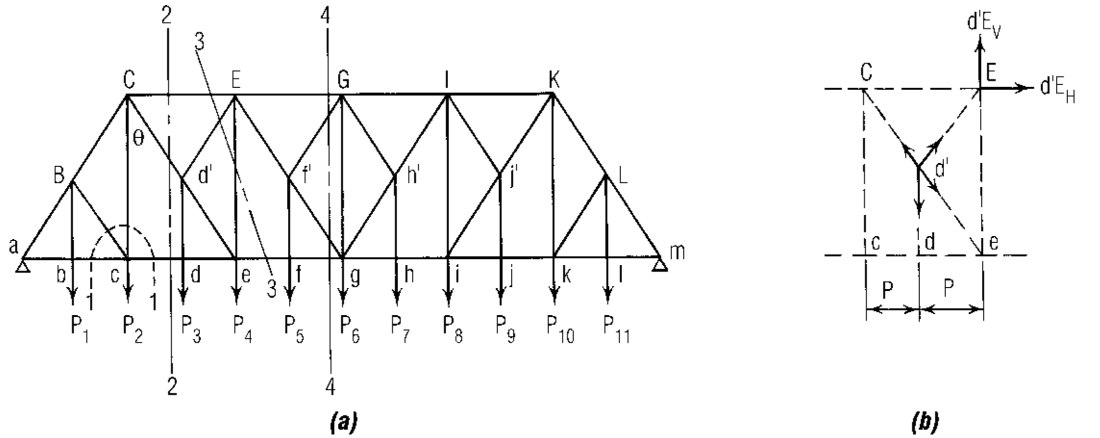


Fig. 6.54 Sections taken through truss with subdivided panels for finding stresses in web members.

6.47 Moving Loads on Trusses and Girders

To minimize bending stresses in truss members, framing is arranged to transmit loads to panel points. Usually, in bridges, loads are transmitted from a slab to stringers parallel to the trusses, and the stringers carry the load to transverse floor beams, which bring it to truss panel points. Similar framing generally is used for bridge girders.

In many respects, analysis of trusses and girders is similar to that for beams—determination of maximum end reaction for moving loads, for example, and use of influence lines (Art. 6.25). For girders, maximum bending moments and shears at various sections must be determined for moving loads, as for beams; and as indicated in Art. 6.46, stresses in truss members may be determined by taking moments about convenient points or from the shear in a panel. But girders and trusses differ from beams in that analysis must take into account the effect at critical sections of loads between panel points since such loads are distributed to the nearest panel points; hence, in some cases, influence lines differ from those for beams.

Stresses in Verticals ▪ The maximum total stress in a load-bearing stiffener of a girder or in a truss vertical, such as *Bb* in Fig. 6.55*a*, equals the maximum reaction of the floor beam at the panel point. The influence line for the reaction at *b* is shown in Fig. 6.55*b* and indicates that for maximum reaction, a uniform load of *w* lb/lin ft should extend a distance of $2p$, from *a* to *c*, where *p* is the length of a panel. In that case, the stress in *Bb* equals *wp*.

Maximum floor beam reaction for concentrated moving loads occurs when the total load between *a* and *c*, W_1 (Fig. 6.55*c*), equals twice the load between *a* and *b*. Then, the maximum live-load stress in *Bb* is

$$r_b = \frac{W_1 g - 2Pg'}{p} = \frac{W_1(g - g')}{p} \qquad (6.82)$$

where *g* is the distance of W_1 from *c*, and g' is the distance of *P* from *b*.

Stresses in Diagonals ▪ For a truss with parallel chords and single-web system, stress in a diagonal, such as *Bc* in Fig. 6.55*a*, equals the shear in the panel multiplied by the secant of the angle θ the diagonal makes with the vertical. The influence diagram for stresses in *Bc*, then, is the shear influence diagram for the panel multiplied by sec θ, as indicated in Fig. 6.55*d*. For maximum tension in *Bc*, loads should be placed only in the portion of the span for which the influence diagram is positive (crosshatched in Fig. 6.55*d*). For maximum compression, the loads should be placed where the diagram is negative (minimum shear).

A uniform load, however, cannot be placed over the full positive or negative portions of the span to get a true maximum or minimum. Any load in the panel is transmitted to the panel points at both

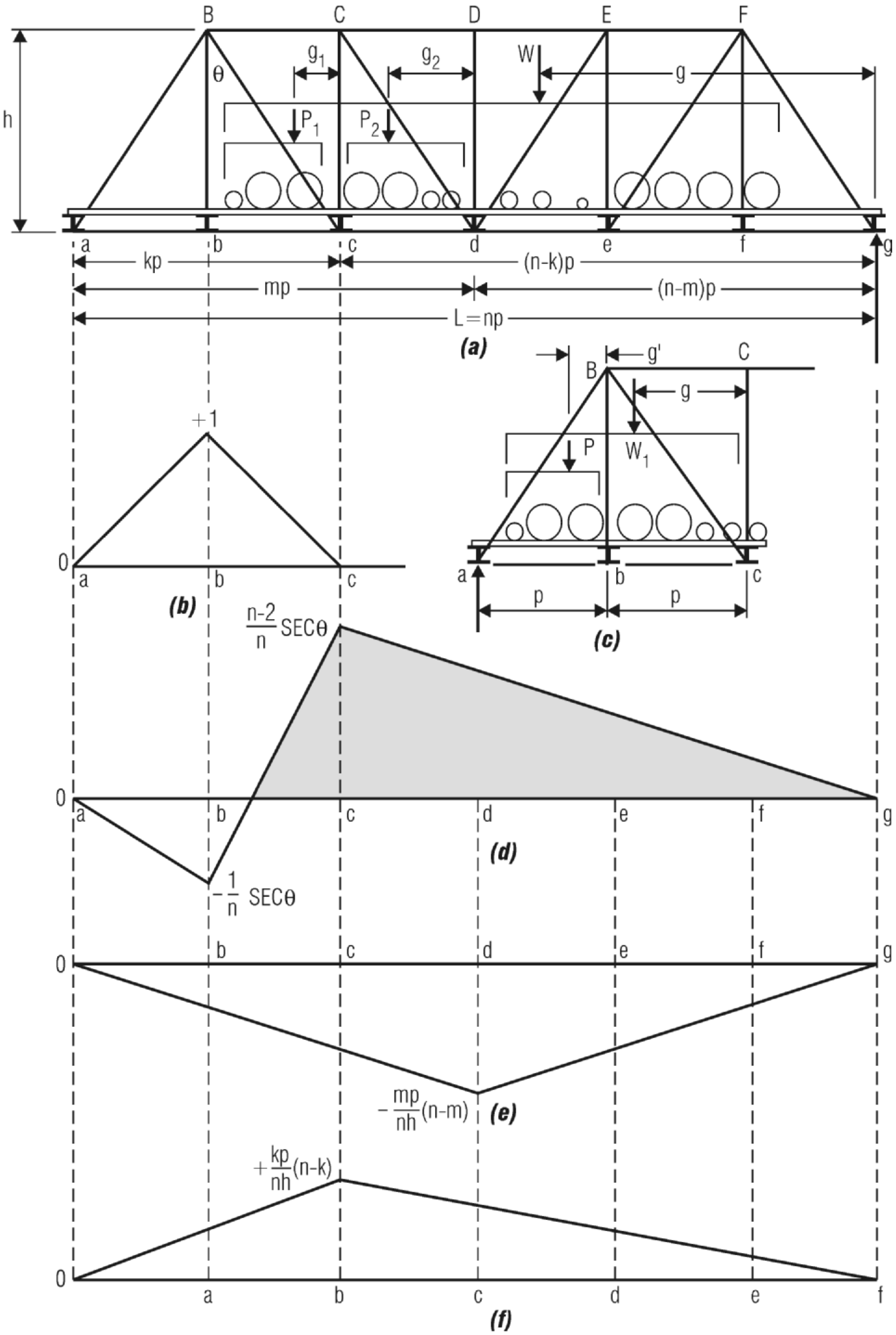


Fig. 6.55 Stresses produced in a truss by moving loads are determined with influence lines.

ends of the panel and decreases the shear. True maximum shear occurs for *Bc* when the uniform load extends into the panel a distance *x* from *c* equal to $(n - k)p/(n - 1)$, where *n* is the number of panels in the truss and *k* the number of panels from the left end of the truss to *c*.

For maximum stress in *Bc* caused by moving concentrated loads, the loads must be placed to produce maximum shear in the panel, and this may require several trials with different wheels placed at *c* (or, for minimum shear, at *b*). When the wheel producing maximum shear is at *c*, the loading will satisfy the following criterion: When the wheel is just to the right of *c*, W/n is greater than P_1, where *W* is the total load on the span and P_1 the load in the panel (Fig. 6.55*a*); when the wheel is just to the left of *c*, W/n is less than P_1.

Stresses in Chords ▪ Stresses in truss chords, in general, can be determined from the bending moment at a panel point, so the influence diagram for chord stress has the same shape as that for bending moment at an appropriate panel point. For example, Fig. 6.55*e* shows the influence line for stress in upper chord *CD* (minus signifies compression). The ordinates are proportional to the bending moment at *d* since the stress in *CD* can be computed by considering the portion of the truss just to the left of *d* and taking moments about *d*. Figure 6.55*f* similarly shows the influence line for stress in bottom chord *cd*.

For maximum stress in a truss chord under uniform load, the load should extend the full length of the truss.

For maximum chord stress caused by moving concentrated loads, the loads must be placed to produce maximum bending moment at the appropriate panel point, and this may require several trials with different wheels placed at the panel point. Usually, maximum moment will be produced with the heaviest grouping of wheels about the panel point.

In all trusses with verticals, the loading producing maximum chord stress will satisfy the following criterion: When the critical wheel is just to the right of the panel point, Wm/n is greater than *P*, where *mp* is the distance of the panel point from the left end of the truss with span *np* and *P* is the sum of the loads to the left of the panel point; when the wheel is just to the left of the panel point, Wm/n is less than *P*.

In a truss without verticals, the maximum stress in the loaded chord is determined by a different criterion. For example, the moment center for the lower chord *bc* (Fig. 6.56) is panel point *C*, at a

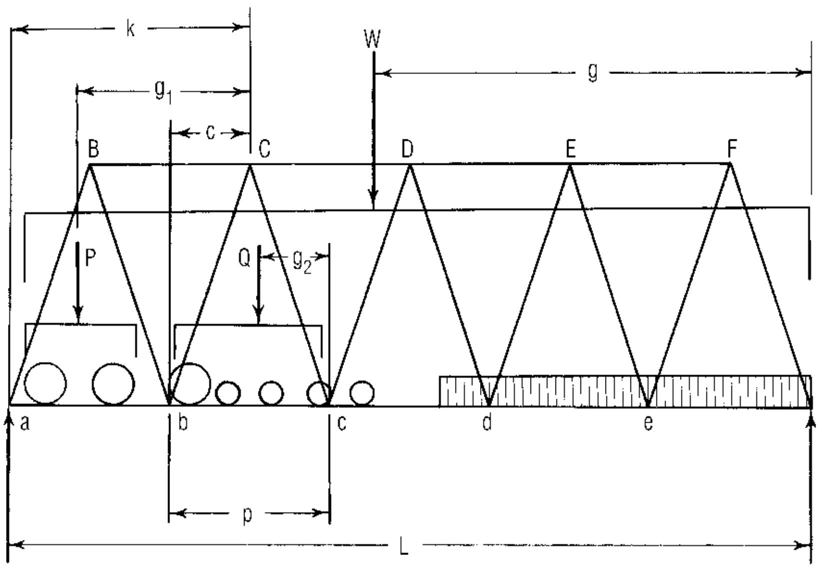


Fig. 6.56 Moving loads on a truss without verticals.

distance c from b. When the critical load is at b or c, the following criterion will be satisfied: When the wheel is just to the right of b or c, Wk/L is greater than $P + Qc/p$; when the wheel is just to the left of b or c, Wk/L is less than $P + Qc/p$, where W is the total load on the span, Q the load in panel bc, P the load to the left of bc, and k the distance of the center of moments C from the left support. The moment at C is $Wgk/L - Pg_1 - Qcg_2/p$, where g is the distance of the center of gravity of the loads W from the right support, g_1 the distance of the center of gravity of the loads P from C, and g_2 the distance of the center of gravity of the loads Q from c, the right end of the panel.

6.48 Counters

For long-span bridges, it often is economical to design the diagonals of trusses for tension only. But in the panels near the center of a truss, maximum shear due to live loads plus impact may exceed and be opposite in sign to the dead-load shear, thus inducing compression in the diagonal. If the tension diagonal is flexible, it will buckle. Hence, it becomes necessary to place in such panels another diagonal crossing the main diagonal (Fig. 6.57). Such diagonals are called *counters.*

Designed only for tension, a counter is assumed to carry no stress under dead load because it would buckle slightly. It comes into action only when the main diagonal is subjected to compression. Hence, the two diagonals never act together.

Although the maximum stresses in the main members of a truss are the same whether or not counters are used, the minimum stresses in the verticals are affected by the presence of counters. In most trusses, however, the minimum stresses in the verticals where counters are used are of the same sign as the maximum stresses and hence have no significance.

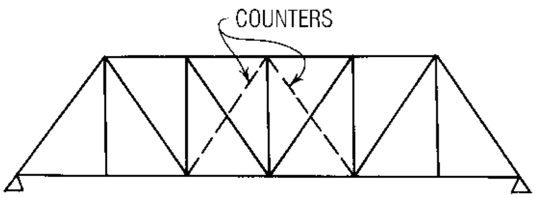


Fig. 6.57 Truss with counters.

6.49 Stresses in Trusses Due to Lateral Forces

To resist lateral forces on bridge trusses, trussed systems are placed in the planes of the top and bottom chords, and the ends, or **portals**, also are braced as low down as possible without impinging on headroom needed for traffic (Fig. 6.58). In stress analysis of lateral trusses, wind loads may be assumed as all applied on the windward chord or as applied equally on the two chords. In the former case, the stresses in the lateral struts are one-half panel load greater than if the latter assumption were made, but this is of no practical consequence.

Where the diagonals are considered as tension members only, counter stresses need not be computed since reversal of wind direction gives greater stresses in the members concerned than any partial loading from the opposite direction. When a rigid system of diagonals is used, the two diagonals of a panel may be assumed equally stressed. Stresses in the chords of the lateral truss should be combined with those in the chords of the main trusses due to dead and live loads.

In computation of stresses in the lateral system for the loaded chords of the main trusses, the wind on the live load should be added to the wind on the trusses. Hence, the wind on the live load should be positioned for maximum stress on the lateral truss. Methods described in Art. 6.46 can be used to compute the stresses on the assumption that each diagonal takes half the shear in each panel.

When the main trusses have inclined chords, the lateral systems between the sloping chords lie in several planes, and the exact determination of all the wind stresses is rather difficult. The stresses in the lateral members, however, may be determined without significant error by considering the lateral truss flattened into one plane. Panel lengths will vary, but the panel loads will be equal and may be determined from the horizontal panel length.

Since some of the lateral forces are applied considerably above the horizontal plane of the end supports of the bridge, these forces tend to overturn the structure (Fig. 6.58*e*). The lateral forces of the upper lateral system (Fig. 6.58*a*) are carried to the portal struts, and the horizontal loads at these points produce an overturning moment about the horizontal plane of the supports. In Fig. 6.58*e*, P represents the horizontal load brought to each portal strut by the upper lateral bracing, h the depth of the truss,

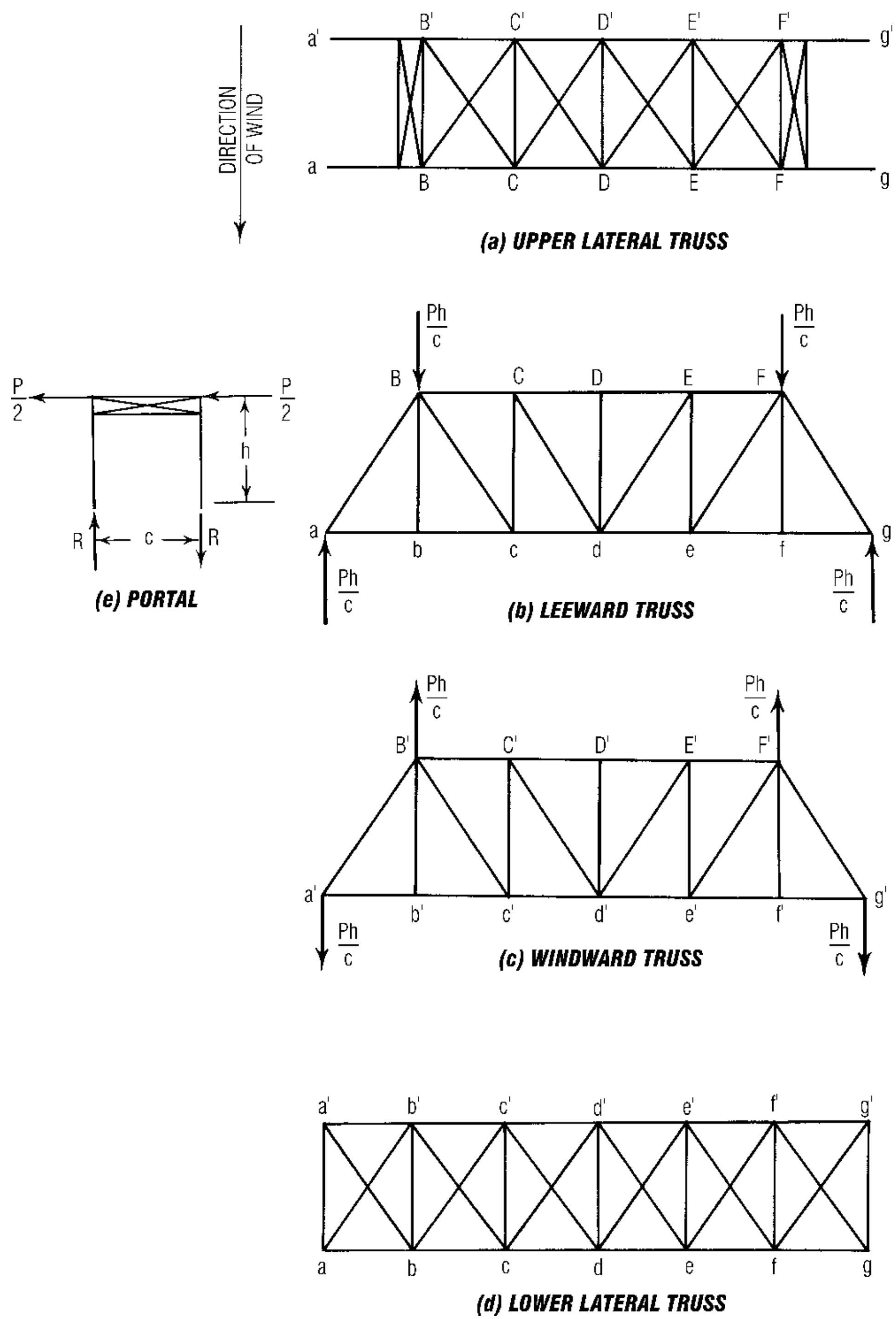


Fig. 6.58 Lateral trusses for bracing top and bottom chords of bridge trusses.

and c the distance between trusses. The overturning moment produced at each end of the structure is Ph, which is balanced by a reaction couple Rc. The value of the reaction R is then Ph/c. An equivalent effect is achieved on the main trusses if loads equal to Ph/c are applied at B and F and at B' and F', as shown in Fig. 6.58*b* and *c*. These loads produce stresses in the end posts and in the lower-chord members, but the web members are not stressed.

The lateral force on the live load also causes an overturning moment, which may be treated in a similar manner. But there is a difference as far as the web members of the main truss are concerned. Since the lateral force on the live load produces an effect corresponding to the position of the live load on the bridge, equivalent panel loads, rather than equivalent reactions, must be computed. If the distance from the resultant of the wind force to the plane of the loaded chord is h', the equivalent vertical panel load is Ph'/c, where P is the horizontal panel load due to the lateral force.

6.50 Complex Trusses

The method of sections may not provide a direct solution for some trusses with inclined chords and multiple-web systems. But if the truss is stable and statically determinate, a solution can be obtained by applying the equations of equilibrium to a section taken around each joint. The stresses in the truss members are obtained by solution of the simultaneous equations.

Since two equations of equilibrium can be written for the forces acting at a joint (Art. 6.46), the total number of equations available for a truss is $2n$, where n is the number of joints. If r is the number of horizontal and vertical components of the reactions, and s the number of stresses, $r + s$ is the number of unknowns.

If $r + s = 2n$, the unknowns can be obtained from solution of the simultaneous equations. If $r + s$ is less than $2n$, the structure is unstable (but the structure may be unstable even if $r + s$ exceeds $2n$). If $r + s$ is greater than $2n$, there are too many unknowns; the structure is statically indeterminate.

General Tools for Structural Analysis

For some types of structures, the equilibrium equations are not sufficient to determine the reactions or the internal stresses. These structures are called *statically indeterminate*.

For the analysis of such structures, additional equations must be written based on a knowledge of the elastic deformations. Hence, methods of analysis that enable deformations to be evaluated for unknown forces or stresses are important for the solution of problems involving statically indeterminate structures. Some of these methods, like the method of virtual work, also are useful in solving complicated problems involving statically determinate systems.

6.51 Virtual Work

A virtual displacement is an imaginary, small displacement of a particle consistent with the constraints upon it. Thus, at one support of a simply supported beam, the virtual displacement could be an infinitesimal rotation $d\theta$ of that end, but not a vertical movement. However, if the support is replaced by a force, then a vertical virtual displacement may be applied to the beam at that end.

Virtual work is the product of the distance a particle moves during a virtual displacement and the component in the direction of the displacement of a force acting on the particle. If the displacement and the force are in opposite directions, the virtual work is negative. When the displacement is normal to the force, no work is done.

Suppose a rigid body is acted on by a system of forces with a resultant R. Given a virtual displacement ds at an angle α with R, the body will have virtual work done on it equal to $R \cos \alpha \, ds$. (No work is done by internal forces. They act in pairs of equal magnitude but opposite direction, and the virtual work done by one force of a pair is equal and opposite in sign to the work done by the other force.) If the body is in equilibrium under the action of the forces, then $R = 0$, and the virtual work also is zero.

Thus, the principle of virtual work may be stated:

If a rigid body in equilibrium is given a virtual displacement, the sum of the virtual work of the forces acting on it must be zero.

As an example of how the principle may be used, let us apply it to the determination of the reaction R of the simple beam in Fig. 6.59*a*. First, replace the support by an unknown force R. Next,

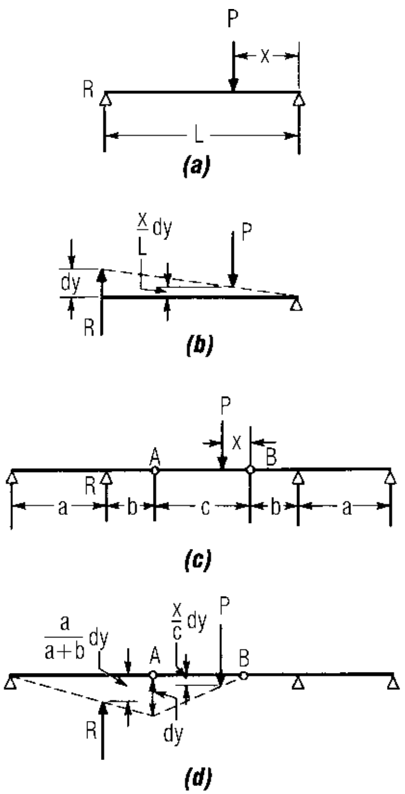


Fig. 6.59 Virtual work applied to determination of a simple-beam reaction (*a*) and (*b*) and the reaction of a beam with suspended span (*c*) and (*d*).

move the end of the beam upward a small amount dy as in Fig. 6.59*b*. The displacement under the load P will be $x\,dy/L$, upward. Then, the virtual work is $R\,dy - Px\,dy/L = 0$, from which $R = Px/L$.

The principle also may be used to find the reaction R of the more complex beam in Fig. 6.59*c*. Again, the first step is to replace one support by an unknown force R. Next, apply a virtual downward displacement dy at hinge A (Fig. 6.59*d*). The displacement under the load P will be $x\,dy/c$ and at the reaction R will be $a\,dy/(a+b)$. According to the principle of virtual work, $-Ra\,dy/(a+b) + Px\,dy/c = 0$; thus, $R = Px(a+b)/ac$. In this type of problem, the method has the advantage that only one reaction need be considered at a time and internal forces are not involved.

6.52 Strain Energy

When an elastic body is deformed, the virtual work done by the internal forces equals the corresponding increment of the strain energy dU, in accordance with the principle of virtual work.

Assume a constrained elastic body acted on by forces $P_1, P_2, \ldots,$ for which the corresponding deformations are $e_1, e_2, \ldots$ Then, $\Sigma P_n de_n = dU$. The increment of the strain energy due to the increments of the deformations is given by

$$dU = \frac{\partial U}{\partial e_1} de_1 + \frac{\partial U}{\partial e_2} de_2 + \cdots$$

When solving a specific problem, a virtual displacement that is most convenient in simplifying the solution should be chosen. Suppose, for example, a virtual displacement is selected that affects only the deformation e_n corresponding to the load P_n, other deformations being unchanged. Then, the principle of virtual work requires that

$$P_n de_n = \frac{\partial U}{\partial e_n} de_n$$

This is equivalent to

$$\frac{\partial U}{\partial e_n} = P_n \qquad (6.83)$$

which states that the partial derivative of the strain energy with respect to a specific deformation gives the corresponding force.

Suppose, for example, the stress in the vertical bar in Fig. 6.60 is to be determined. All bars are

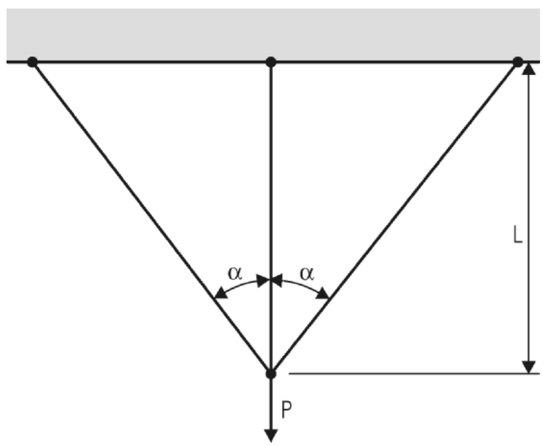


Fig. 6.60 Indeterminate truss.

made of the same material and have the same cross section A. If the vertical bar stretches an amount e under the load P, the inclined bars will each stretch an amount $e\cos\alpha$. The strain energy in the system is [from Eq. (6.23*a*)]

$$U = \frac{AE}{2L}(e^2 + 2e^2\cos^3\alpha)$$

and the partial derivative of this with respect to e must be equal to P; that is,

$$P = \frac{AE}{2L}(2e + 4e\cos^3\alpha) = \frac{AEe}{L}(1 + 2\cos^3\alpha)$$

Noting that the force in the vertical bar equals AEe/L, we find from the above equation that the required stress equals $P/(1 + 2\cos^3\alpha)$.

Castigliano's Theorems ▪ If strain energy is expressed as a function of statically independent forces, the partial derivative of the strain energy with respect to a force gives the deformation corresponding to that force:

$$\frac{\partial U}{\partial P_n} = e_n \tag{6.84}$$

This is known as Castigliano's first theorem. (His second theorem is the principle of least work.)

6.53 Method of Least Work

Castigliano's second theorem, also known as the principle of least work, states:

The strain energy in a statically indeterminate structure is the minimum consistent with equilibrium.

As an example of the use of the method of least work, an alternative solution will be given for the stress in the vertical bar in Fig. 6.60 (see Art. 5.52). Calling this stress X, we note that the stress in each of the inclined bars must be $(P - X)/2\cos\alpha$. Using Eq. (6.23*a*), we can express the strain energy in the system in terms of X:

$$U = \frac{X^2L}{2AE} + \frac{(P - X)^2L}{4\,AE\cos^3\alpha}$$

Hence, the internal work in the system will be a minimum when

$$\frac{\partial U}{\partial X} = \frac{XL}{AE} - \frac{(P - X)L}{2\,AE\cos^3\alpha} = 0$$

Solving for X gives the stress in the vertical bar as $P/(1 + 2\cos^3\alpha)$, as in Art. 5.52.

6.54 Dummy Unit-Load Method for Displacements

The strain energy for pure bending is $U = M^2L/2EI$ [see Eq. (6.23*d*)]. To find the strain energy due to bending stress in a beam, we can apply this equation to a differential length dx of the beam and integrate over the entire span. Thus,

$$U = \int_0^L \frac{M^2dx}{2EI} \tag{6.85}$$

If we let M represent the bending moment due to a generalized force P, the partial derivative of the strain energy with respect to P is the deformation d corresponding to P. Differentiating Eq. (6.85) gives

$$d = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}dx \tag{6.86}$$

The partial derivative in this equation is the rate of change of bending moment with the load P. It equals the bending moment m produced by a unit generalized load applied at the point where the deformation is to be measured and in the direction of the deformation. Hence, Eq. (6.86) can also be written as

$$d = \int_0^L \frac{Mm}{EI}dx \tag{6.87}$$

To find the vertical deflection of a beam, we apply a dummy unit load vertically at the point where the deflection is to be measured and substitute the bending moments due to this load and the actual loading in Eq. (6.87). Similarly, to compute a rotation, we apply a dummy unit moment.

Beam Deflections ▪ As a simple example, let us apply the dummy unit-load method to the determination of the deflection at the center of a simply supported, uniformly loaded beam of constant moment of inertia (Fig. 6.61*a*). As indicated in Fig. 6.61*b*, the bending moment at a distance x from one end is $(wL/2)x - (w/2)x^2$. If we apply a dummy

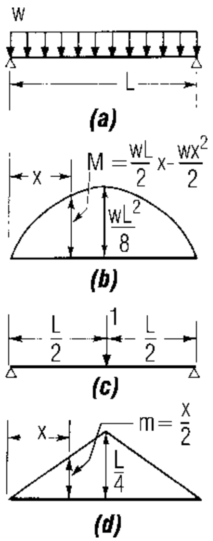


Fig. 6.61 Dummy unit-load method applied to a uniformly loaded beam (*a*) to find the midspan deflection; (*b*) moment diagram for the uniform load; (*c*) unit load at midspan; (*d*) moment diagram for the unit load.

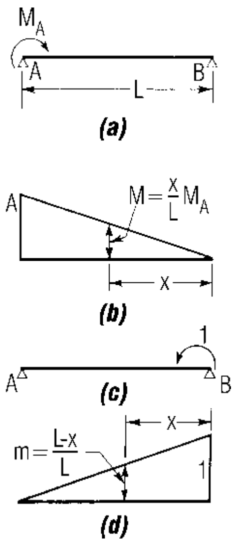


Fig. 6.62 End rotation at *B* in beam *AB* (*a*) caused by end moment at *A* is determined by dummy unit-load method; (*b*) moment diagram for the end moment; (*c*) unit moment applied at beam end; (*d*) moment diagram for that moment.

unit load vertically at the center of the beam (Fig. 6.61*c*), where the vertical deflection is to be determined, the moment at x is $x/2$, as indicated in Fig. 6.61*d*. Substituting in Eq. (6.87) and taking advantage of symmetry of loading gives

$$d = 2\int_0^{L/2}\left(\frac{wL}{2}x - \frac{w}{2}x^2\right)\frac{x\,dx}{2\,EI} = \frac{5wL^4}{384EI}$$

Beam-End Rotations ▪ As another example, let us apply the method to finding the end rotation at one end of a simply supported, prismatic beam produced by a moment applied at the other end. In other words, the problem is to find the end rotation at B, θ_B in Fig. 6.62*a*, due to M_A. As indicated in Fig. 6.62*b*, the bending moment at a distance x from B due to M_A is $M_A x/L$. If we apply a dummy unit moment at B (Fig. 6.62*c*), it will produce a moment at x of $(L - x)/L$ (Fig. 6.62*d*).

Substituting in Eq. (6.87) gives

$$\theta_B = \int_0^L M_A \frac{x}{L}\frac{L-x}{L}\frac{dx}{EI} = \frac{M_A L}{6EI} \qquad (6.88)$$

Shear Deflections ▪ To determine the deflection of a beam due to shear, Castigliano's first theorem can be applied to the strain energy in shear:

$$U = \iint \frac{v^2}{2G}\,dA\,dx \qquad (6.89)$$

where v = shearing unit stress

G = modulus of rigidity

A = cross-sectional area

Truss Deflections ▪ The dummy unit-load method also may be adapted to computation of truss deformations. The strain energy in a truss is given by

$$U = \sum \frac{S^2 L}{2AE} \qquad (6.90)$$

which represents the sum of the strain energy for all the members of the truss. S is the stress in each member due to the loads, L the length of each, A the cross-sectional area, and E the modulus of elasticity. Application of Castigliano's first theorem (Art. 6.52) and differentiation inside the summation sign yield the deformation:

$$d = \sum \frac{SL}{AE}\frac{\partial S}{\partial P} \tag{6.91}$$

where, as in Art. 6.54, P represents a generalized load. The partial derivative in this equation is the rate of change of axial stress with P. It equals the axial stress u produced in each member of the truss by a unit load applied at the point where the deformation is to be measured and in the direction of the deformation. Consequently, Eq. (6.91) also can be written

$$d = \sum \frac{SuL}{AE} \tag{6.92}$$

To find the vertical deflection at any point of a truss, apply a dummy unit vertical load at the panel point where the deflection is to be measured. Substitute in Eq. (6.92) the stresses in each member of the truss due to this load and the actual loading. Similarly, to find the rotation of any joint, apply a dummy unit moment at the joint, compute the stresses in each member of the truss, and substitute in Eq. (6.92).

When it is necessary to determine the relative movement of two panel points in the direction of a member connecting them, apply dummy unit loads in opposite directions at those points.

Note that members not stressed by the actual loads or the dummy loads do not enter into the calculation of a deformation.

As an example of the application of Eq. (6.92), let us compute the midspan deflection of the truss in Fig. 6.63*a*. The stresses in kips due to the 20-kip load at every lower-chord panel point are given in Fig. 6.63*a* and Table 6.2. Also, the ratios of length of members in inches to their cross-sectional areas in square inches are given in Table 6.2. We apply a dummy unit vertical load at L_2, where the deflection is required. Stresses u due to this load are shown in Fig. 6.63*b* and Table 6.2.

Table 6.2 also contains the computations for the deflection. Members not stressed by the 20-kip loads or the dummy unit loads are not included. Taking advantage of the symmetry of the truss, the values are tabulated for only half the truss and the sum is doubled. Also, to reduce the amount of calculation, the modulus of elasticity E, which is equal to 30,000 is not included until the very last step since it is the same for all members.

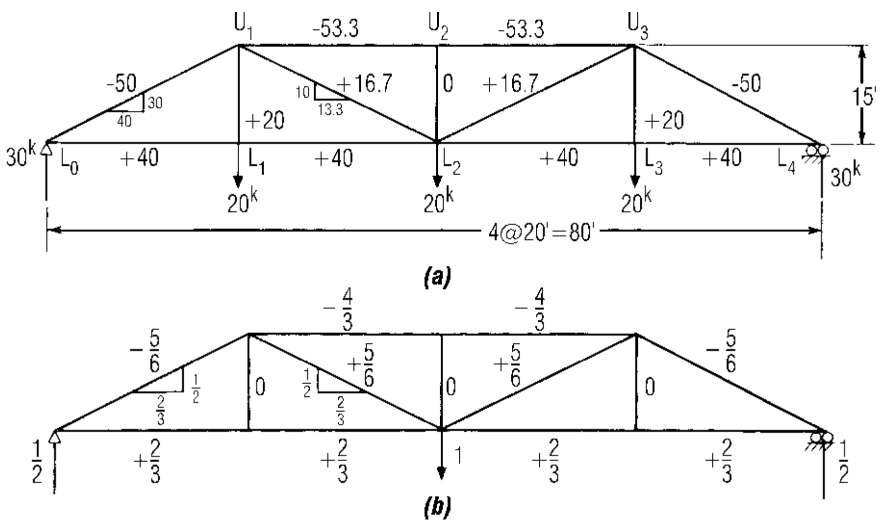


Fig. 6.63 Dummy unit-load method applied to a loaded truss to find (*a*) midspan deflection; (*b*) stresses produced by a unit load applied at midspan.

Table 6.2 Midspan Deflection of Truss of Fig. 6.63

Member	L/A	S	u	SuL/A
L_0L_2	160	+40	+2/3	4,267
L_0U_1	75	−50	−5/6	3,125
U_1U_2	60	−53.3	−4/3	4,267
U_1L_2	150	+16.7	+5/6	2,083
				13,742

Division of the summation of the last column by the modulus of elasticity $E = 30{,}000$ ksi yields the midspan deflection.

$$d = \sum \frac{SuL}{AE} = \frac{2 \times 13{,}742}{30{,}000} = 0.916\,\text{in}$$

6.55 Reciprocal Theorem and Influence Lines

Consider a structure loaded by a group of independent forces A, and suppose that a second group of forces B is added. The work done by the forces A acting over the displacements due to B will be W_{AB}.

Now, suppose the forces B had been on the structure first and then load A had been applied. The work done by the forces B acting over the displacements due to A will be W_{BA}.

The reciprocal theorem states that $W_{AB} = W_{BA}$.

Some very useful conclusions can be drawn from this equation. For example, there is the reciprocal deflection relationship:

The deflection at a point A due to a load at B equals the deflection at B due to the same load applied at A. Also, the rotation at A due to load (or moment) at B equals the rotation at B due to the same load (or moment) applied to A.

Another consequence is that deflection curves also may be influence lines, to some scale, for reactions, shears, moments, or deflections (**Mueller-Breslau principle**). For example, suppose the influence line for a reaction is to be found; that is, we wish to plot the reaction R as a unit load moves over the structure, which may be statically indeterminate. For loading condition A, we analyze the structure with a unit load on it at a distance x from some reference point. For loading condition B, we apply a dummy unit vertical load upward at the place where the reaction is to be determined,

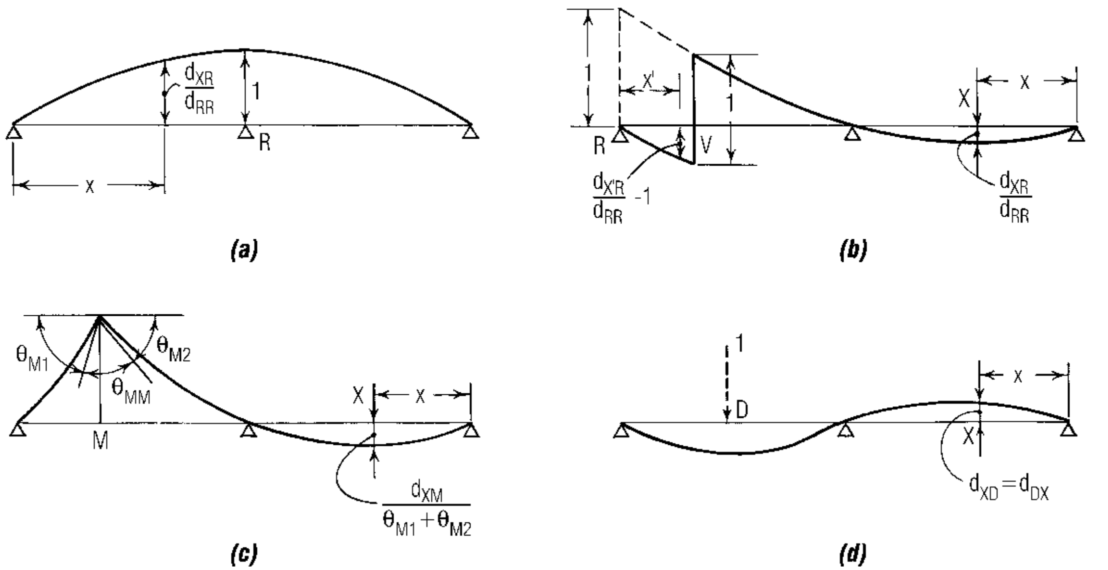


Fig. 6.64 Influence lines for a continuous beam are obtained from deflection curves. (*a*) Reaction at R; (*b*) shear at V; (*c*) bending moment at M; (*d*) deflection at D.

deflecting the structure off the support. At a distance x from the reference point, the displacement is d_{xR}, and over the support the displacement is d_{RR}. Hence, $W_{AB} = -1d_{xR} + Rd_{RR}$. On the other hand, W_{BA} is zero since loading condition A provides no displacement for the dummy unit load at the support in condition B. Consequently, from the reciprocal theorem, $W_{AB} = W_{BA} = 0$; hence

$$R = \frac{d_{xR}}{d_{RR}} \tag{6.93}$$

Since d_{RR}, the deflection at the support due to a unit load applied there, is a constant, R is proportional to d_{XR}. So the influence line for a reaction can be obtained from the deflection curve resulting from a displacement of the support (Fig. 6.64*a*). The magnitude of the reaction is obtained by dividing each ordinate of the deflection curve by d_{RR}.

Similarly, the influence line for shear can be obtained from the deflection curve produced by cutting the structure and shifting the cut ends vertically at the point for which the influence line is desired (Fig. 6.64*b*).

The influence line for bending moment can be obtained from the deflection curve produced by cutting the structure and rotating the cut ends at the point for which the influence line is desired (Fig. 6.64*c*).

Finally, it may be noted that the deflection curve for a load of unity is also the influence line for deflection at that point (Fig. 6.64*d*).

6.56 Superposition Methods

The principle of superposition states that, if several loads are applied to a linearly elastic structure, the displacement at each point of the structure equals the sum of the displacements induced at the point when the loads are applied individually in any sequence. Furthermore, the bending moment (or shear) at each point equals the sum of the bending moments (or shears) induced at the point by the loads applied individually in any sequence.

The principle holds only when the displacement (deflection or rotation) at every point of the structure is directly proportional to applied loads. Also, it is required that unit stresses be proportional to unit strains and that displacements be very small so that calculations can be based on the undeformed configuration of the structure without significant error.

As a simple example, consider a bar with length L and cross-sectional area A loaded with n axial loads $P_1, P_2, \ldots, P_n$. Let F equal the sum of the loads. From Eq. (6.8), F causes an elongation $\delta = FL/AE$, where E is the modulus of elasticity of the bar. According to the principle of superposition, if e_1 is the elongation caused by P_1 alone, e_2 by P_2 alone, ... and e_n by P_n alone, then regardless of the sequence in which the loads are applied, when all the loads are on the bar,

$$\delta = e_1 + e_2 + \cdots + e_n$$

This simple case can be easily verified by substituting $e_1 = P_1L/AE$, $e_2 = P_2L/AE, \ldots$, and $e_n = P_nL/AE$ in this equation and noting that $F = P_1 + P_2 + \cdots + P_n$:

$$\delta = \frac{P_1L}{AE} + \frac{P_2L}{AE} + \cdots + \frac{P_nL}{AE}$$
$$= (P_1 + P_2 + \cdots + P_n)\frac{L}{AE} = \frac{FL}{AE}$$

In the preceding equations, L/AE represents the elongation induced by a unit load and is called the **flexibility** of the bar.

The reciprocal, AE/L, represents the force that causes a unit elongation and is called the **stiffness** of the bar.

Analogous properties of beams, columns, and other structural members and the principle of superposition are useful in analysis of many types of structures. Calculation of stresses and displacements of statically indeterminate structures, for example, often can be simplified by resolution of bending moments, shears, and displacements into components chosen to supply sufficient equations for the solution from requirements for equilibrium of forces and compatibility of displacements.

Consider the continuous beam $ALRBC$ shown in Fig. 6.65*a*. Under the loads shown, member LR is subjected to end moments M_L and M_R (Fig. 6.65*b*) that are initially unknown. The bending-moment diagram for LR for these end moments is shown at the left in Fig. 6.65*c*. If these end moments were known, LR would be statically determinate; that is, LR could be treated as a simply supported beam subjected to known end moments, M_L and M_R. The analysis can be further simplified by resolution of the bending-moment diagram into the three components shown to the right of the equals sign

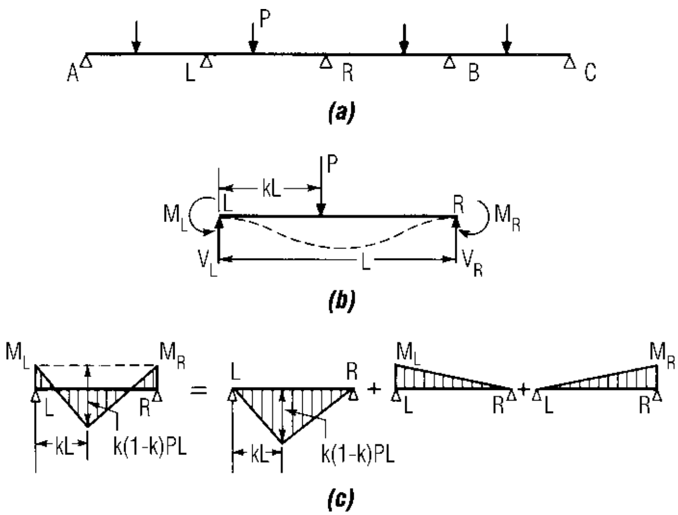


Fig. 6.65 Any span of a continuous beam (*a*) can be treated as a simple beam, as shown in (*b*) and (*c*). In (*c*), the moment diagram is resolved into basic components.

in Fig. 6.65*c*. This example leads to the following conclusion.

The bending moment at any section of a span *LR* of a continuous beam or frame equals the simple-beam moment due to the applied loads, plus the simple-beam moment due to the end moment at *L*, plus the simple-beam moment due to the end moment at *R*.

When the moment diagrams for all the spans of *ALRBC* in Fig. 6.65 have been resolved into components so that the spans may be treated as simple beams, all the end moments (moments at supports) can be determined from two basic requirements:

1. The sum of the moments at every support equals zero.
2. The end rotation (angular change at the support) of each member rigidly connected at the support is the same.

6.57 Influence-Coefficient Matrices

A matrix is a rectangular array of numbers in rows and columns that obeys certain mathematical rules known generally as matrix algebra and matrix calculus. A matrix consisting of only a single column is called a **vector**. In this book, matrices and vectors are represented by boldface letters, and their elements by lightface symbols, with appropriate subscripts. It often is convenient to use numbers for the subscripts to indicate the position of an element in the matrix. Generally, the first digit indicates the row, and the second digit, the column. Thus, in matrix **A**, A_{23} represents the element in the second row and third column:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{6.94}$$

Methods based on matrix representations often are advantageous for structural analysis and design of complex structures. One reason is that matrices provide a compact means of representing and manipulating large quantities of numbers. Another reason is that computers can perform matrix operations automatically and speedily. Computer programs are widely available for the purpose.

Matrix Equations ▪ Matrix notation is especially convenient in representing the solution of simultaneous linear equations, which arise frequently in structural analysis. For example, suppose a set of equations is represented in matrix notation by $\mathbf{AX} = \mathbf{B}$, where **X** is the vector of variables $X_1, X_2, \ldots, X_n$, **B** is the vector of the constants on the right-hand side of the equations, and **A** is a matrix of the coefficients of the variables. Multiplication of both sides of the equation by $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$, yields $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$.

Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, the identity matrix, and $\mathbf{IX} = \mathbf{X}$, the solution of the equations is represented by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. The matrix inversion $\mathbf{A}^{-1}$ can be readily performed by computers. For large matrices, however, it often is more practical to solve the equations; for example, by the Gaussian procedure of eliminating one unknown at a time.

In the application of matrices to structural analysis, loads and displacements are considered applied at the intersection of members (joints, or nodes). The loads may be resolved into moments, torques, and horizontal and vertical components. These may be assembled for each node into a vector and then all the node vectors may be combined into a force vector $\mathbf{P}$ for the whole structure.

$$\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \tag{6.95}$$

Similarly, displacements corresponding to those forces may be resolved into rotations, twists, and horizontal and vertical components and assembled for the whole structure into a vector $\mathbf{\Delta}$.

$$\mathbf{\Delta} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} \tag{6.96}$$

If the structure meets requirements for application of the principle of superposition (Art. 6.56) and forces and displacements are arranged in the proper sequence, the vectors of forces and displacements are related by

$$\mathbf{P} = \mathbf{K}_{\mathbf{\Delta}} \tag{6.97a}$$

$$\mathbf{\Delta} = \mathbf{FP} \tag{6.97b}$$

where $\mathbf{K}$ = stiffness matrix of the whole structure

$\mathbf{F}$ = flexibility matrix of the whole structure = $\mathbf{K}^{-1}$

The stiffness matrix $\mathbf{K}$ transforms displacements into loads. The flexibility matrix $\mathbf{F}$ transforms loads into displacements. The elements of $\mathbf{K}$ and $\mathbf{F}$ are functions of material properties, such as the modulus of elasticity; geometry of the structure; and sectional properties of members of the structure, such as area and moment of inertia. $\mathbf{K}$ and $\mathbf{F}$ are square matrices; that is, the number of rows in each equals the number of columns. In addition, both matrices are symmetrical; that is, in each matrix, the columns and rows may be interchanged without changing the matrix. Thus, $K_{ij} = K_{ji}$, and $F_{ij} = F_{ji}$, where i indicates the row in which an element is located, and j, the column.

Influence Coefficients ▪ Elements of the stiffness and flexibility matrices are influence coefficients. Each element is derived by computing the displacements (or forces) occurring at nodes when a unit displacement (or force) is imposed at one node, while all other displacements (or forces) are taken as zero.

Let Δ_i be the ith element of matrix $\mathbf{\Delta}$. Then, a typical element F_{ij} of $\mathbf{F}$ gives the displacement of a node i in the direction of Δ_i when a unit force acts at a node j in the direction of force P_j and no other forces are acting on the structure. The jth column of $\mathbf{F}$, therefore, contains all the nodal displacements induced by a unit force acting at node j in the direction of P_j.

Similarly, let P_i be the ith element of matrix $\mathbf{P}$. Then, a typical element K_{ij} of $\mathbf{K}$ gives the force at a node i in the direction of P_i when a node j is given a unit displacement in the direction of displacement Δ_j and no other displacements are permitted. The jth column of $\mathbf{K}$, therefore, contains all the nodal forces caused by a unit displacement of node j in the direction of Δ_j.

Application to a Beam ▪ A general method for determining the forces and moments in a continuous beam is as follows: Remove as many supports or members as necessary to make the structure statically determinate. (Such supports and members are often referred to as redundant.) Compute for the actual loads the deflections or rotations of the statically determinate structure in the direction of the unknown forces and couples exerted by the removed supports or members. Then, in terms of these forces and couples, treated as variables, compute the corresponding deflections or rotations the forces and couples produce in the statically determinate structure (see Arts. 6.32 and 6.54). Finally, for each redundant support or member, write equations that give the known rotations and deflections of the original structure in terms of the deformations of the statically determinate structure.

For example, one method of finding the reactions of the continuous beam *AC* in Fig. 6.66*a* is to remove supports 1, 2, and 3 temporarily. The beam is now simply supported between *A* and *C*. Hence, the reactions and the bending moments throughout can be computed from the laws of equilibrium. Beam *AC* deflects at points 1, 2, and 3, whereas we know that the continuous beam is prevented from deflecting at those points by the supports there. This information enables us to write three equations in terms of the three unknown reactions.

To determine the equations, assume that nodes exist at the location of the supports 1, 2, and 3. Then, for the actual loads, compute the vertical deflections d_1, d_2, and d_3 of simple beam *AC* at nodes 1, 2, and 3, respectively (Fig. 6.66*b*). Next, form two vectors, **d** with elements d_1, d_2, d_3, and **R** with the unknown reactions R_1 at node 1, R_2 at node 2, and R_3 at node 3 as elements. Since the beam may be assumed to be linearly elastic, set $\mathbf{d} = \mathbf{FR}$, where **F** is the flexibility matrix for simple beam *AC*. The elements y_{ij} of **F** are influence coefficients. To determine them, calculate column 1 of **F** as the deflections y_{11}, y_{21}, and y_{31} at nodes 1, 2, and 3, respectively, when a unit force is applied at node 1 (Fig. 6.66*c*). Similarly, compute column 2 of **F** for a unit force at node 2 (Fig. 6.66*d*) and column 3 for a unit force at node 3 (Fig. 6.66*e*). The three equations then are given by

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{6.98}$$

The solution may be represented by $\mathbf{R} = \mathbf{F}^{-1}\mathbf{d}$ and obtained by matrix or algebraic methods. See also Art. 6.66.

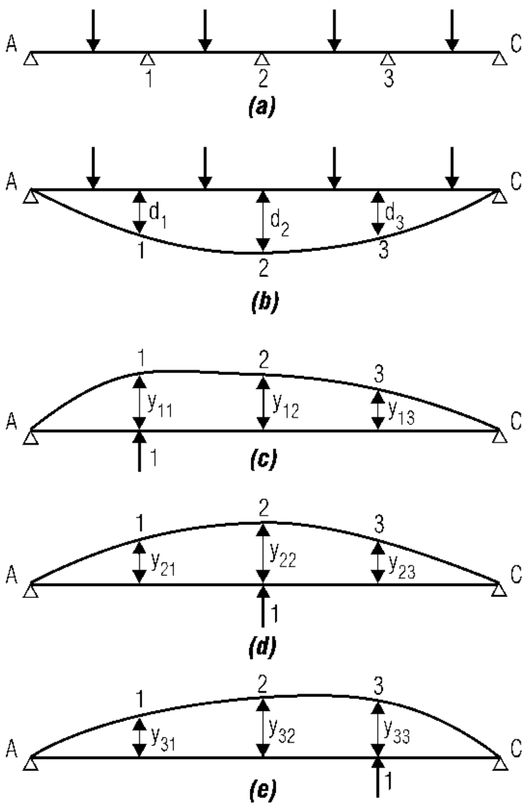


Fig. 6.66 Continuous beam (*a*) is converted into a simple beam (*b*) by temporary removal of interior supports. Reactions are then computed by equating the deflections due to the actual loads (*b*) to the sum of the deflections produced by the unknown reactions and the deflections due to the unit loads (*c*), (*d*), and (*e*).

Continuous Beams and Frames

Continuous beams and frames are statically indeterminate. Bending moments in them are functions of the geometry, moments of inertia, and modulus of elasticity of individual members as well as of loads and spans. Although these moments can be determined by the methods described in Arts. 6.51 to 6.55, there are methods specially developed for beams and frames that often make analysis simpler. The following articles describe some of these methods.

6.58 Carry-Over and Fixed-End Moments

When a member of a continuous beam or frame is loaded, bending moments are induced at the ends of the member as well as between the ends. The magnitude of the end moments in the member depends on the magnitude and location of the loads, the geometry of the member, and the amount of restraint offered to end rotation of the member by other members connected to it. Connections are assumed to be rigid; that is, all members at a joint rotate through the same angle. As a result, end moments are induced in the connecting members, in addition to end moments that may be induced by loads on those spans.

Computation of end moments in a continuous beam or frame requires that the geometry and elastic properties of the members be known or assumed. (If these characteristics have to be assumed, computations may have to be repeated when they become known.)

Loads on any span, as well as the displacement of any joint, induce moments at the ends of the other members of the structure. As a result, an originating end moment may be considered distributed to the other members. The ratio of the end moment in an unloaded span to the originating end moment in the loaded span is a constant.

Sign Convention ▪ For computation of end moments, the following sign convention is most convenient: A moment acting at an end of a member or at a joint is positive if it tends to rotate the end or joint clockwise; it is negative, if it tends to rotate the end or joint counterclockwise.

Similarly, the angular rotation at the end of a member is positive if in a clockwise direction, negative if counterclockwise. Thus, a positive end moment produces a positive end rotation in a simple beam.

For ease in visualizing the shape of the elastic curve under the action of loads and end moments, plot bending-moment diagrams on the tension side of each member. Hence, if an end moment is represented by a curved arrow, the arrow will point in the direction in which the moment is to be plotted.

Carry-over Moments ▪ If a span of a continuous beam is loaded and if the far end of a connecting member is restrained by support conditions against rotation, a resisting moment is induced at the far end. That moment is called a carry-over moment. The ratio of the carry-over moment to the other end moment in the span is called carry-over factor. It is a constant for the member, independent of the magnitude and sign of the moments to be carried over. Every beam has two carry-over factors, one directed toward each end.

As pointed out in Art. 6.56, analysis of a span of a continuous beam or frame can be simplified by treating it as a simple beam subjected to applied end moments. Thus, it is convenient to express the equations for carry-over factors in terms of the end rotations of simple beams: Convert a continuous member *LR* to a simple beam with the same span *L*. Apply a unit moment to one end (Fig. 6.67). The end rotation at the support where the moment is applied is α, and at the far end, the rotation is β. By the dummy-load method (Art. 6.54), if x is measured from the β end,

$$\alpha = \frac{1}{L^2}\int_0^L \frac{x^2}{EI_x}\,dx \tag{6.99}$$

$$\beta = \frac{1}{L^2}\int_0^L \frac{x(L-x)}{EI_x} \tag{6.100}$$

in which I_x is the moment of inertia at a section a distance of x from the β end, and E is the modulus of elasticity. In accordance with the reciprocal theorem (Art. 6.55), β has the same value regardless of the beam end to which the unit moment is applied (Fig. 6.67). For prismatic beams

$$\alpha_L = \alpha_R = \frac{L}{3EI} \tag{6.101}$$

$$\beta = \frac{L}{6EI} \tag{6.102}$$

The preceding equations can be used to determine carry-over factors for any magnitude of end restraint. The carry-over factors toward ends fixed against rotation, however, are of special importance

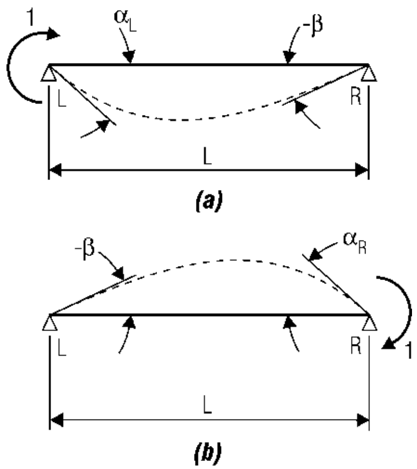


Fig. 6.67 End rotations of simple beam *LR* produced by a unit end moment (*a*) at *L*; (*b*) at *R*.

for moment distribution by converging approximations. For a span LR with ends L and R assumed to be fixed, the carry-over factor toward R is given by

$$C_R = \frac{\beta}{\alpha_R} \tag{6.103}$$

Similarly, the carry-over factor toward support L is given by

$$C_L = \frac{\beta}{\alpha_L} \tag{6.104}$$

If an end of a beam is free to rotate, the carry-over factor toward that end is zero.

Since the carry-over factors are positive, the moment carried over has the same sign as the applied moment.

Carry-over Factors for Prismatic Beams ▪ For prismatic beams, $\beta = L/6EI$ and $\alpha = L/3EI$. Hence,

$$C_L = C_R = \frac{L}{6EI} \cdot \frac{3EI}{L} = \frac{1}{2} \tag{6.105}$$

For beams with variable moment of inertia, β and α can be determined from Eqs. (6.99) and (6.100) and the carry-over factors from Eqs. (6.103) and (6.104).

Fixed-End Stiffness ▪ The fixed-end stiffness of a beam is defined as the moment that is required to induce a unit rotation at the support where it is applied while the other end of the beam is fixed against rotation. Stiffness is important because it determines the proportion of the total moment applied at a joint, or intersection of members, that is distributed to each member of the joint.

In Fig. 6.68*a*, the fixed-end stiffness of beam LR at end R is represented by K_R. When K_R is applied to beam LR at R, a moment $M_L = C_L K_R$ is carried over to end L, where C_L is the carry-over factor toward L. K_R induces an angle change α_R at R, where α_R is given by Eq. (6.99). The carry-over moment induces at R an angle change $-C_L K_R \beta$, where β is given by Eq. (6.100). Since, by the definition of stiffness, the total angle change at R is unity, $K_R \alpha_R - C_L K_R \beta = 1$, from which

$$K_R = \frac{I/\alpha_R}{1 - C_R C_L} \tag{6.106}$$

when C_R is substituted for β/α_R [see Eq. (6.103)].

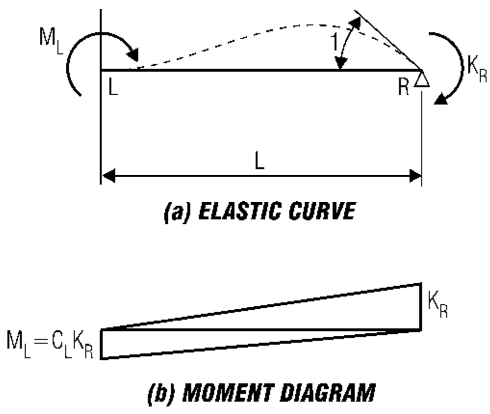


Fig. 6.68 Fixed-end stiffness.

In a similar manner, the stiffness at L is found to be

$$K_L = \frac{1/\alpha_L}{1 - C_R C_L} \tag{6.107}$$

Stiffness of Prismatic Beams ▪ With the use of Eqs. (6.101) and (6.105), the stiffness of a beam with constant moment of inertia is given by

$$K_L = K_R = \frac{3EI/L}{1 - \frac{1}{2} \times \frac{1}{2}} = \frac{4EI}{L} \tag{6.108}$$

where L = span of the beam

E = modulus of elasticity

I = moment of inertia of beam cross section

Beam with Hinge ▪ The stiffness of one end of a beam when the other end is free to rotate can be obtained from Eq. (6.106) or (6.107) by setting the carry-over factor toward the hinged end equal to zero. Thus, for a prismatic beam with one end hinged, the stiffness of the beam at the other end is given by

$$K = \frac{3EI}{L} \tag{6.109}$$

This equation indicates that a prismatic beam hinged at only one end has three-fourths the stiffness, or resistance to end rotation, as a beam fixed at both ends.

Fixed-End Moments ▪ A beam so restrained at its ends that no rotation is produced there by the loads is called a fixed-end beam, and the end moments are called fixed-end moments. Actually, it would be very difficult to construct a beam with ends that are truly fixed. The concept of fixed ends, however, is useful in determining the moments in continuous beams and frames.

Fixed-end moments may be expressed as the product of a coefficient and WL, where W is the total load on the span L. The coefficient is independent of the properties of other members of the structure. Thus, any member of a continuous beam or frame can be isolated from the rest of the structure and its fixed-end moments computed. Then, the actual moments in the beam can be found by applying a correction to each fixed-end moment.

Assume, for example, that the fixed-end moments for the loaded beam in Fig. 6.69*a* are to be determined. Let M_L^F be the moment at the left end L, and M_R^F the moment at the right end R of the beam. Based on the condition that no rotation is permitted at either end and that the reactions at the supports are in equilibrium with the applied loads, two equations can be written for the end moments in terms of the simple-beam end rotations, θ_L at L and θ_R at R for the specific loading.

Let K_L be the fixed-end stiffness at L and K_R the fixed-end stiffness at R, as given by Eqs. (6.106) and (6.107). Then, by resolution of the moment diagram into simple-beam components, as indicated in Figs. 6.69*f* to *h*, and application of the superposition principle (Art. 6.56), the fixed-end moments are found to be

$$M_L^F = -K_L(\theta_L + C_R\theta_R) \tag{6.110}$$

$$M_R^F = -K_R(\theta_R + C_L\theta_L) \tag{6.111}$$

where C_L and C_R are the carry-over factors to L and R, respectively [Eqs. (6.103) and (6.104)]. The end rotations θ_L and θ_R can be computed by a method described in Art. 6.32 or 6.54.

Moments for Prismatic Beams ▪ The fixed-end moments for beams with constant moment of inertia can be derived from the equations given above with the use of Eqs. (6.105) and (6.108):

$$M_L^R = -\frac{4EI}{L}\left(\theta_L + \frac{1}{2}\theta_R\right) \tag{6.112}$$

$$M_R^F = -\frac{4EI}{L}\left(\theta_R + \frac{1}{2}\theta_L\right) \tag{6.113}$$

where L = span of the beam
E = modulus of elasticity
I = moment of inertia

For horizontal beams with gravity loads only, θ_R is negative. As a result, M_L^F is negative and M_R^F positive.

For propped beams (one end fixed, one end hinged) with variable moment of inertia, the fixed-end moments are given by

$$M_L^F = -\frac{\theta_L}{\alpha_L} \quad \text{or} \quad M_R^F = -\frac{\theta_R}{\alpha_R} \tag{6.114}$$

where α_L and α_R are given by Eq. (6.99). For prismatic propped beams, the fixed-end moments are

$$M_L^F = -\frac{3EI\theta_L}{L} \quad \text{or} \quad M_R^F = -\frac{3EI\theta_R}{L} \tag{6.115}$$

Deflection of Supports ▪ Fixed-end moments for loaded beams when one support is displaced vertically with respect to the other support may be computed with the use of Eqs. (6.110) to (6.115) and the principle of superposition: Compute the fixed-end moments induced by the deflection of the beam when not loaded and add them to the fixed-end moments for the loaded condition with immovable supports.

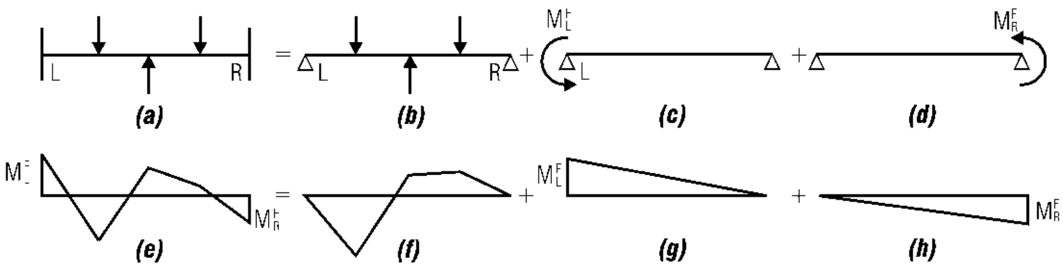


Fig. 6.69 Loads on fixed-end beam LR shown in (*a*) are resolved into component loads on a simple beam (*b*), (*c*), and (*d*). The corresponding moment diagrams are shown in (*e*) to (*h*).

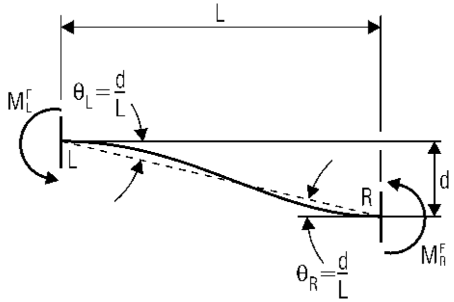


Fig. 6.70 End moments caused in a fixed-end beam by displacement *d* of one end.

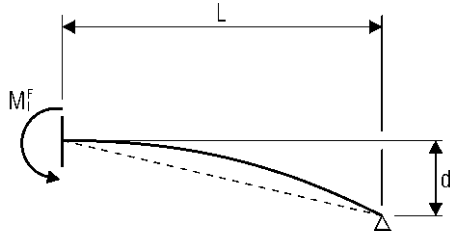


Fig. 6.71 End moment caused in a propped beam by displacement *d* of an end.

The fixed-end moments for the unloaded condition can be determined directly from Eqs. (6.110) and (6.111). Consider beam LR in Fig. 6.70, with span L and support R deflected a distance d vertically below its original position. If the beam were simply supported, the angle change caused by the displacement of R would be very nearly d/L. Hence, to obtain the fixed-end moments for the deflected condition, set $\theta_L = \theta_R = d/L$ and substitute these simple-beam end rotations in Eqs. (6.110) and (6.111):

$$M_L^F = -K_L(1 + C_R)\frac{d}{L} \qquad (6.116)$$

$$M_R^F = -K_R(1 + C_L)\frac{d}{L} \qquad (6.117)$$

If end L is displaced downward with respect to R, d/L would be negative and the fixed-end moments positive.

For beams with constant moment of inertia, the fixed-end moments are given by

$$M_L^F = M_R^F = -\frac{6EI}{L} \cdot \frac{d}{L} \qquad (6.118)$$

The fixed-end moments for a propped beam, such as beam LR shown in Fig. 6.71, can be obtained similarly from Eq. (6.114). For variable moment of inertia,

$$M^F = -\left(\frac{d}{L}\right)\left(\frac{1}{\alpha_L}\right) \qquad (6.119)$$

For a prismatic propped beam,

$$M^F = -\frac{3EI}{L} \cdot \frac{d}{L} \qquad (6.120)$$

Reverse signs for downward displacement of end L.

Computation Aids for Prismatic Beams ▪ Fixed-end moments for several common types of loading on beams of constant moment of inertia (prismatic beams) are given in Fig. 6.72. Also, the curves in Fig. 6.74 enable fixed-end moments to be computed easily for any type of loading on a prismatic beam. Before the curves can be entered, however, certain characteristics of the loading must be calculated. These include $\bar{x}L$, the location of the center of gravity of the loading with respect to one of the loads; $G^2 = \Sigma b_n{}^2 P_n/W$, where $b_n L$ is the distance from each load P_n to the center of gravity of the loading (taken positive to the right); and $S^3 = \Sigma b_n{}^3 P_n/W$ (see case 8, Fig. 6.73). These values are given in Fig. 6.73 for some common types of loading.

The curves in Fig. 6.74 are entered at the bottom with the location a of the center of gravity of the loading with respect to the left end of the span. At the intersection with the proper G curve, proceed horizontally to the left to the intersection with the proper S line, then vertically to the horizontal scale indicating the coefficient m by which to multiply WL to obtain the fixed-end moment. The curves solve the equations:

$$m_L = \frac{M_L^F}{WL} = G^2[1 - 3(1 - a)] + a(1 - a)^2 + S^3 \qquad (6.121)$$

$$m_R = \frac{M_R^F}{WL} = G^2(1 - 3a) + a^2(1 - a) - S^3 \qquad (6.122)$$

where M_L^F is the fixed-end moment at the left support and M_R^F at the right support.

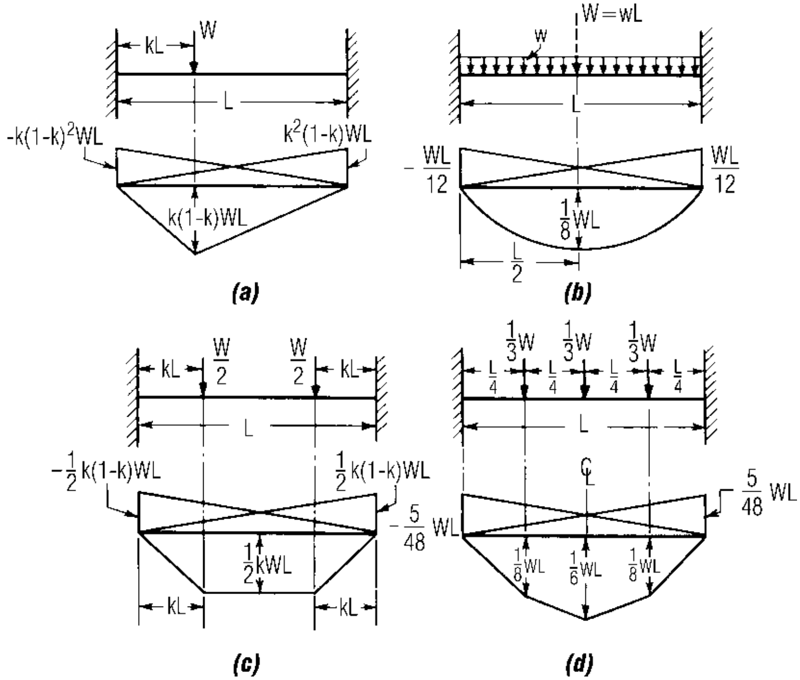


Fig. 6.72 Fixed-end moments for a prismatic beam: (*a*) for concentrated load; (*b*) for a uniform load; (*c*) for two equal concentrated loads; (*d*) for three equal concentrated loads.

As an example of the use of the curves, find the fixed-end moments in a prismatic beam of 20-ft span carrying a triangular loading of 100 kips, similar to the loading shown in case 4, Fig. 6.73, distributed over the entire span, with the maximum intensity at the right support.

Case 4 gives the characteristics of the loading: $y = 1$; the center of gravity is $L/3$ from the right support; so $a = 0.67$, $G^2 = 1/18 = 0.056$, and $S^3 = -1/135 = -0.007$. To find M_R^F, we enter Fig. 6.74 at the bottom with $a = 0.67$ on the upper scale and proceed vertically to the estimated location of the intersection of the coordinate with the $G^2 = 0.06$ curve. Then we move horizontally to the intersection with the line for $S^3 = -0.007$, as indicated by the dashed line in Fig. 6.74. Referring to the scale at the top of the diagram, we find the coefficient m_R to be 0.10. Similarly, with = 0.67 on the lowest scale, we find the coefficient m_L to be 0.07. Hence, the fixed-end moment at the right support is $0.10 \times 100 \times 20 = 200$ ft-kips, and at the left support $-0.07 \times 100 \times 20 = -140$ ft-kips.

6.59 Slope-Deflection Equations

In Arts. 6.56 and 6.58, moments and displacements in a member of a continuous beam or frame are obtained by addition of their simple-beam components. Similarly, moments and displacements can be determined by superposition of fixed-end-beam components. This method, for example, can be used to derive relationships between end moments and end rotations of a beam known as slope-deflection equations. These equations can be used to compute end moments in continuous beams.

Consider a member *LR* of a continuous beam or frame (Fig. 6.75). *LR* may have a moment of inertia that varies along its length. The support *R* is displaced vertically downward a distance *d* from its original position. Because of this and the loads on the member and adjacent members, *LR* is subjected to end moments M_L at *L* and M_R at *R*. The total end rotation at *L* is θ_L, and at *R*, θ_R. All displacements are so small that the member can be considered to

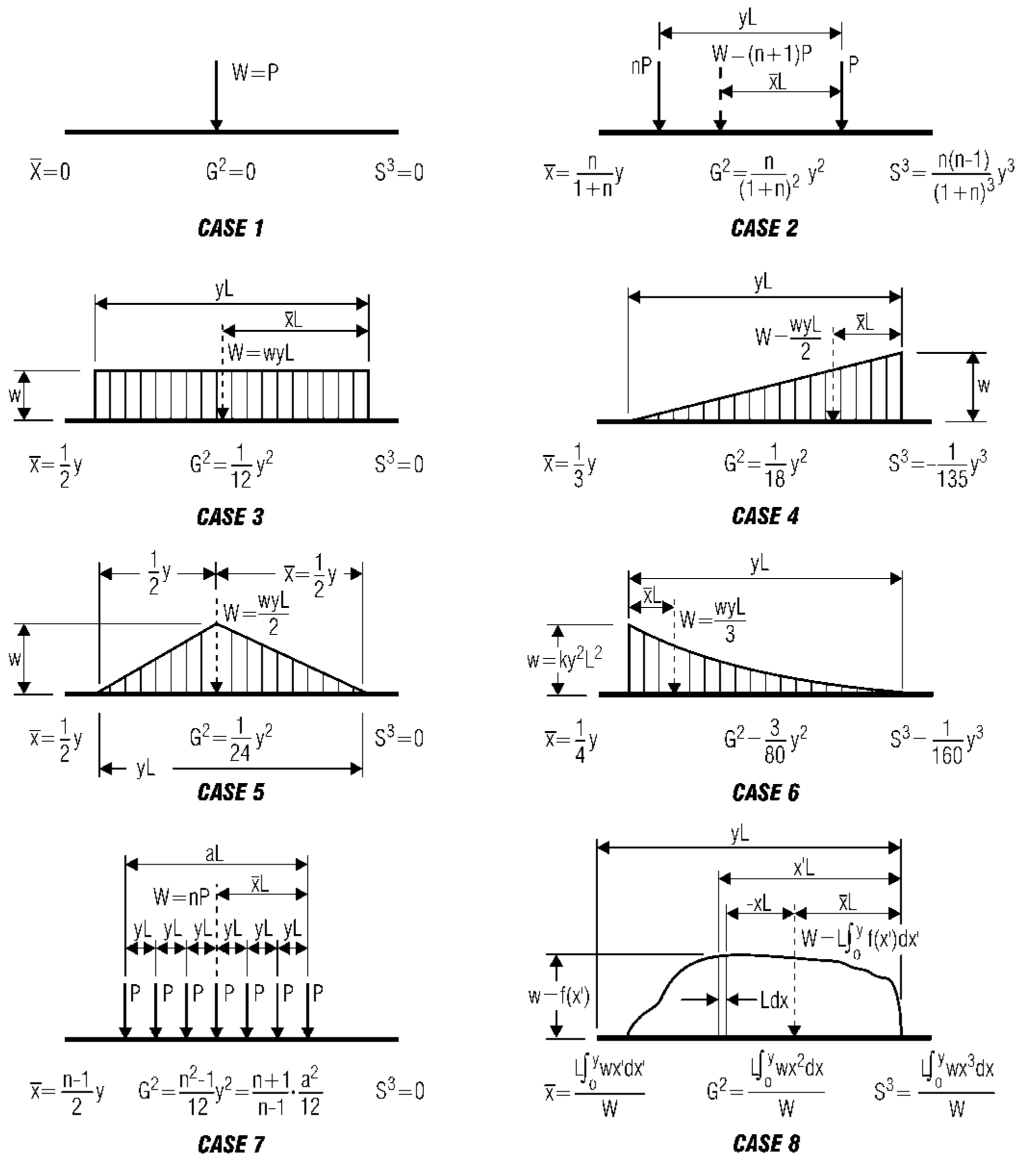


Fig. 6.73 Characteristics of loadings.

rotate clockwise through an angle nearly equal to d/L, where L is the span of the beam.

Assume that rotation is prevented at ends L and R by end moments m_L at L and m_R at R. Then, by application of the principle of superposition (Art. 6.56) and Eqs. (6.116) and (6.117),

$$m_L = M_L^F - K_L(1 + C_R)\frac{d}{L} \tag{6.123}$$

$$m_R = M_R^F - K_R(1 + C_L)\frac{d}{L} \tag{6.124}$$

where M_L^F = fixed-end moment at L due to the load on LR

M_R^F = fixed-end moment at R due to the load on LR

K_L = fixed-end stiffness at end L

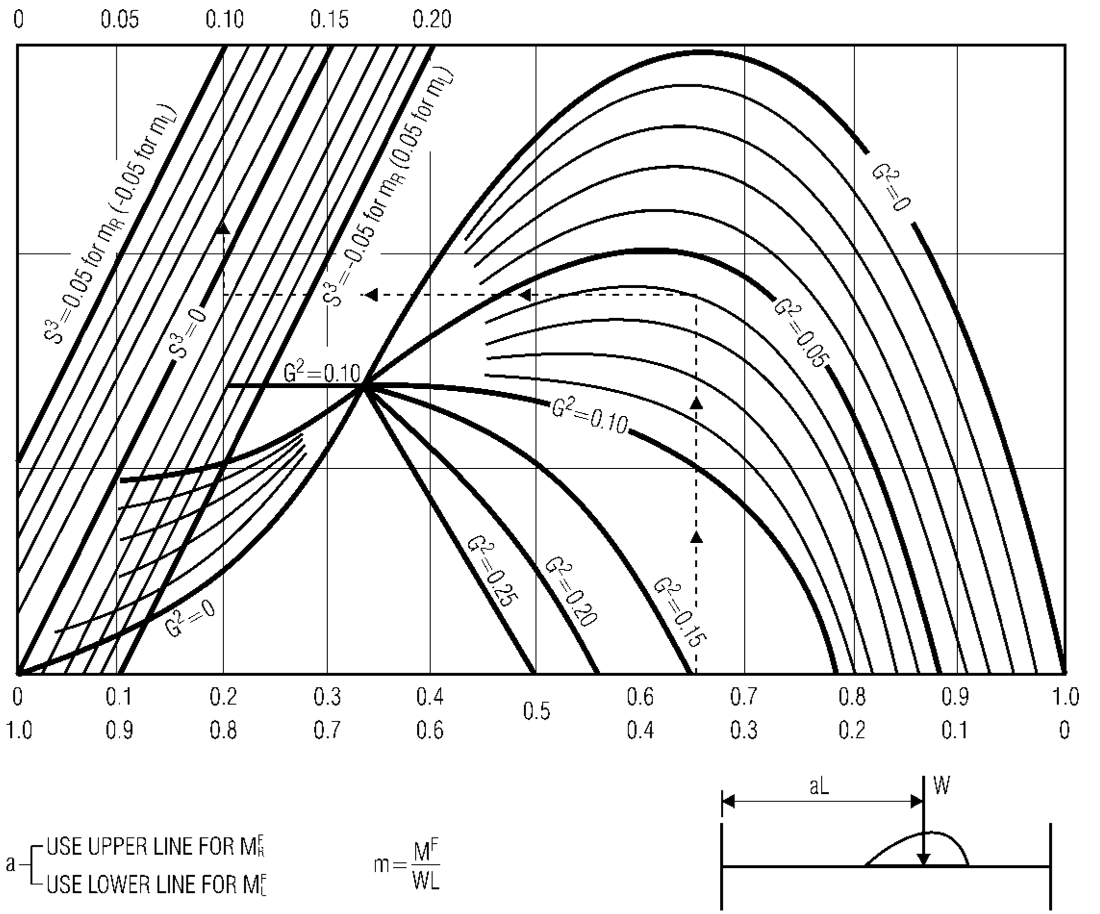


Fig. 6.74 Chart for fixed-end moments caused by any type of loading.

K_R = fixed-end stiffness at end R

C_L = carry-over factor toward end L

C_R = carry-over factor toward end R

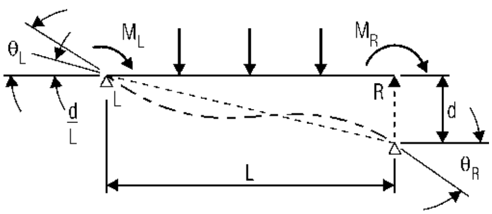


Fig. 6.75 End moments M_L and M_R restrain against rotation the ends of loaded span LR of a continuous beam when one end is displaced.

Since ends L and R are not fixed but actually undergo angle changes θ_L and θ_R at L and R, respectively, the joints must now be permitted to rotate while an end moment M'_L is applied at L and an end moment M'_R, at R to produce those angle changes (Fig. 6.76). With the use of the definitions of carry-over factor and fixed-end stiffness (Art. 6.58), these moments are found to be

$$M'_L = K_L(\theta_L + C_R\theta_R) \qquad (6.125)$$

$$M'_R = K_R(\theta_R + C_L\theta_L) \qquad (6.126)$$

The slope-deflection equations for LR then result from addition of M'_L to m_L, which yields M_L, and of M'_R to m_R, which yields M_R,

$$M_L = K_L(\theta_L + C_R\theta_R) + M^F_L - K_L(1 + C_R)\frac{d}{L} \qquad (6.127)$$

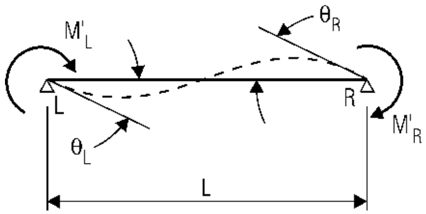


Fig. 6.76 Moments applied to the ends of a simple beam produce end rotations there.

$$M_R = K_R(\theta_R + C_L\theta_L) + M_R^F - K_R(1 + C_L)\frac{d}{L} \quad (6.128)$$

For beams with constant moment of inertia, the slope-deflection equations become

$$M_L = \frac{4EI}{L}\left(\theta_L + \frac{1}{2}\theta_R\right) + M_L^F - \frac{6EI}{L}\cdot\frac{d}{L} \quad (6.129)$$

$$M_R = \frac{4EI}{L}\left(\theta_R + \frac{1}{2}\theta_L\right) + M_R^F - \frac{6EI}{L}\frac{d}{L} \quad (6.130)$$

where E = modulus of elasticity

I = moment of inertia of the cross section

Note that if end L moves downward with respect to R, the sign for d in the preceding equations is changed.

If the end moments M_L and M_R are known and the end rotations are to be determined, Eqs. (6.125) to (6.128) can be solved for θ_L and θ_R or derived by superposition of simple-beam components, as is done in Art. 6.58. For beams with moment of inertia varying along the span:

$$\theta_L = (M_L - M_L^F)\alpha L - (M_R - M_R^F)\beta + \frac{d}{L} \quad (6.131)$$

$$\theta_R = (M_R - M_R^F)\alpha_R - (M_L - M_L^F)\beta + \frac{d}{L} \quad (6.132)$$

where α is given by Eq. (6.99) and β by Eq. (6.100). For beams with constant moment of inertia:

$$\theta_L = \frac{L}{3EI}(M_L - M_L^F) - \frac{L}{6EI}(M_R - M_R^F) + \frac{d}{L} \quad (6.133)$$

$$\theta_R = \frac{L}{3EI}(M_R - M_R^F) - \frac{L}{6EI}(M_L - M_L^F) + \frac{d}{L} \quad (6.134)$$

The slope-deflection equations can be used to determine end moments and rotations of the spans of continuous beams by writing compatibility and equilibrium equations for the conditions at each support. For example, the sum of the moments at each support must be zero. Also, because of continuity, the ends of all members at a support must rotate through the same angle. Hence, M_L for one span, given by Eq. (6.127) or (6.129), must be equal to $-M_R$ for the adjoining span, given by Eq. (6.128) or (6.130), and the end rotation θ at that support must be the same on both sides of the equation. One such equation, with the end rotations at the supports as the unknowns can be written for each support. With the end rotations determined by solution of the simultaneous equations, the end moments can be computed from the slope-deflection equations and the continuous beam can now be treated as statically determinate.

See also Arts. 6.60 and 6.66.

(C. H. Norris et al., "Elementary Structural Analysis,", McGraw-Hill Book Company, New York.)

6.60 Moment Distribution

The properties of fixed-end beams presented in Art. 6.58 enable the computation of end moments in continuous beams and frames by moment distribution, in which end moments induced by loads or displacements of joints are distributed to all the spans. The distribution is based on the assumption that translation is prevented at all joints and supports, rotation of the ends of all members of a joint is the same, and the sum of the end moments at every joint is zero.

The frame in Fig. 6.77 consists of four prismatic members rigidly connected together at O and fixed at ends A, B, C, and D. If an external moment U is applied at O, the sum of the end moments in each member at O must be equal to U. Furthermore, all members must rotate at O through the same angle θ since they are assumed to be rigidly connected there. Hence, by the definition of fixed-end stiffness (Art. 6.58), the proportion of U induced in or "distributed" to the end of each member at O equals the ratio of the stiffness of that member to the sum of the stiffnesses of all the members at O. This ratio is called the **distribution factor** at O for the member.

Suppose a moment of 100 ft-kips is applied at O, as indicated in Fig. 6.77*b*. The relative stiffness (or I/L) is assumed as shown in the circle on each

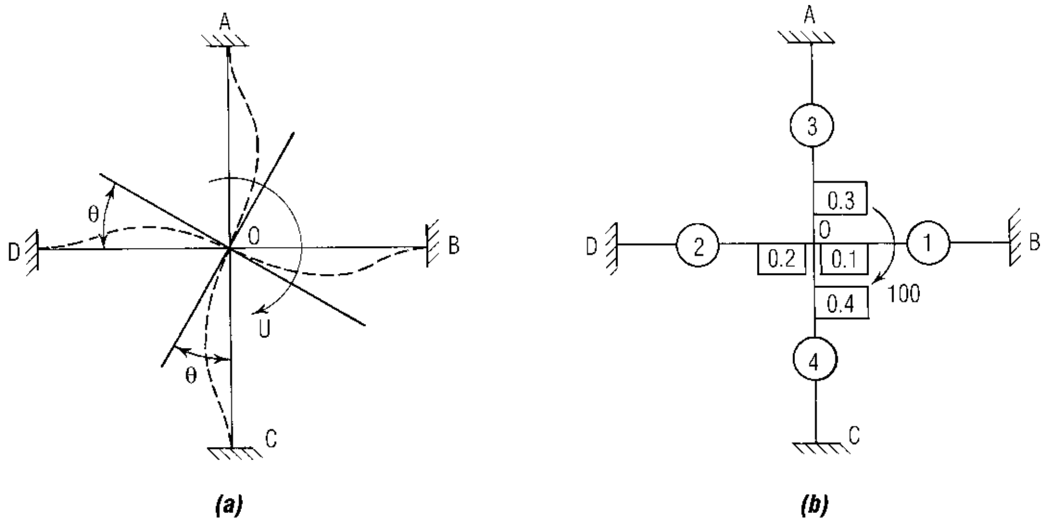


Fig. 6.77 Joint between four members of a simple frame is rotated by an applied moment. (*a*) Elastic curve; (*b*) stiffness and moment distribution factors.

member. The distribution factors for the moment at *O* are computed from the stiffnesses and shown in the boxes. For example, the distribution factor for *OA* equals its stiffness divided by the sum of the stiffnesses of all the members at the joint: $3/(3+1+4+2) = 0.3$. Hence, the moment induced in *OA* at *O* is $0.3 \times 100 = 30$ ft-kips. Similarly, *OB* gets 10 ft-kips, *OC* 40 ft-kips, and *OD* 20 ft-kips.

Because the far ends of these members are fixed, one-half of these moments are carried over to them (Art. 6.58). Thus $M_{AO} = 0.5 \times 30 = 15$; $M_{BO} = 0.5 \times 10 = 5$; $M_{CO} = 0.5 \times 40 = 20$; and $M_{DO} = 0.5 \times 20 = 10$.

Most structures consist of frames similar to the one in Fig. 6.77, or even simpler, joined together. Although the ends of the members may not be fixed, the technique employed for the frame in Fig. 6.77 can be applied to find end moments in such continuous structures.

Span with Simple Support ▪ Before the general method is presented, one shortcut is worth noting. Advantage can be taken when a member has a hinged end to reduce the work in distributing moments. This is done by using the true stiffness of the member instead of the fixed-end stiffness. (For a prismatic beam, the stiffness of a member with a hinged end is three-fourths the fixed-end stiffness; for a beam with variable moment of inertia, it is equal to the fixed-end stiffness times $1 - C_L C_R$, where C_L and C_R are the fixed-end carry-over factors to each end of the beam.) Naturally, the carry-over factor toward the hinge is zero.

Moment Release and Distribution ▪ When beam ends are neither fixed nor pinned but restrained by elastic members, moments can be distributed by a series of converging approximations. At first, all joints are locked against rotation. As a result, the loads will create fixed-end moments at the ends of every loaded member (Art. 6.58). At each joint, the unbalanced moment, a moment equal to the algebraic sum of the fixed-end moments at the joint, is required to hold it fixed. But if the joint actually is not fixed, the unbalanced moment does not exist. It must be removed by applying an equal but opposite moment. One joint at a time is unlocked by applying a moment equal but opposite in sign to the unbalanced moment. The unlocking moment must be distributed to the members at the joint in proportion to their fixed-end stiffnesses. As a result, the far end of each member should receive a "carry-over" moment equal to the distributed moment times a carry-over factor (Art. 6.58).

After all joints have been released at least once, it generally will be necessary to repeat the process—sometimes several times—before the corrections to the fixed-end moments become negligible. To reduce the number of cycles, start the unlocking of joints with those having the greatest unbalanced moments. Also, include carry-over moments with fixed-end moments in computing unbalanced moments.

Example ▪ Suppose the end moments are to be found for the continuous beam *ABCD* in Fig. 6.78, given the fixed-end moments on the first line of the figure. The I/L values for all spans are given equal; therefore, the relative fixed-end stiffness for all members is unity. But since *A* is a hinged end, the computation can be shortened by using the actual relative stiffness, which is ¾. Relative stiffnesses for all members are shown in the circle on each member. The distribution factors are shown in the boxes at each joint.

Begin the computation with removal of the unbalance in fixed-end moments (first line in Fig. 6.78). The greatest unbalanced moment, by inspection, occurs at hinged end *A* and is −400, so unlock this joint first. Since there are no other members at the joint, distribute the full unlocking moment of +400 to *AB* at *A* and carry over one-half to *B*. The unbalance at *B* now is +400−480 plus the carry-over of +200 from *A*, or a total of +120. Hence, a moment of −120 must be applied and distributed to the members at *B* by multiplying by the distribution factors in the corresponding boxes.

The net moment at *B* could be found now by adding the fixed-end and distributed moments at the joint. But it generally is more convenient to delay the summation until the last cycle of distribution has been completed.

After *B* is unlocked, the moment distributed to *BA* need not be carried over to *A* because the carry-over factor toward the hinged end is zero. But half the moment distributed to *BC* is carried over to *C*. Similarly, unlock joint *C* and carry over half the distributed moments to *B* and *D*, respectively. Joint *D* should not be unlocked since it actually is a fixed end. Thus, the first cycle of moment distribution has been completed.

Carry out the second cycle in the same manner. Release joint *B*, and carry over to *C* half the distributed moment in *BC*. Finally, unlock *C* to complete the cycle. Add the fixed-end and distributed moments to obtain the final moments.

6.61 Maximum Moments in Continuous Frames

In continuous frames, maximum end moments and maximum interior moments are produced by

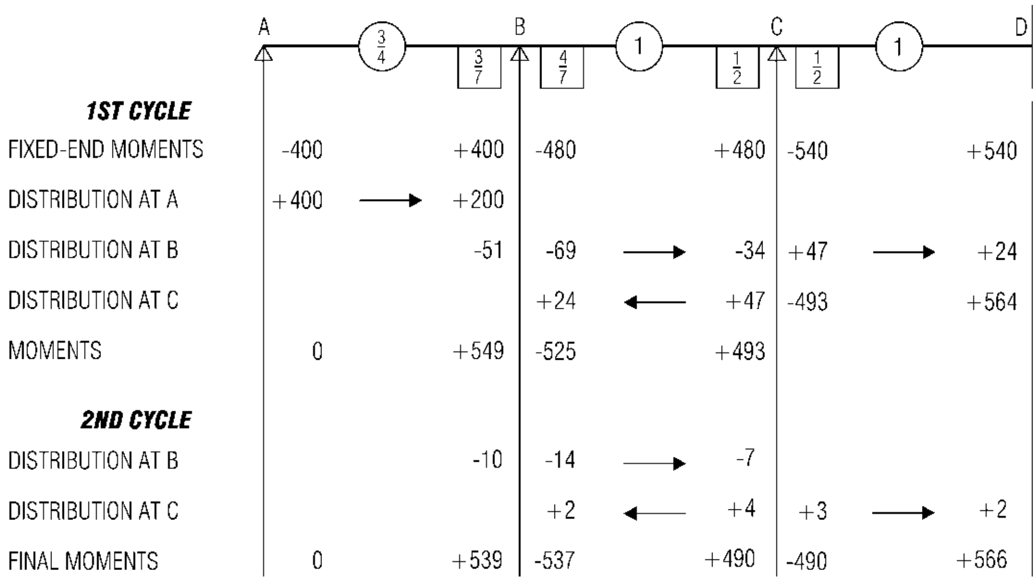


Fig. 6.78 Moment distribution in a beam by converging approximations.

different combinations of loadings. For maximum end moment in a beam, live load should be placed on that beam and on the beam adjoining the end for which the moment is to be computed. Spans adjoining these two should be assumed to be carrying only dead load.

For maximum midspan moments, the beam under consideration should be fully loaded, but adjoining spans should be assumed to be carrying only dead load.

The work involved in distributing moments due to dead and live loads in continuous frames in buildings can be greatly simplified by isolating each floor. The tops of the upper columns and the bottoms of the lower columns can be assumed fixed. Furthermore, the computations can be condensed considerably by following the procedure recommended in "Continuity in Concrete Building Frames," EB033D Portland Cement Association, Skokie, Ill. 60077 (www.portcement.org), and illustrated in Fig. 6.79.

Figure 6.79 presents the complete calculation for maximum end and midspan moments in four floor beams *AB*, *BC*, *CD*, and *DE*. Columns are assumed to be fixed at the story above and below. None of the beam or column sections is known to begin with; so as a start, all members will be assumed to have a fixed-end stiffness of unity, as indicated on the first line of the calculation.

Column Moments ▪ The second line gives the distribution factors (Art. 6.60) for each end of the beams; column moments will not be computed until moment distribution to the beams has been completed. Then, the sum of the column moments at each joint may be easily computed since they are the moments needed to make the sum of the end moments at the joint equal to zero. The sum of the column moments at each joint can then be distributed to each column there in proportion to its stiffness. In this example, each column will get one-half the sum of the column moments.

Fixed-end moments at each beam end for dead load are shown on the third line, just above the horizontal line, and fixed-end moments for live plus dead loads on the fourth line. Corresponding midspan moments for the fixed-end condition also are shown on the fourth line, and like the end moments will be corrected to yield actual midspan moments.

Maximum End Moments ▪ For maximum end moment at *A*, beam *AB* must be fully loaded, but *BC* should carry dead load only. Holding *A* fixed, we first unlock joint *B*, which has a total-load fixed-end moment of +172 in *BA* and a dead-load fixed-end moment of −37 in *BC*. The releasing

	A			B			C			D		E
1. STIFFNESS		1			1			1			1	
2. DISTRIBUTION FACTOR	0.33		0.25	0.25		0.25	0.25		0.25	0.25		0.33
3. F.E.M. DEAD LOAD	—		+91	-37		+37	-70		+70	-59		—
4. F.E.M. TOTAL LOAD	-172	+99	+172	-78	+73	+78	-147	+85	+147	-126	+63	+126
5. CARRY-OVER	-17	+11	+29	-1	+1	-2	-11	+7	+14	-21	+13	+7
6. ADDITION	-189	+18	+201	-79	-1	+76	-158	+9	+161	-147	+5	+133
7. DISTRIBUTION	+63		-30	-30		+21	+21		-4	-4		-44
8. MAX. MOMENTS	-126	+128	+171	-109	+73	+97	-137	+101	+157	-151	+81	+89

Fig. 6.79 Moment distribution in a continuous frame by converging approximations.

moment required, therefore, is $-(172-37)$, or -135. When B is released, a moment of -135×0.25 is distributed to BA. One-half of this is carried over to A, or $-135 \times 0.25 \times 0.5 = -17$. This value is entered as the carry-over at A on the fifth line in Fig. 6.79. Joint B then is relocked.

At A, for which we are computing the maximum moment, we have a total-load fixed-end moment of -172 and a carry-over of -17, making the total -189, shown on the sixth line. To release A, a moment of $+189$ must be applied to the joint. Of this, 189×0.33, or 63, is distributed to AB, as indicated on the seventh line. Finally, the maximum moment at A is found by adding lines 6 and 7: $-189 + 63 = -126$.

For maximum moment at B, both AB and BC must be fully loaded, but CD should carry only dead load. We begin the determination of the maximum moment at B by first releasing joints A and C, for which the corresponding carry-over moments at BA and BC are $+29$ and $-(+78-70) \times 0.25 \times 0.5 = -1$, shown on the fifth line in Fig. 6.79. These bring the total fixed-end moments in BA and BC to $+201$ and -79, respectively. The releasing moment required is $-(201-79) = -122$. Multiplying this by the distribution factors for BA and BC when joint B is released, we find the distributed moments, -30, entered on line 7. The maximum end moments finally are obtained by adding lines 6 and 7: $+171$ at BA and -109 at BC. Maximum moments at C, D, and E are computed and entered in Fig. 6.79 in a similar manner. This procedure is equivalent to two cycles of moment distribution.

Maximum Midspan Moments ▪ The computation of maximum midspan moments in Fig. 6.79 is based on the assumption that in each beam the midspan moment is the sum of the simple-beam midspan moment and one-half the algebraic difference of the final end moments (the span carries full load but adjacent spans only dead load). Instead of starting with the simple-beam moment, however, we begin, for convenience, with the midspan moment for the fixed-end condition and then apply two corrections. In each span, these corrections equal the carry-over moments entered on line 5 for the two ends of the beam multiplied by a factor.

For beams with variable moment of inertia, the factor is $\pm \frac{1}{2}(1/C + D - 1)$, where C is the fixed-end carry-over factor toward the end for which the correction factor is being computed and D the distribution factor for that end. The plus sign is used for correcting the carry-over at the right end of a beam and the minus sign for the carry-over at the left end. For prismatic beams, the correction factor becomes $\pm \frac{1}{2}(1 + D)$.

For example, to find the corrections to the midspan moment in AB, we first multiply the carry-over at A on line 5, -17, by $-\frac{1}{2}(1 + 0.33)$. The correction, $+11$, also is entered on the fifth line. Then we multiply the carry-over at B, $+29$, by $+\frac{1}{2}(1 + 0.25)$ and enter the correction, $+18$, on line 6. The final midspan moment is the sum of lines 4, 5, and 6: $+99 + 11 + 18 = +128$. Other midspan moments in Fig. 6.79 are obtained in a similar manner.

Approximate methods for determining wind and seismic stresses in tall buildings are given in Arts. 15.4, 15.9, and 15.10.

6.62 Moment-Influence Factors

For certain types of structures, particularly those for which different types of loading conditions must be investigated, it may be more convenient to find maximum end moments from a table of moment-influence factors. This table is made up by listing for the end of each member in a structure the moment induced in that end when a moment (for convenience, $+1000$) is applied to each joint successively. Once this table has been prepared, no additional moment distribution is necessary for computing the end moments due to any loading condition.

For a specific loading pattern, the moment at any beam end M_{AB} may be obtained from the moment-influence table by multiplying the entries under AB for the various joints by the actual

Table 6.3 Moment-Influence Factors for Fig. 6.80

Member	+1,000 at *B*	+1,000 at *C*
AB	351	−105
BA	702	−210
BC	298	210
CB	70	579
CD	−70	421
DC	−35	210

Table 6.4 Moment Collection Table for Fig. 6.80

Remarks	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
1. Sidesway *FEM*	−3,000*M*	−3,000*M*			−1,000*M*	−1,000*M*
2. Distribution for *B*	+1,053*M*	−2,106*M*	+894*M*	+210*M*	−210*M*	−105*M*
3. Distribution for *C*	−105*M*	−210*M*	+210*M*	+579*M*	+421*M*	+210*M*
4. Final sidesway *M*	−2,052*M*	−1,104*M*	+1,104*M*	+789*M*	−789*M*	−895*M*
5. For 2,000-lb horizontal	−17,000	−9,100	+9,100	+6,500	−6,500	−7,400
6. 4,000-lb vertical FEM			−12,800	+3,200		
7. Distribution for *B*	+4,490	+8,980	+3,820	+897	−897	−448
8. Distribution for *C*	+336	+672	−672	−1,853	−1,347	−673
9. Moments with no sidesway	+4,826	+9,652	−9,652	+2,244	−2,244	−1,121
10. Sidesway *M*	−4,710	−2,540	+2,540	+1,810	−1,810	−2,060
11. For 4,000-lb vertical	+116	+7,112	−7,112	+4,054	−4,054	−3,181

unbalanced moments at those joints divided by 1000 and summing. (See also Art. 6.64 and Tables 6.3 and 6.4.)

6.63 Procedure for Sidesway

For some structures, it is convenient to know the effect of a movement of a support normal to the original position. But the moment-distribution method is based on the assumption that such movement of a support does not occur. The method, however, can be modified to evaluate end moments resulting from a support movement.

The procedure is to distribute moments as usual, assuming no deflection at the supports. This implies that additional external forces are exerted at the supports to prevent movement. These forces can be computed. Then, equal and opposite forces are applied to the structure to produce the final configuration, and the moments that they induce are distributed as usual. These moments added to those obtained with undeflected supports yield the final moments.

Example—Horizontal Axial Load ▪ Suppose the rigid frame in Fig. 6.80 is subjected to a 2000-lb horizontal load acting to the right at the level of beam *BC*. The first step is to compute the moment-influence factors by applying moments of +1000 at joints *B* and *C* (Art. 6.62), assuming sidesway is prevented, and enter the distributed moments in Table 6.3.

Since there are no intermediate loads on the beam and columns, the only fixed-end moments that need be considered are those in the columns due to lateral deflection of the frame.

This deflection, however, is not known initially. So we assume an arbitrary deflection, which produces a fixed-end moment of −1000*M* at the

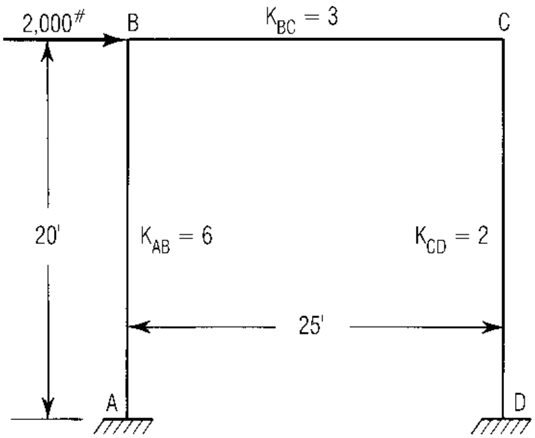


Fig. 6.80 Laterally loaded rigid frame.

top of column *CD*. *M* is an unknown constant to be determined from the fact that the sum of the shears in the deflected columns must equal the 2000-lb load. The deflection also produces a moment of $-1000M$ at the bottom of *CD* [see Eq. (6.118)].

From the geometry of the structure, we furthermore note that the deflection of *B* relative to *A* equals the deflection of *C* relative to *D*. Then, according to Eq. 6.118, the fixed-end moments of the columns of this frame are proportional to the stiffnesses of the columns and hence are equal in *AB* to $-1000M \times \frac{6}{2} = -3000M$. The column fixed-end moments are entered in the first line of Table 6.4, the moment-collection table for Fig. 6.80.

In the deflected position of the frame, joints *B* and *C* are unlocked in succession. First, we apply a releasing moment of $+3000M$ at *B*. We distribute it by multiplying by 3 the entries in the columns marked "+1000 at *B*" in Table 6.3. Similarly, a releasing moment of $+1000M$ is applied at *C* and distributed with the aid of the moment-influence factors. The distributed moments are entered in the second and third lines of the moment-collection table. The final moments are the sum of the fixed-end moments and the distributed moments and are given in the fourth line of Table 6.4, in terms of *M*.

Isolating each column and taking moments about one end, we find that the overturning moment due to the shear equals the sum of the end moments. We have one such equation for each column. Adding these equations, noting that the sum of the shears equals 2000 lb, we obtain

$$-M(2052 + 1104 + 789 + 895) = -2000 \times 20$$

from which we find $M = 8.26$. This value is substituted in the sidesway totals (line 4) in the moment-collection table to yield the end moments for the 2000-lb horizontal load (line 5).

Example—Vertical Load on Beam ▪ Suppose a vertical load of 4000 lb is applied to *BC* of the rigid frame in Fig. 6.80, 5 ft from *B*. The same moment-influence factors and moment-collection table can again be used to determine the end moments with a minimum of labor.

The fixed-end moment at *B*, with sidesway prevented, is $-12{,}800$, and at *C* $+3200$ (Fig. 6.72*a*). With the joints still locked, the frame is permitted to move laterally an arbitrary amount, so that in addition to the fixed-end moments due to the 4000-lb load, column fixed-end moments of $-3000M$ at *A* and *B* and $-1000M$ at *C* and *D* are induced. The moment-collection table already indicates in line 4 the effect of relieving these column moments by unlocking joints *B* and *C*. We now have to superimpose the effect of releasing joints *B* and *C* to relieve the fixed-end moments for the vertical load. This we can do with the aid of the moment-influence factors. The distribution is shown in lines 7 and 8 of Table 6.4, the moment-collection table. The sums of the fixed-end moments and distributed moments for the 4000-lb load are shown in line 9.

The unknown *M* can be evaluated from the fact that the sum of the horizontal forces acting on the columns must be zero. This is equivalent to requiring that the sum of the column end moments equal zero:

$$-M(2052 + 1104 + 789 + 895) + 4826 + 9652 - 2244 - 1121 = 0$$

from which $M = 2.30$. This value is substituted in line 4 of Table 6.4 to yield the sidesway moments for the 4000-lb load (line 10). Addition of these moments to the totals for no sidesway (line 9) gives the final moments (line 11).

Multistory Frames ▪ This procedure permits analysis of one-story bents with straight beams by solution of one equation with one unknown, regardless of the number of bays. If the frame is multistory, the procedure can be applied to each story. Since an arbitrary horizontal deflection is introduced at each floor or roof level, there are as many unknowns and equations as there are stories. (For approximate methods for determining wind and seismic stresses in tall buildings, see Arts. 15.9 and 15.10.)

Arched Bents ▪ The procedure is more difficult to apply to bents with curved or polygonal members between the columns. The effect of the change in the horizontal projection of the curved or polygonal portion of the bent must be included in the calculations. In many cases, it may be easier to analyze the bent as a curved beam (arch).

6.64 Load Distribution to Bents and Shear Walls

Provision should be made for all structures to transmit lateral loads, such as those from wind, earthquakes, and traction and braking of vehicles, to foundations and their supports that have high resistance to displacement. For the purpose, various types of bracing may be used, including struts, tension ties, diaphragms, trusses, and shear walls.

The various bracing members are usually designed to interact as a system. Structural analysis then is necessary to determine the distribution of the lateral loads on the system to the bracing members. The analysis may be based on the principles presented in the preceding articles but it requires a knowledge or assumption of the structural characteristics of the system components. For example, suppose a horizontal diaphragm, such as a concrete floor, is to be used to distribute horizontal forces to several parallel vertical trusses. In this case, the distribution would depend not only on the relative resistance of the trusses to the horizontal forces but also on the rigidity (or flexibility) of the diaphragm.

In tall buildings, bents or shear walls, which act as vertical cantilevers and generally are often also used to support some of the gravity loads, usually are spaced at appropriate intervals to transmit lateral loads to the foundations. A **bent** consists of vertical trusses or continuous rigid frames located in a plane (Fig. 6.81*a*). The trusses usually are an assemblage of columns, horizontal girders, and diagonal bracing (Fig. 6.81*b* to *e*). The rigid frames are composed of girders and columns, with so-called wind connections between them to establish continuity (Fig. 6.81*f*). **Shear walls** are thin cantilevers, usually constructed of concrete but sometimes of masonry or steel plates (Fig. 6.81*g*). They require bracing normal to their plane.

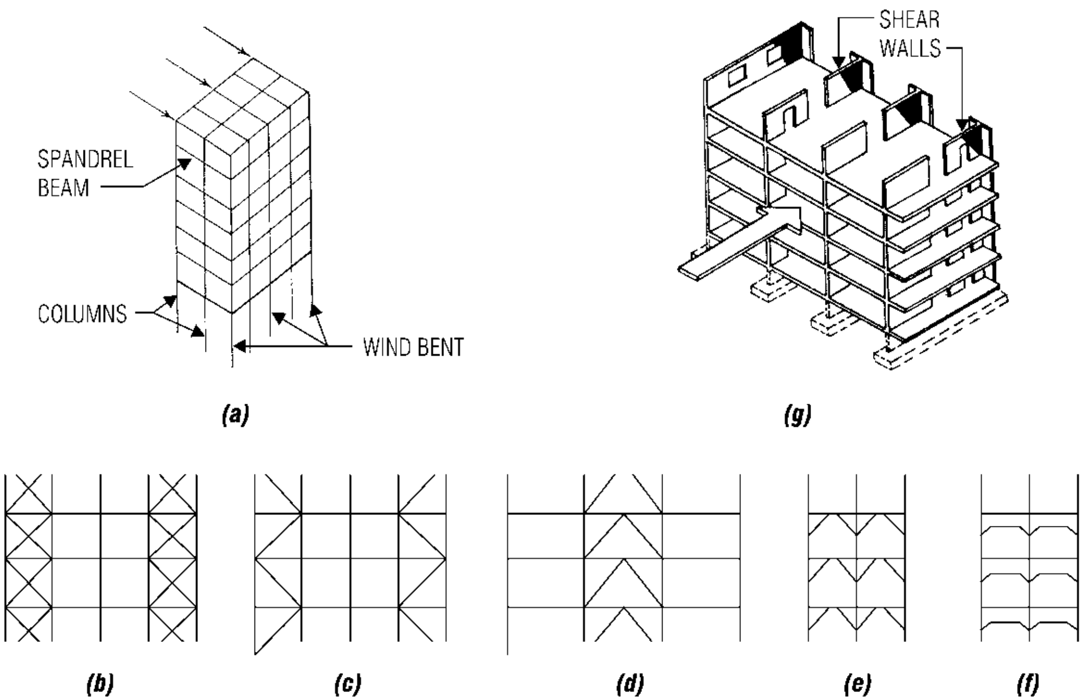


Fig. 6.81 Building frame resists lateral forces with (*a*) wind bents or (*g*) shear walls or a combination of the two. Bents may be braced in any of several ways, including (*b*) X bracing, (*c*) K bracing, (*d*) inverted V bracing, (*e*) knee bracing, and (*f*) rigid connections.

Where bents or shear walls are connected by rigid diaphragms so that they must deflect equally under horizontal loads, the proportion of the total horizontal load at any level carried by a bent or shear wall that is parallel to the load depends on the relative rigidity, or stiffness, of the bent or wall. Rigidity of this bracing is inversely proportional to its deflection under a unit horizontal load.

When the line of action of the resultant of the lateral forces does not pass through the center of rigidity of the vertical, lateral-force-resisting system, distribution of rotational forces must be considered as well as distribution of the translational forces. If relatively rigid diaphragms are used, the torsional forces may be distributed to the bents or shear walls in proportion to their relative rigidities and their distance from the center of rigidity. A flexible diaphragm should not be considered capable of distributing torsional forces.

Deflections of Bents and Shear Walls ▪ Horizontal deflections in the planes of bents and shear walls can be computed on the assumption that they act as cantilevers. Deflections of braced bents can be calculated by the dummy-unit-load method (Art. 6.54) or a matrix method. Deflections of rigid frames can be computed by adding the drifts of the stories, as determined by moment distribution (Art. 6.60) or a matrix method. And deflections of shear walls can be calculated from formulas given in Art. 6.32, the dummy-unit-load method, or a matrix method.

For a shear wall, the deflection in its plane induced by a load in its plane is the sum of the flexural deflection as a cantilever and the deflection due to shear. Thus, for a wall with solid rectangular cross section, the deflection at the top due to uniform load is

$$\delta = \frac{1.5wH}{Et}\left[\left(\frac{H}{L}\right)^3 + \frac{H}{L}\right] \tag{6.135}$$

where w = uniform lateral load

H = height of the wall

E = modulus of elasticity of the wall material

t = wall thickness

L = length of wall

For a shear wall with a concentrated load P at the top, the deflection at the top is

$$\delta_c = \frac{4P}{Et}\left[\left(\frac{H}{L}\right)^3 + 0.75\frac{H}{L}\right] \tag{6.136}$$

If the wall is fixed against rotation at the top, however, the deflection is

$$\delta_f = \frac{P}{Et}\left[\left(\frac{H}{L}\right)^3 + 3\frac{H}{L}\right] \tag{6.137}$$

Where shear walls contain openings, such as those for doors, corridors, or windows, computations for deflection and rigidity are more complicated. Approximate methods, however, may be used.

(F. S. Merritt and Jonathan T. Ricketts, "Building Design and Construction Handbook," 5th ed., McGraw-Hill Publishing Co., New York, books.mcgraw-hill.com.)

6.65 Beams Stressed into the Plastic Range

When an elastic material, such as structural steel, is loaded with a gradually increasing load, stresses are proportional to strains nearly to the yield point. If the material, like steel, also is ductile, then it continues to carry load beyond the yield point, although strains increase rapidly with little increase in load (Fig. 6.82*a*).

Similarly, a beam made of a ductile material continues to carry more load after the stresses in the outer surfaces reach the yield point. The stresses, however, will no longer vary with distance from the neutral axis; so the flexure formula [Eq. (6.44)] no longer holds. But if simplifying assumptions are made, approximating the stress-strain relationship beyond the elastic limit, the load-carrying capacity of the beam can be computed with satisfactory accuracy.

Modulus of rupture is defined as the stress computed from the flexure formula for the maximum bending moment a beam sustains at failure. This is not a true stress but it is sometimes used to compare the strength of beams.

For a ductile material, the idealized stress-strain relationship in Fig. 6.82*b* may be assumed. Stress is proportional to strain until the yield-point stress

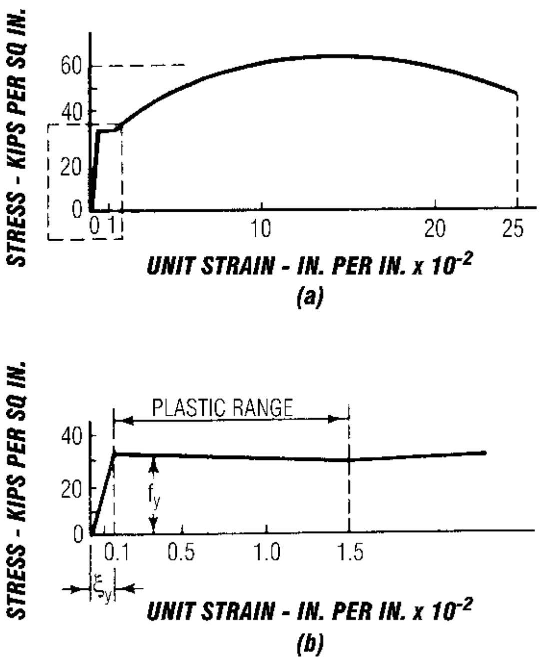


Fig. 6.82 Stress-strain relationship for a ductile material generally is similar to the curve in (*a*). To simplify plastic analysis, the portion of (*a*) enclosed by the dashed lines is approximated by the curve in (*b*), which extends to the range where strain hardening begins.

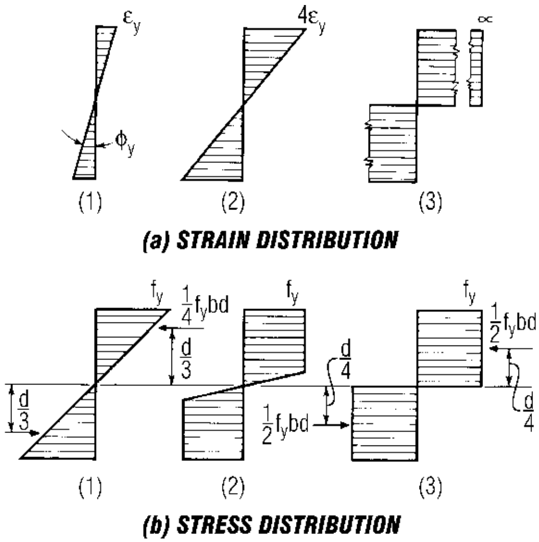


Fig. 6.83 Strain distribution is shown in (*a*) and stress distribution in (*b*) for a cross section of a rectangular beam as it is loaded beyond the yield point, assuming the idealized stress-strain relationship in Fig. 6.82*b*. Stage (1) shows the conditions at the yield point of the outer surfaces; (2) after yielding starts; and (3) at ultimate load.

f_y is reached, after which strain increases at a constant stress.

For a beam of this material, it is also assumed that:

1. Plane sections remain plane, strains thus being proportional to distance from the neutral axis.
2. Properties of this material in tension are the same as those in compression.
3. Its fibers behave the same in flexure as in tension.
4. Deformations remain small.

Strain distribution across the cross section of a rectangular beam, based on these assumptions, is shown in Fig. 6.83*a*. At the yield point, the unit strain is ε_y and the curvature ϕ_y, as indicated in (1). In (2), the strain has increased several times, but the section still remains plane. Finally, at failure, (3), the strains are very large and nearly constant across the lower and upper halves of the section.

Corresponding stress distributions are shown in Fig. 6.83*b*. At the yield point (1), stresses vary linearly and the maximum is f_y. With increase in load, more and more fibers reach the yield point, and the stress distribution becomes nearly constant, as indicated in (2). Finally, at failure (3), the stresses are constant across the top and bottom parts of the section and equal to the yield-point stress.

The resisting moment at failure for a rectangular beam can be computed from the stress diagram for stage 3. If b is the width of the member and d its depth, then the ultimate moment for a rectangular beam is

$$M_p = \frac{bd^2}{4} f_y \qquad (6.138)$$

Since the resisting moment at stage 1 is $M_y = f_y bd^2/6$, the beam carries 50% more moment before failure than when the yield-point stress is first reached in the outer fibers ($M_P/M_y = 1.5$).

A circular section has an M_P/M_y ratio of about 1.7, while a diamond section has a ratio of 2. The average wide-flange rolled-steel beam has a ratio of about 1.14.

The relationship between moment and curvature in a beam can be assumed to be similar to the stress-strain relationship in Fig. 6.82*b*. Curvature ϕ varies linearly with moment until $M_y = M_P$ is reached, after which ϕ increases indefinitely at constant moment. That is, a plastic hinge forms.

Moment Redistribution ▪ This ability of a ductile beam to form plastic hinges enables a fixed-end or continuous beam to carry more load after M_P occurs at a section because a redistribution of moments takes place. Consider, for example, a uniformly loaded fixed-end beam. In the elastic range, the end moments are $M_L = M_R = WL/12$, while the midspan moment M_C is $WL/24$. The load when the yield point is reached in the outer fibers is $W_y = 12M_y/L$. Under this load, the moment capacity of the ends of the beam is nearly exhausted; plastic hinges form there when the moment equals M_P. As load is increased, the ends then rotate under constant moment and the beam deflects as a simply supported beam. The moment at midspan increases until the moment capacity at that section is exhausted and a plastic hinge forms. The load causing that condition is the ultimate load W_u since, with three hinges in the span, a link mechanism is formed and the member continues to deform at constant load. At the time the third hinge is formed, the moments at ends and center are all equal to M_P. Therefore, for equilibrium, $2M_P = W_uL/8$, from which $W_u = 16M_P/L$. Since, for the idealized moment-curvature relationship, M_P was assumed equal to M_y, the carrying capacity due to redistribution of moments is 33% greater.

Finite-Element Methods

From the basic principles given in preceding articles, systematic procedures have been developed for determining the behavior of a structure from a knowledge of the behavior under load of its components. In these methods, called finite-element methods, a structural system is considered an assembly of a finite number of finite-size components, or elements. These are assumed to be connected to each other only at discrete points, called nodes. From the characteristics of the elements, such as their stiffness or flexibility, the characteristics of the whole system can be derived. With these known, internal stresses and strains throughout can be computed.

Choice of elements to be used depends on the type of structure. For example, for a truss with joints considered hinged, a natural choice of element would be a bar, subjected only to axial forces. For a rigid frame, the elements might be beams subjected to bending and axial forces, or to bending, axial forces, and torsion. For a thin plate or shell, elements might be triangles or rectangles, connected at vertices. For three-dimensional structures, elements might be beams, bars, tetrahedrons, cubes, or rings.

For many structures, because of the number of finite elements and nodes, analysis by a finite-element method requires mathematical treatment of large amounts of data and solution of numerous simultaneous equations. For this purpose, use of computers is advisable. The mathematics of such analyses is usually simpler and more compact when the data are handled in matrix form. (See also Art. 6.57.)

6.66 Force and Displacement Methods

The methods used for analyzing structures generally may be classified as force (flexibility) or displacement (stiffness) methods.

In analysis of statically indeterminate structures by force methods, forces are chosen as redundants, or unknowns. The choice is made in such a way that equilibrium is satisfied. These forces are then determined from the solution of equations that insure compatibility of all displacements of elements at each node. After the redundants have been computed, stresses and strains throughout the structure can be found from equilibrium equations and stress-strain relations.

In displacement methods, displacements are chosen as unknowns. The choice is made in such a way that geometric compatibility is satisfied. These displacements are then determined from the solution of equations that insure that forces acting at each node are in equilibrium. After the unknowns have been computed, stresses and strains throughout the structure can be found from equilibrium equations and stress-strain relations.

When choosing a method, the following should be kept in mind: In force methods, the number of unknowns equals the degree of indeterminancy. In displacement methods, the number of unknowns equals the degrees of freedom of displacement at nodes. The fewer the unknowns, the fewer the calculations required.

Both methods are based on the force-displacement relations and utilize the stiffness and flexibility matrices described in Art. 6.57. In these methods, displacements and external forces are resolved into components—usually horizontal, vertical, and rotational—at nodes, or points of connection of finite elements. In accordance with Eq. (6.97*a*), the stiffness matrix transforms displacements into forces. Similarly, in accordance with Eq. (6.97*b*), the flexibility matrix transforms forces into displacements. To accomplish the transformation, the nodal forces and displacements must be assembled into correspondingly positioned elements of force and displacement vectors. Depending on whether the displacement or the force method is chosen, stiffness or flexibility matrices are then established for each of the finite elements and these matrices are assembled to form a square matrix, from which the stiffness or flexibility matrix for the structure as a whole is derived. With that matrix known and substituted into equilibrium and compatibility equations for the structure, all nodal forces and displacements of the finite elements can be determined from the solution of the equations. Internal stresses and strains in the elements can be computed from the now known nodal forces and displacements.

6.67 Element Flexibility and Stiffness Matrices

The relationship between *independent* forces and displacements at nodes of finite elements in a structure is determined by flexibility matrices **f** or stiffness matrices **k** of the elements. In some cases, the components of these matrices can be developed from the defining equations:

The jth column of a flexibility matrix of a finite element contains all the nodal displacements of the element when one force S_j is set equal to unity and all other independent forces are set equal to zero.

The jth column of a stiffness matrix of a finite element consists of the forces acting at the nodes of the element to produce a unit displacement of the node at which displacement δ_j occurs and in the direction of δ_j but no other nodal displacements of the element.

Bars with Axial Stress Only ▪ As an example of the use of the definitions of flexibility and stiffness, consider the simple case of an elastic bar under tension applied by axial forces P_i and P_j at nodes i and j, respectively (Fig. 6.84). The bar might be the finite element of a truss, such as a diagonal or a hanger. Connections to other members are made at nodes i and j, which can transmit only forces in the directions i to j or j to i.

For equilibrium, $P_i = P_j = P$. Displacement of node j relative to node i is e. From Eq. (6.8), $e = PL/AE$, where L is the initial length of the bar, A the bar cross-sectional area, and E the modulus of elasticity. Setting $P = 1$ yields the flexibility of the bar,

$$f = \frac{L}{AE} \tag{6.139}$$

Setting $e = 1$ gives the stiffness of the bar,

$$k = \frac{AE}{L} \tag{6.140}$$

Beams with Bending Only ▪ As another example of the use of the definition to determine element flexibility and stiffness matrices, consider the simple case of an elastic prismatic beam in bending applied by moments M_i and M_j at nodes i and j, respectively (Fig. 6.85). The beam might be a finite element of a rigid frame. Connecztions to other members are made at nodes i and j, which can transmit moments and forces normal to the beam.

Nodal displacements of the element can be sufficiently described by rotations θ_i and θ_j relative to the straight line between nodes i and j. For equilibrium, forces $V_j = -V_i$ normal to the beam are required at nodes j and i, respectively, and $V_j = (M_i + M_j)/L$, where L is the span of the beam. Thus, M_i and M_j are the only independent forces

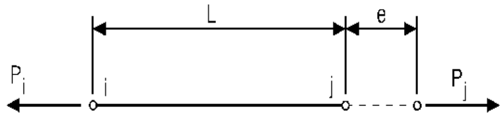


Fig. 6.84 Elastic bar in tension.

acting. Hence, the force-displacement relationship can be written for this element as

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} = \mathbf{f} \begin{bmatrix} M_i \\ M_j \end{bmatrix} = \mathbf{fM} \qquad (6.141)$$

$$\mathbf{M} = \begin{bmatrix} M_i \\ M_j \end{bmatrix} = \mathbf{k} \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} = \mathbf{k}\boldsymbol{\theta} \qquad (6.142)$$

The flexibility matrix **f** then will be a 2 × 2 matrix. The first column can be obtained by setting $M_i = 1$ and $M_j = 0$ (Fig. 6.85*b*). The resulting angular rotations are given by Eqs. (6.101) and (6.102). For a beam with constant moment of inertia I and modulus of elasticity E, the rotations are $\alpha = L/3EI$ and $\beta = -L/6EI$. Similarly, the second column can be developed by setting $M_i = 0$ and $M_j = 1$.

The flexibility matrix for a beam in bending then is

$$\mathbf{f} = \begin{bmatrix} \frac{L}{3EI} & -\frac{L}{6EI} \\ -\frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad (6.143)$$

The stiffness matrix, obtained in a similar manner or by inversion of **f**, is

$$\mathbf{k} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} = \frac{2EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (6.144)$$

Beams Subjected to Bending and Axial Forces ▪ For a beam subjected to nodal moments M_i and M_j and axial forces P, flexibility and stiffness are represented by 3 × 3 matrices. The load-displacement relations for a beam of span L, constant moment of inertia I, modulus of elasticity E, and cross-sectional area A are given by

$$\begin{bmatrix} \theta_i \\ \theta_j \\ e \end{bmatrix} = \mathbf{f} \begin{bmatrix} M_i \\ M_j \\ P \end{bmatrix} \qquad \begin{bmatrix} M_i \\ M_j \\ P \end{bmatrix} = \mathbf{k} \begin{bmatrix} \theta_i \\ \theta_j \\ e \end{bmatrix} \qquad (6.145)$$

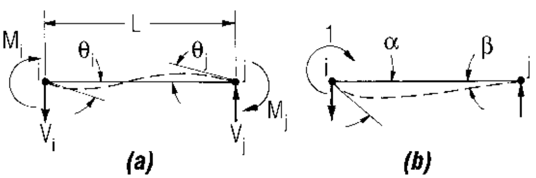


Fig. 6.85 Beam subjected to end moments and shears.

where e = axial displacement. In this case, the flexibility matrix is

$$\mathbf{f} = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \eta \end{bmatrix} \qquad (6.146)$$

where $\eta = 6I/A$, and the stiffness matrix, with $\psi = A/I$, is

$$\mathbf{k} = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & \psi \end{bmatrix} \qquad (6.147)$$

6.68 Displacement (Stiffness) Method

With the stiffness or flexibility matrix of each finite element of a structure known, the stiffness or flexibility matrix for the whole structure can be determined, and with that matrix, forces and displacements throughout the structure can be computed (Art. 6.67). To illustrate the procedure, the steps in the displacement, or stiffness, method are described in the following. The steps in the flexibility method are similar. For the stiffness method:

Step 1. Divide the structure into interconnected elements and assign a number, for identification purposes, to every node (intersection and terminal of elements). It may also be useful to assign an identifying number to each element.

Step 2. Assume a right-handed cartesian coordinate system, with axes x, y, z. Assume also at each node of a structure to be analyzed a system of base unit vectors, $\mathbf{e}_1$ in the direction of the x axis, $\mathbf{e}_2$ in the direction of the y axis, and $\mathbf{e}_3$ in the direction of the z axis. Forces and moments acting at a node are resolved into components in the directions of the base vectors. Then, the forces and moments at the node may be represented by the vector $P_i\mathbf{e}_i$, where P_i is the magnitude of the force or moment acting in the direction of $\mathbf{e}_i$. This vector, in turn, may be conveniently represented by a column matrix **P**. Similarly, the displacements—translations and rotation—of the node may be represented by the vector $\Delta_i\mathbf{e}_i$, where Δ_i is the magnitude of the displacement acting in the direction of $\mathbf{e}_i$. This vector, in turn, may be represented by a column matrix **Δ**.

For compactness, and because, in structural analysis, similar operations are performed on all nodal forces, all the loads, including moments, acting on all the nodes may be combined into a single column matrix **P**. Similarly, all the nodal displacements may be represented by a single column matrix Δ.

When loads act along a beam, they could be replaced by equivalent forces at the nodes—simple-beam reactions and fixed-end moments, both with signs reversed from those induced by the loads. The final element forces are then determined by adding these moments and reactions to those obtained from the solution with only the nodal forces.

Step 3. Develop a stiffness matrix $\mathbf{k}_i$ for each element i of the structure (see Art. 6.67). By definition of stiffness matrix, nodal displacements and forces for the ith element are related by

$$\mathbf{S}_i = \mathbf{k}_i \boldsymbol{\delta}_i \qquad i = 1, 2, \ldots, n \tag{6.148}$$

where $\mathbf{S}_i$ = matrix of forces, including moments and torques acting at the nodes of the ith element

$\boldsymbol{\delta}_i$ = matrix of displacements of the nodes of the ith element

Step 4. For compactness, combine this relationship between nodal displacements and forces for each element into a single matrix equation applicable to all the elements:

$$\mathbf{S} = \mathbf{k}\boldsymbol{\delta} \tag{6.149}$$

where **S** = matrix of all forces acting at the nodes of all elements

$\boldsymbol{\delta}$ = matrix of all nodal displacements for all elements

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{k}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbf{k}_n \end{bmatrix} \tag{6.150}$$

Step 5. Develop a matrix $\mathbf{b}_o$ that will transform the displacements $\boldsymbol{\Delta}$ of the nodes of the structure into the displacement vector $\boldsymbol{\delta}$ while maintaining geometric compatibility:

$$\boldsymbol{\delta} = \mathbf{b}_o \boldsymbol{\Delta} \tag{6.151}$$

$\mathbf{b}_o$ is a matrix of influence coefficients. The jth column of $\mathbf{b}_o$ contains the element nodal displacements when the node where Δ_j occurs is given a unit displacement in the direction of Δ_j, and no other nodes are displaced.

Step 6. Compute the stiffness matrix **K** for the whole structure from

$$\mathbf{K} = \mathbf{b}_o^T \mathbf{k} \mathbf{b}_o \tag{6.152}$$

where $\mathbf{b}_o^T$ = transpose of $\mathbf{b}_o$ = matrix $\mathbf{b}_o$ with rows and columns interchanged.

This equation may be derived as follows: From energy relationships, $\mathbf{P} = \mathbf{b}_o^T \mathbf{S}$. Substitution of $\mathbf{k}\boldsymbol{\delta}$ for **S** [Eq. (6.149)] and then substitution of $\mathbf{b}_o\boldsymbol{\Delta}$ for $\boldsymbol{\delta}$ [Eq. (6.151)] yields $\mathbf{P} = \mathbf{b}_o^T \mathbf{k} \mathbf{b}_o \boldsymbol{\Delta}$. Comparison of this with Eq. (6.97*a*), $\mathbf{P} = \mathbf{k}\boldsymbol{\Delta}$, leads to Eq. (6.152).

Step 7. With the stiffness matrix **K** now known, solve the simultaneous equations

$$\boldsymbol{\Delta} = \mathbf{K}^{-1}\mathbf{P} \tag{6.153}$$

for the nodal displacements $\boldsymbol{\Delta}$. With these determined, calculate the member forces from

$$\mathbf{S} = \mathbf{k}\mathbf{b}_o\boldsymbol{\Delta} \tag{6.154}$$

(N. M. Baran, "Finite Element Analysis on Microcomputers," and H. Kardestuncer and D. H. Norris, "Finite Element Handbook," McGraw-Hill Publishing Company, New York, books.mcgraw-hill.com; K. Bathe, "Finite Element Procedures in Engineering Analysis," T. R. Hughes, "The Finite Element Method," and H. T. Y. Yang, "Finite Element Structural Analysis," Prentice-Hall, Upper Saddle River, N.J.; W. Weaver, Jr., and J. M. Gere, "Matrix Analysis of Framed Structures," Van Nostrand Reinhold, New York.)

First- And Second-Order Analysis of Frames ▪ The deformation of the brace frame due to the lateral forces is usually small and is not normally taken into account. Consequently, moments in the columns are amplified only by the moment produced by the axial force acting through the deflections along the member. These moments are called Pδ moments, where P is the column axial load and δ is the lateral deflection of the member with respect to the chord connecting its end points.

Unbraced frames subjected to unsymmetrical loads and/or lateral forces undergo lateral displacements. As a result of these displacements, columns in the frame are subjected to additional moments PΔ, where Δ is the lateral displacement of one end of a column with respect to the other end. In multistory structures, the PΔ moment for the columns in any one story is (ΣP)Δ, where ΣP is the total vertical load on the story and Δ is the lateral deflection of the story with respect to the one below.

Analyses based on the dimensions of an undeformed frame are called first-order analyses, while those based on the deformed frame, taking into account both the Pδ and the PΔ effects, are called second-order analyses. Second-order analyses are basically geometrical nonlinear problems that required the use of computer programs. But not all programs that are advertised as for second-order analysis consider the Pδ moments. In the practical design, the PΔ effect is generally taken into account in the structural analysis through the iteration algorithm of computer programs, while the Pδ effect is considered during member design and only one step approximation is used through the magnification factor $1/(1 - P/Pc)$, as in Eq. (6.59).

(E. H. Gaylord, Jr. et al., "Design of Steel Structures" 3rd edition, McGraw-Hill, books: mcgraw-hill.com.)

Stresses in Arches

An arch is a curved beam, the radius of curvature of which is very large relative to the depth of section. It differs from a straight beam in that: (1) loads induce both bending and direct compressive stress in an arch; (2) arch reactions have horizontal components even though all loads are vertical, and (3) deflections have horizontal as well as vertical components. Names of arch parts are given in Fig. 6.86.

The necessity of resisting the horizontal components of the reactions is an important consideration in arch design. Sometimes these forces are taken by tie rods between the supports, sometimes by heavy abutments or buttresses.

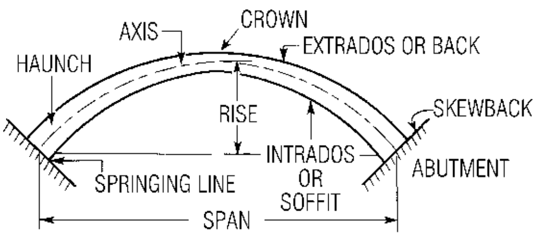


Fig. 6.86 Names of parts of a fixed arch.

Arches may be built with fixed ends, as can straight beams, or with hinges at the supports. They may also be built with an internal hinge, usually located at the uppermost point, or crown.

6.69 Three-Hinged Arches

An arch with an internal hinge and hinges at both supports (Fig. 6.87) is statically determinate. There are four unknowns—two horizontal and two vertical components of the reactions—but four equations based on the laws of equilibrium are available: (1) The sum of the horizontal forces must be zero. (In Fig. 6.87, $H_L = H_R = H$.) (2) The sum of the moments about the left support must be zero. ($V_R = Pk$). (3) The sum of the moments about the right support must be zero. [$V_L = P(1 - k)$.] (4) The bending moment at the crown hinge must be zero (not to be confused with the sum of the moments about the crown, which also must be equal to zero but which would not lead to an independent equation for the solution of the reactions). Hence, for the right half of the arch in Fig. 6.87*a*, $Hh - V_R b = 0$, and $H = V_R b/h$. The influence line for H is a straight line, varying from zero for loads over the supports to the maximum of Pab/Lh for a load at C.

Reactions and stresses in three-hinged arches can be determined graphically by taking advantage of the fact that the bending moment at the crown hinge is zero. For example, in Fig. 6.87*a*, the load P is applied to segment AC of the arch. Then, since the bending moment at C must be zero, the line of action of the reaction R_R at B must pass through the crown hinge. It intersects the line of action of P at X. The line of action of the reaction R_L at A also must pass through X since P and the two reactions are in equilibrium. By constructing a force triangle with the load P and the lines of action of the reactions thus determined, you can obtain the magnitude of the reactions (Fig. 6.87*b*). After the reactions have been found, the stresses can be computed from the laws of statics or, in the case of a trussed arch, determined graphically.

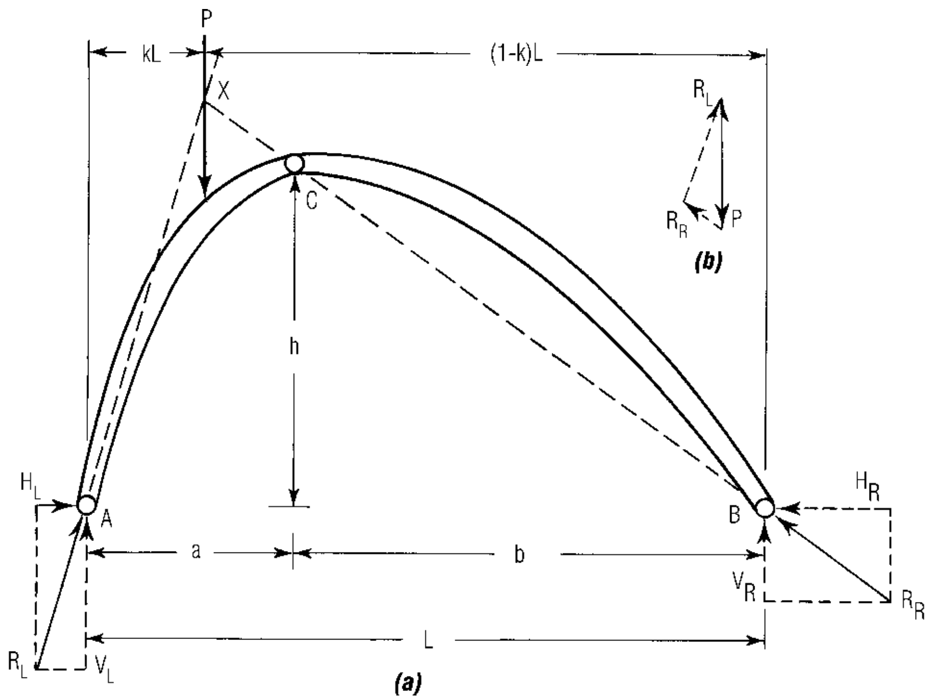


Fig. 6.87 Three-hinged arch.

6.70 Two-Hinged Arches

When an arch has hinges at the supports only (Fig. 6.88*a*), it is statically indeterminate; there is one more unknown reaction component than can be determined by the three equations of equilibrium. Another equation can be written from knowledge of the elastic behavior of the arch. One procedure is to assume that one of the supports is on rollers. The arch then is statically determinate, and the reactions and horizontal movement of the support can be computed for this condition (Fig. 6.88*b*). Next, the horizontal force required to return the movable support to its original position can be calculated (Fig. 6.88*c*). Finally, the reactions for the two-hinged arch (Fig. 6.88*d*) are obtained by superimposing the first set of reactions on the second.

For example, if δx is the horizontal movement of the support due to the loads on the arch, and if $\delta x'$ is the horizontal movement of the support due to a unit horizontal force applied to the support, then

$$\delta x + H\delta x' = 0 \qquad (6.155)$$

$$H = -\frac{\delta x}{\delta x'} \qquad (6.156)$$

where H is the unknown horizontal reaction. (When a tie rod is used to take the thrust, the right-hand side of Eq. (6.155) is not zero but the elongation of the rod HL/A_sE_s, where L is the length of the rod, A_s its cross-sectional area, and E_s its modulus of elasticity. To account for the effect of an increase in temperature t, add to the left-hand side $EctL$, where E is the modulus of elasticity of the arch, c the coefficient of expansion.)

The dummy-unit-load method can be used to compute δx and $\delta x'$ (Art. 6.54):

$$\delta x = \int_A^B \frac{My\,ds}{EI} - \int_A^B \frac{N\,dx}{AE} \qquad (6.157)$$

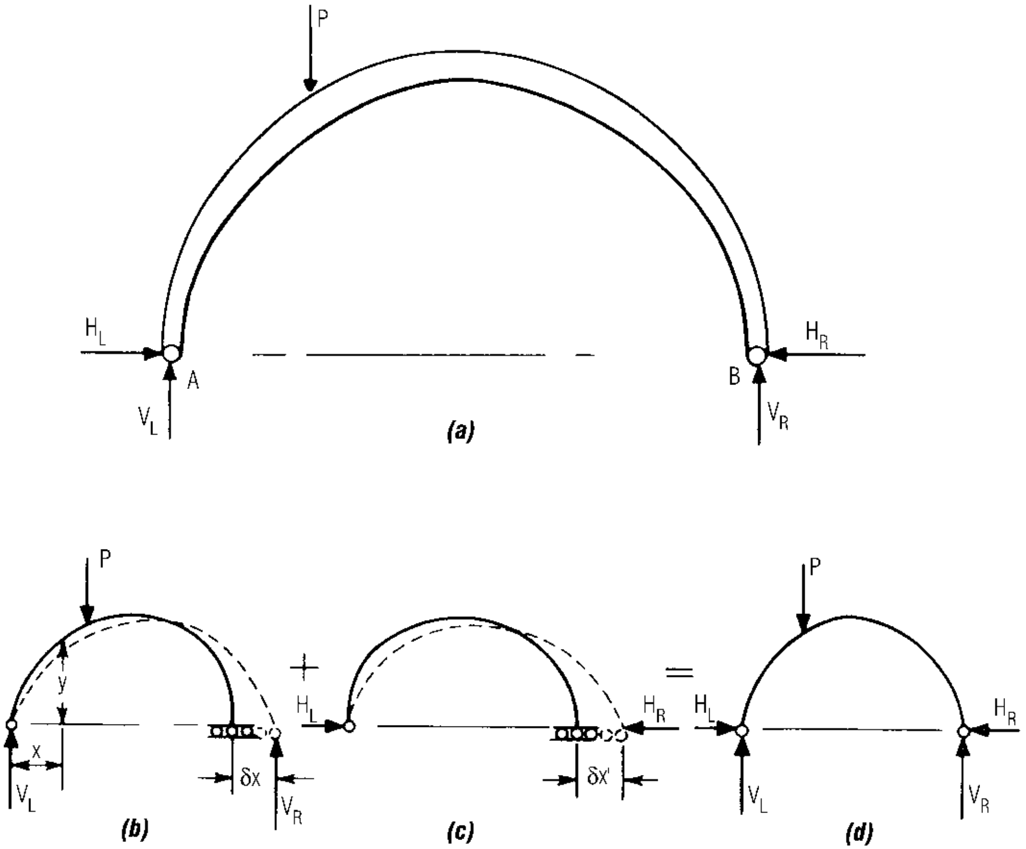


Fig. 6.88 Two-hinged arch.

where M = bending moment at any section due to loads

y = ordinate of section measured from immovable end of arch

I = moment of inertia of arch cross section

A = cross-sectional area of arch

ds = differential length along arch axis

dx = differential length along the horizontal

N = normal thrust on cross section due to loads

$$\delta x' = -\int_A^B \frac{y^2\, ds}{EI} - \int_A^B \frac{\cos^2 \alpha\, dx}{AE} \qquad (6.158)$$

where α = the angle the tangent to the axis at the section makes with the horizontal.

Equations (6.157) and (6.158) do not include the effects of shear deformation and curvature, which usually are negligible. Unless the thrust is very large, the second term on the right-hand side of Eq. (6.157) also can be dropped.

In most cases, integration is impracticable. The integrals generally must be evaluated by approximate methods. The arch axis is divided into a convenient number of elements of length Δs, and the functions under the integral sign are evaluated for each element. The sum of these terms is approximately equal to the integral. Thus, for the usual two-hinged arch

$$H = \frac{\sum_A^B (My\ \Delta s/EI)}{\sum^B (y^2\ \Delta s/EI) + \sum^B (\cos^2\ \alpha\ \Delta x/AE)} \qquad (6.159)$$

(S. Timoshenko and D. H. Young, "Theory of Structures," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; S. F. Borg and J. J. Gennaro, "Modern Structural Analysis," Van Nostrand Reinhold Company, New York.)

6.71 Stresses in Arch Ribs

When the reactions have been found for an arch (Arts. 6.69 to 6.70), the principal forces acting on any cross section can be found by applying the equations of equilibrium. For example, consider the portion of an arch in Fig. 6.89, where the forces acting at an interior section X are to be found. The load P, H_L (or H_R), and V_L (or V_R) may be resolved into components parallel to the axial thrust N and the shear S at X, as indicated in Fig. 6.89. Then, by equating the sum of the forces in each direction to zero, we get

$$N = V_L \sin \theta_x + H_L \cos \theta_x + P \sin(\theta_x - \theta) \tag{6.160}$$

$$S = V_L \cos \theta_x - H_L \sin \theta_x + P \cos(\theta_x - \theta) \tag{6.161}$$

And the bending moment at X is

$$M = V_L x - H_L y - Pa \cos \theta - Pb \sin \theta \tag{6.162}$$

The shearing unit stress on the arch cross section at X can be determined from S with the aid of Eq. (6.49). The normal unit stresses can be calculated from N and M with the aid of Eq. (6.57).

When designing an arch, it may be necessary to compute certain secondary stresses, in addition to those caused by live, dead, wind, and snow loads. Among the secondary stresses to be considered are those due to temperature changes, rib shortening due to thrust or shrinkage, deformation of tie rods, and unequal settlement of footings. The procedure is the same as for loads on the arch, with the deformations producing the secondary stresses substituted for or treated the same as the deformations due to loads.

Also, the stability of arches should be considered. According to the mode of failure, there exist in-plane stability and out-of-plane buckling issues. (Theodore V. Galambos, "Guide to Stability Design Criteria for Metal Structures", 5th edition, John Wiley & Sons. Inc., www.wiley.com)

Thin-Shell Structures

A structural shell is a curved surface structure. Usually, it is capable of transmitting loads in more than two directions to supports. It is highly efficient structurally when it is so shaped, proportioned, and supported that it transmits the loads without bending or twisting.

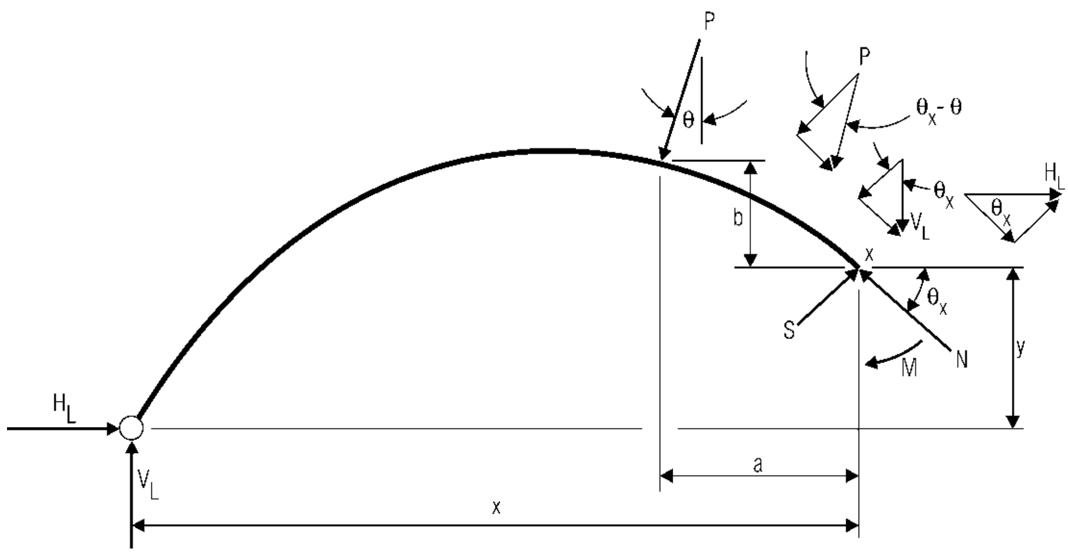


Fig. 6.89 Stresses in an arch rib.

A shell is defined by its middle surface, halfway between its extrados, or outer surface, and intrados, or inner surface. Thus, depending on the geometry of the middle surface, it might be a type of dome, barrel arch, cone, or hyperbolic paraboloid. Its thickness is the distance, normal to the middle surface, between extrados and intrados.

6.72 Thin-Shell Analysis

A thin shell is a shell with a thickness relatively small compared with its other dimensions. But it should not be so thin that deformations would be large compared with the thickness.

The shell should also satisfy the following conditions: Shearing stresses normal to the middle surface are negligible. Points on a normal to the middle surface before it is deformed lie on a straight line after deformation. And this line is normal to the deformed middle surface.

Calculation of the stresses in a thin shell generally is carried out in two major steps, both usually involving the solution of differential equations. In the first, bending and torsion are neglected (membrane theory, Art. 6.73). In the second step, corrections are made to the previous solution by superimposing the bending and shear stresses that are necessary to satisfy boundary conditions (bending theory, Art. 6.74).

6.73 Membrane Theory for Thin Shells

Thin shells usually are designed so that normal shears, bending moments, and torsion are very small, except in relatively small portions of the shells. In the membrane theory, these stresses are ignored.

Despite the neglected stresses, the remaining stresses are in equilibrium, except possibly at boundaries, supports, and discontinuities. At any interior point, the number of equilibrium conditions equals the number of unknowns. Thus, in the membrane theory, a thin shell is statically determinate.

The membrane theory does not hold for concentrated loads normal to the middle surface, except possibly at a peak or valley. The theory does not apply where boundary conditions are incompatible with equilibrium; and it is inexact where there is geometric incompatibility at the boundaries. The last is a common condition, but the error is very small if the shell is not very flat. Usually, disturbances of membrane equilibrium due to incompatibility with deformations at boundaries, supports, or discontinuities are appreciable only in a narrow region about each source of disturbance. Much larger disturbances result from incompatibility with equilibrium conditions.

To secure the high structural efficiency of a thin shell, select a shape, proportions, and supports for the specific design conditions that come as close as possible to satisfying the membrane theory. Keep the thickness constant; if it must change, use a gradual taper. Avoid concentrated and abruptly changing loads. Change curvature gradually. Keep discontinuities to a minimum. Provide reactions that are tangent to the middle surface. At boundaries, insure, to the extent possible, compatibility of shell deformations with deformations of adjoining members, or at least keep restraints to a minimum. Make certain that reactions along boundaries are equal in magnitude and direction to the shell forces there.

Means usually adopted to satisfy these requirements at boundaries and supports are illustrated in Fig. 6.90. In Fig. 6.90*a*, the slope of the support and provision for movement normal to the middle surface insure a reaction tangent to the middle surface. In Fig. 6.90*b*, a stiff rib, or ring girder, resists unbalanced shears and transmits normal forces to columns below. The enlarged view of the ring girder in Fig. 6.90*c* shows gradual thickening of the shell to reduce the abruptness of the change in section. The stiffening ring at the lantern in Fig. 6.90*d*, extending around the opening at the crown, projects above the middle surface, for compatibility of strains, and connects through a transition curve with the shell; often, the rim need merely be thickened when the edge is upturned, and the ring can be omitted. In Fig. 6.90*e*, the boundary of the shell is a thickened edge. In Fig. 6.90*f*, a scalloped shell provides gradual tapering for transmitting the loads to the supports, at the same time providing access to the shell enclosure. And in Fig. 6.90*g*, a column is flared widely at the top to support a thin shell at an interior point.

Even when the conditions for geometric compatibility are not satisfactory, the membrane theory is a useful approximation. Furthermore, it yields a particular solution to the differential equations of the bending theory.

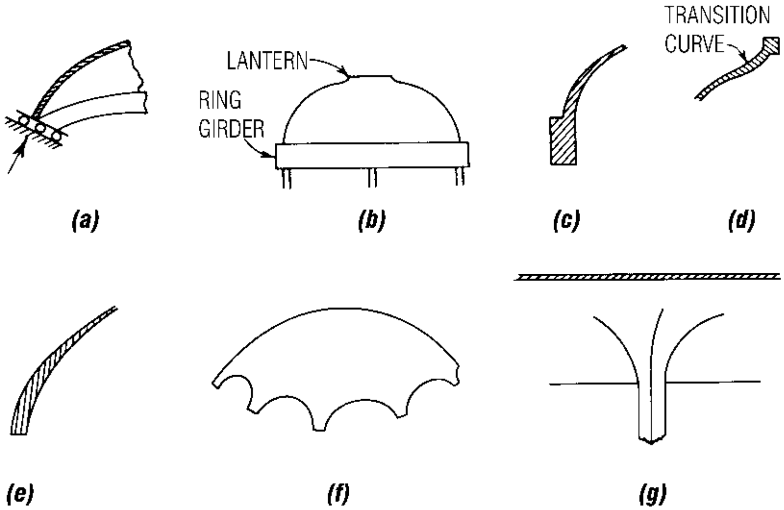


Fig. 6.90 Special provisions made at supports and boundaries of thin shells to meet requirements of the membrane theory include (*a*) a device to ensure a reaction tangent to the middle surface, (*b*) stiffened edges, such as the ring girder at the base of a dome, (*c*) gradually increased shell thickness at a stiffening member, (*d*) a transition curve at changes in section, (*e*) a stiffened edge obtained by thickening the shell, (*f*) scalloped edges, and (*g*) a flared support.

(D. P. Billington, "Thin-Shell Concrete Structures," 2nd ed., and S. Timoshenko and S. Woinowsky-Krieger, "Theory of Plates and Shells," McGraw-Hill Publishing Company, New York, books.mcgraw-hill.com; V. S. Kelkar and R. T. Sewell, "Fundamentals of the Analysis and Design of Shell Structures," Prentice-Hall, Englewood Cliffs, N.J., www.prenhall.com.)

6.74 Bending Theory for Thin Shells

When equilibrium conditions are not satisfied or incompatible deformations exist at boundaries, bending and torsion stresses arise in the shell. Sometimes, the design of the shell and its supports can be modified to reduce or eliminate these stresses (Art. 6.73). When the design cannot eliminate them, provision must be made for the shell to resist them.

But even for the simplest types of shells and loading, the stresses are difficult to compute. In bending theory, a thin shell is statically indeterminate; deformation conditions must supplement equilibrium conditions in setting up differential equations for determining the unknown forces and moments. Solution of the resulting equations may be tedious and time-consuming, if indeed solution is possible.

In practice, therefore, shell design relies heavily on the designer's experience and judgment. The designer should consider the type of shell, material of which it is made, and support and boundary conditions, and then decide whether to apply a bending theory in full, use an approximate bending theory, or make a rough estimate of the effects of bending and torsion. (Note that where the effects of a disturbance are large, these change the normal forces and shears computed by the membrane theory.) For domes, for example, the usual procedure is to use as a support a deep, thick girder or a heavily reinforced or prestressed tension ring, and the shell is gradually thickened in the vicinity of this support (Fig. 6.90*c*).

Circular barrel arches, with ratio of radius to distance between supporting arch ribs less than 0.25, may be designed as beams with curved cross section. Secondary stresses, however, must be taken into account. These include stresses due to volume change of rib and shell, rib shortening, unequal settlement of footings, and temperature differentials between surfaces.

Bending theory for cylinders and domes is given in W. Flügge, "Stresses in Shells," Springer-Verlag, New York, www.springer-ny.com; S. Timoshenko and S. Woinowsky-Krieger, "Theory of Plates and Shells," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; "Design of Cylindrical Concrete Shell Roofs," Manual of Practice no. 31, American Society of Civil Engineers. www.asce.org.

6.75 Stresses in Thin Shells

The results of the membrane and bending theories are expressed in terms of unit forces and unit moments, acting per unit of length over the thickness of the shell. To compute the unit stresses from these forces and moments, usual practice is to assume normal forces and shears to be uniformly distributed over the shell thickness and bending stresses to be linearly distributed.

Then, normal stresses can be computed from equations of the form

$$f_x = \frac{N_x}{t} + \frac{M_x}{t^3/12} z \tag{6.163}$$

where z = distance from middle surface

t = shell thickness

M_x = unit bending moment about an axis parallel to direction of unit normal force N_x

Similarly, shearing stresses produced by central shears T and twisting moments D may be calculated from equations of the form

$$v_{xy} = \frac{T}{t} \pm \frac{D}{t^3/12} z \tag{6.164}$$

Normal shearing stresses may be computed on the assumption of a parabolic stress distribution over the shell thickness:

$$v_{xz} = \frac{V}{t^3/6} \left(\frac{t^2}{4} - z^2 \right) \tag{6.165}$$

where V = unit shear force normal to middle surface.

For axes rotated with respect to those used in the thin-shell analysis, use Eqs. (6.27) and (6.28) to transform stresses or unit forces and moments from the given to the new axes.

Folded Plates

A folded-plate structure consists of a series of thin planar elements, or flat plates, connected to

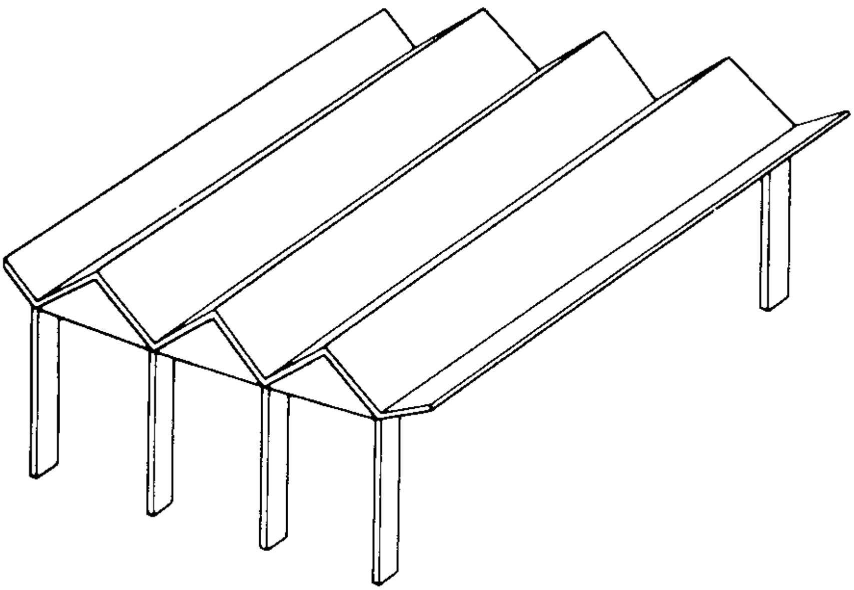


Fig. 6.91 Folded-plate structure.

one another along their edges. Usually used on long spans, especially for roofs, folded plates derive their economy from the girder action of the plates and the mutual support they give one another.

Longitudinally, the plates may be continuous over their supports. Transversely, there may be several plates in each bay (Fig. 6.91). At the edges, or folds, they may be capable of transmitting both moment and shear or only shear.

6.76 Folded-Plate Theory

A folded-plate structure has a two-way action in transmitting loads to its supports. Transversely, the elements act as slabs spanning between plates on either side. The plates then act as girders in carrying the load from the slabs longitudinally to supports, which must be capable of resisting both horizontal and vertical forces.

If the plates are hinged along their edges, the design of the structure is relatively simple. Some simplification also is possible if the plates, though having integral edges, are steeply sloped or if the span is sufficiently long with respect to other dimensions that beam theory applies. But there are no criteria for determining when such simplification is possible with acceptable accuracy. In general, a reasonably accurate analysis of folded-plate stresses is advisable.

Several good methods are available (D. Yitzhaki, "The Design of Prismatic and Cylindrical Shell Roofs," North Holland Publishing Company, Amsterdam, "Phase I Report on Folded-Plate Construction," *Proceedings Paper* 3741, *Journal of the Structural Division*, *ASCE*, December 1963; and A. L. Parme and J. A. Sbarounis, "Direct Solution of Folded Plate Concrete Roofs," EB021D Portland Cement Association, Skokie, IL. 60077). They all take into account the effects of plate deflections on the slabs and usually make the following assumptions:

The material is elastic, isotropic, and homogeneous. The longitudinal distribution of all loads on all plates is the same. The plates carry loads transversely only by bending normal to their planes and longitudinally only by bending within their planes. Longitudinal stresses vary linearly over the depth of each plate. Supporting members, such as diaphragms, frames, and beams, are infinitely stiff in their own planes and completely flexible normal to their own planes. Plates have no torsional stiffness normal to their own planes. Displacements due to forces other than bending moments are negligible.

Regardless of the method selected, the computations are rather involved; so it is wise to carry out the work in a well-organized table. The Yitzhaki method (Art. 6.77) offers some advantages over others in that the calculations can be tabulated, it is relatively simple, it requires the solution of no more simultaneous equations than one for each edge for simply supported plates, it is flexible, and it can easily be generalized to cover a variety of conditions.

6.77 Yitzhaki Method for Folded Plates

Based on the assumptions and general procedure given in Art. 6.76, the Yitzhaki method deals in two ways with the slab and plate systems that comprise a folded-plate structure. In the first, a unit width of slab is considered continuous over supports immovable in the direction of the load (Fig. 6.92*b*). The strip usually is taken where the longitudinal plate stresses are a maximum. Secondly, the slab reactions are taken as loads on the plates, which now are assumed to be hinged along the edges (Fig. 6.92*c*). Thus, the slab reactions cause angle changes in the plates at each fold. Continuity is restored by applying an unknown moment to the plates at each edge. The moments can be determined from the fact that at each edge the sum of the angle changes due to the loads and to the unknown moments must equal zero.

The angle changes due to the unknown moments have two components. One is the angle change at each slab end, now hinged to an adjoining slab, in the transverse strip of unit width. The second is the angle change due to deflection of the plates. The method assumes that the angle change at each fold varies in the same way longitudinally as the angle changes along the other folds.

For more details, see D. Yitzhaki and Max Reiss, "Analysis of Folded Plates," *Proceedings Paper* 3303, *Journal of the Structural Division*, *ASCE*, October 1962; F. S. Merritt and Jonathan T. Ricketts, "Building Design and Construction Handbook," 5th ed., McGraw-Hill Book Company, New York.

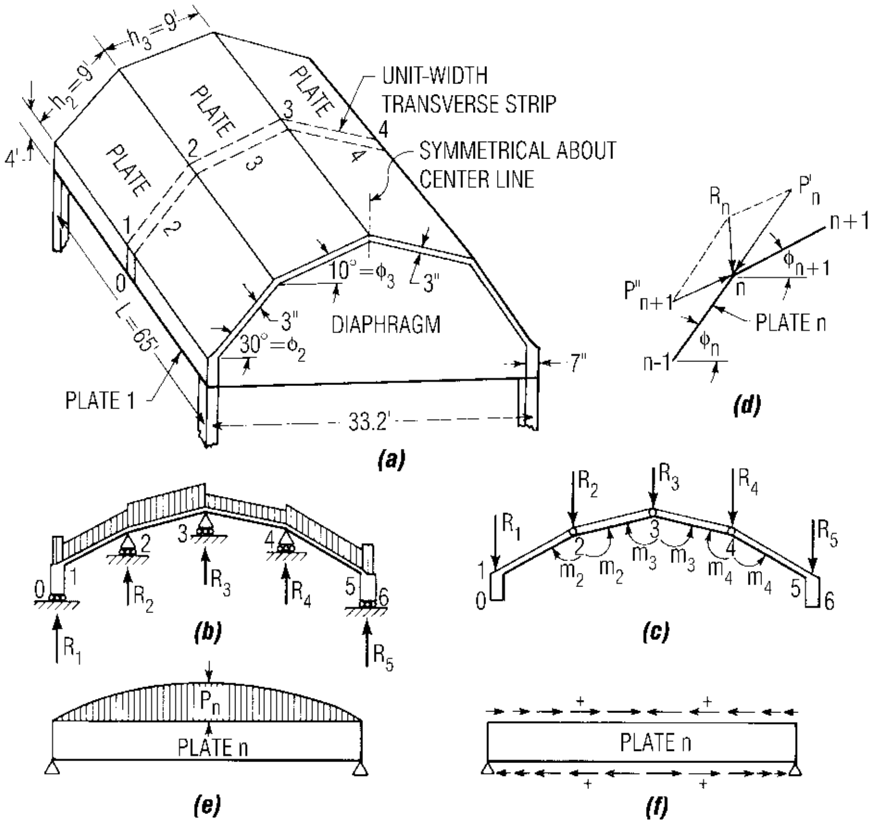


Fig. 6.92 Folded-plate structure is analyzed by first considering a transverse strip (*a*) as a continuous slab on supports that do not deflect (*b*). Then, the slabs are assumed hinged (*c*) and acted upon by the reactions computed in the first step and unknown moments to correct for this assumption. In the longitudinal direction, the plates act as deep girders (*e*) with shears along the edges, positive directions shown in (*f*). Slab reactions are resolved into plate forces, parallel to the planes of the plates (*d*).

Cable-Supported Structures*

A cable is a linear structural member, like a bar of a truss. The cross-sectional dimensions of a cable relative to its length, however, are so small that it cannot withstand bending or compression. Consequently, under loads at an angle to its longitudinal axis, a cable sags and assumes a shape that enables it to develop tensile stresses that resist the loads.

Structural efficiency results from two cable characteristics: (1) Uniformity of tensile stresses over the cable cross section, and (2) usually, small variation of tension along the longitudinal axis. Hence, it is economical to use materials with very high tensile strength for cables.

Cables sometimes are used in building construction as an alternative to such tension members as hangers, ties, or tension chords of trusses. For example, cables are used in a form of long-span cantilever-truss construction in which a horizontal roof girder is supported at one end by a column and near the other end by a cable that extends diagonally upward to the top of a vertical mast above the column support (cable-stayed-girder construction, Fig. 6.93).

*Reprinted with permission from F. S. Merritt, "Structural Steel Designers Handbook," McGraw-Hill Book Company, New York.

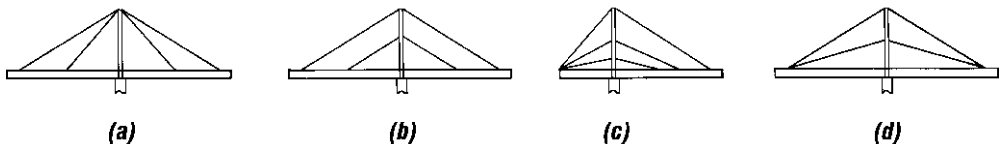


Fig. 6.93 Types of stayed girders: (*a*) Bundles (converging); (*b*) harp; (*c*) fan; (*d*) star.

Cable stress can be computed for this case from the laws of equilibrium. Similarly, cable-stayed girders are used to support bridge decks.

Cables also may be used instead of or with girders, trusses, or membranes to support roofs or bridge decks. For the purpose, cables may be arranged in numerous ways. It is consequently impractical to treat in detail in this book any but the simplest types of such applications of cables. Instead, general procedures for analyzing cable-supported structures are presented in the following. (See also Arts. 17.15 and 17.17).

6.78 Simple Cables

An ideal cable has no resistance to bending. Thus, in analysis of a cable in equilibrium, not only is the sum of the moments about any point equal to zero, but so is the bending moment at any point. Consequently, the equilibrium shape of the cable corresponds to the funicular, or bending-moment, diagram for the loading (Fig. 6.94*a*). As a result, the tensile force at any point of the cable is tangent there to the cable curve.

The point of maximum sag of a cable coincides with the point of zero shear. (Sag in this case should be measured parallel to the direction of the shear forces.)

Stresses in a cable are a function of the deformed shape. Equations needed for analysis, therefore, usually are nonlinear. Also, in general, stresses and deformations cannot be obtained accurately by superimposition of loads. A common procedure in analysis is to obtain a solution in steps by using linear equations to approximate the nonlinear ones and by starting with the initial geometry to obtain better estimates of the final geometry.

For convenience in analysis, the cable tension, directed along the cable curve, usually is resolved into two components. Often, it is advantageous to resolve the tension T into a horizontal component H and a vertical component V (Fig. 6.94*b*). Under vertical loading then, the horizontal component is constant along the cable. Maximum tension occurs at the support. V is zero at the point of maximum sag.

For a general, distributed vertical load q, the cable must satisfy the second-order linear differential equation

$$Hy'' = q \tag{6.166}$$

where y = rise of cable at distance x from low point (Fig. 6.94*b*)

$y'' = d^2y/dx^2$

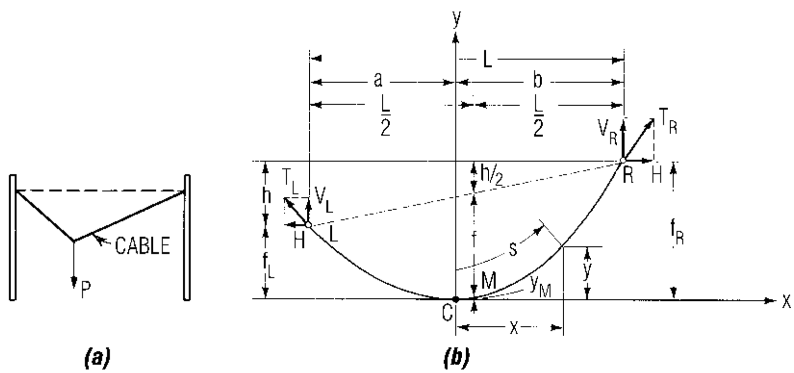


Fig. 6.94 Simple cables: (*a*) Shape of cable with a concentrated load; (*b*) shape of cable with supports at different levels.

6.78.1 Catenary

Weight of a cable of constant cross section represents a vertical loading that is uniformly distributed along the length of cable. Under such a loading, a cable takes the shape of a catenary.

Take the origin of coordinates at the low point C and measure distance s along the cable from C (Fig. 6.94*b*). If q_o is the load per unit length of cable, Eq. (6.166) becomes

$$Hy'' = q_o \frac{ds}{dx} = q_o\sqrt{1 + y'^2} \tag{6.167}$$

where $y' = dy/dx$. Solving for y' gives the slope at any point of the cable

$$y' = \sinh\frac{q_o x}{H} = \frac{q_o x}{H} + \frac{1}{3!}\left(\frac{q_o x}{H}\right)^3 + \cdots \tag{6.168}$$

A second integration then yields the equation for the cable shape, which is called a catenary.

$$\begin{aligned} y &= \frac{H}{q_o}\left(\cosh\frac{q_o x}{H} - 1\right) \\ &= \frac{q_o}{H}\frac{x^2}{2!} + \left(\frac{q_o}{H}\right)^3 + \frac{x^4}{4!} + \cdots \end{aligned} \tag{6.169}$$

If only the first term of the series expansion is used, the cable equation represents a parabola. Because the parabolic equation usually is easier to handle, a catenary often is approximated by a parabola.

For a catenary, length of arc measured from the low point is

$$s = \frac{H}{q_o}\sinh\frac{q_o x}{H} = x + \frac{1}{3!}\left(\frac{q_o}{H}\right)^2 x^3 + \cdots \tag{6.170}$$

Tension at any point is

$$T = \sqrt{H^2 + q_o^2 s^2} = H + q_o y \tag{6.171}$$

The distance from the low point C to the left support L is

$$a = \frac{H}{q_o}\cosh^{-1}\left(\frac{q_o}{H} f_L + 1\right) \tag{6.172}$$

where f_L = vertical distance from C to L. The distance from C to the right support R is

$$b = \frac{H}{q_o}\cosh^{-1}\left(\frac{q_o}{H} f_R + 1\right) \tag{6.173}$$

where f_R = vertical distance from C to R.

Given the sags of a catenary f_L and f_R under a distributed vertical load q_o, the horizontal component of cable tension H may be computed from

$$\frac{q_o l}{H} = \cosh^{-1}\left(\frac{q_o f_L}{H} + 1\right) + \cosh^{-1}\left(\frac{q_o f_R}{H} + 1\right) \tag{6.174}$$

where l = span, or horizontal distance between supports L and $R = a + b$. This equation usually is solved by trial. A first estimate of H for substitution in the right-hand side of the equation may be obtained by approximating the catenary by a parabola. Vertical components of the reactions at the supports can be computed from

$$R_L = H\sinh\frac{q_o a}{H} \qquad R_R = H\sinh\frac{q_o b}{H} \tag{6.175}$$

6.78.2 Parabola

Uniform vertical live loads and uniform vertical dead loads other than cable weight generally may be treated as distributed uniformly over the horizontal projection of the cable. Under such loadings, a cable takes the shape of a parabola.

Take the origin of coordinates at the low point C (Fig. 6.94*b*). If w_o is the load per foot horizontally, Eq. (6.166) becomes

$$Hy'' = w_o \tag{6.176}$$

Integration gives the slope at any point of the cable

$$y' = \frac{w_o x}{H} \tag{6.177}$$

A second integration then yields the parabolic equation for the cable shape

$$y = \frac{w_o x^2}{2H} \tag{6.178}$$

The distance from the low point C to the left support L is

$$a = \frac{1}{2} - \frac{Hh}{w_o l} \tag{6.179}$$

where l = span, or horizontal distance between supports L and $R = a + b$

h = vertical distance between supports

The distance from the low point C to the right support R is

$$b = \frac{1}{2} + \frac{Hh}{w_o l} \tag{6.180}$$

Supports at Different Levels ▪ The horizontal component of cable tension H may be computed from

$$H = \frac{w_o l^2}{h^2}\left(f_R - \frac{h}{2} \pm \sqrt{f_L f_R}\right) = \frac{w_o l^2}{8f} \tag{6.181}$$

where f_L = vertical distance from C to L

f_R = vertical distance from C to R

f = sag of cable measured vertically from chord LR midway between supports (at $x = Hh/w_o l$)

As indicated in Fig. 6.94*b*,

$$f = f_L + \frac{h}{2} - y_M \tag{6.182}$$

where $y_M = Hh^2/2w_o l^2$. The minus sign should be used in Eq. (6.181) when low point C is between supports. If the vertex of the parabola is not between L and R, the plus sign should be used.

The vertical components of the reactions at the supports can be computed from

$$V_L = w_o a = \frac{w_o l}{2} - \frac{Hh}{l}$$
$$V_r = w_o b = \frac{w_o l}{2} + \frac{Hh}{l} \tag{6.183}$$

Tension at any point is

$$T = \sqrt{H^2 + w_o^2 x^2} \tag{6.184}$$

Length of parabolic arc RC is

$$L_{RC} = \frac{b}{2}\sqrt{1 + \left(\frac{w_o b}{H}\right)^2} + \frac{H}{2w_o}\sinh\frac{w_o b}{H} = b + \frac{1}{6}\left(\frac{w_o}{H}\right)^2 b^3 + \cdots \tag{6.185}$$

Length of parabolic arc LC is

$$L_{LC} = \frac{a}{2}\sqrt{1 + \left(\frac{w_o a}{H}\right)^2} + \frac{H}{2w_o}\sinh\frac{w_o a}{H} = a + \frac{1}{6}\left(\frac{w_o}{H}\right)^2 a^3 + \cdots \tag{6.186}$$

Supports at Same Level ▪ In this case, $f_L = f_R = f$, $h = 0$, and $a = b = l/2$. The horizontal component of cable tension H may be computed from

$$H = \frac{w_o l^2}{8f} \tag{6.187}$$

The vertical components of the reactions at the supports are

$$V_L = V_R = \frac{w_o l}{2} \tag{6.188}$$

Maximum tension occurs at the supports and equals

$$T_L = T_R = \frac{w_o l}{2}\sqrt{1 + \frac{l^2}{16f^2}} \tag{6.189}$$

Length of cable between supports is

$$L = \frac{1}{2}\sqrt{1 + \left(\frac{w_o l}{2H}\right)^2} + \frac{H}{w_o}\sinh\frac{w_o l}{2H} = l\left(1 + \frac{8}{3}\frac{f^2}{l^2} - \frac{32}{5}\frac{f^4}{l^4} + \frac{256}{7}\frac{f^6}{l^6} + \cdots\right) \tag{6.190}$$

If additional uniformly distributed load is applied to a parabolic cable, the change in sag is approximately

$$\Delta f = \frac{15}{16}\frac{l}{f}\frac{\Delta L}{5 - 24f^2/l^2} \tag{6.191}$$

For a rise in temperature t, the change in sag is about

$$\Delta f = \frac{15}{16}\frac{l^2 ct}{f(5 - 24f^2/l^2)}\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right) \tag{6.192}$$

where c = coefficient of thermal expansion.

Elastic elongation of a parabolic cable is approximately

$$\Delta L = \frac{Hl}{AE}\left(1 + \frac{16}{3}\frac{f^2}{l^2}\right) \tag{6.193}$$

where A = cross-sectional area of cable

E = modulus of elasticity of cable steel

H = horizontal component of tension in cable

If the corresponding change in sag is small, so that the effect on H is negligible, this change may be computed from

$$\Delta f = \frac{15}{16}\frac{Hl^2}{AEf}\frac{1 + 16f^2/3l^2}{5 - 24f^2/l^2} \tag{6.194}$$

For the general case of vertical dead load on a cable, the initial shape of the cable is given by

$$y_D = \frac{M_D}{H_D} \tag{6.195}$$

where M_D = dead-load bending moment that would be produced by load in a simple beam

H_D = horizontal component of tension due to dead load

For the general case of vertical live load on the cable, the final shape of the cable is given by

$$y_D + \delta = \frac{M_D + M_L}{H_D + H_L} \tag{6.196}$$

where δ = vertical deflection of cable due to live load

M_L = live-load bending moment that would be produced by live load in simple beam

H_L = increment in horizontal component of tension due to live load

Subtraction of Eq. (6.195) from Eq. (6.196) yields

$$\delta = \frac{M_L - H_L y_D}{H_D + H_L} \tag{6.197}$$

If the cable is assumed to take a parabolic shape, a close approximation to H_L may be obtained from

$$\frac{H_L}{AE} K = \frac{w_D}{H_D} \int_0^l \delta \, dx - \frac{1}{2} \int_0^l \delta'' \delta \, dx \tag{6.198}$$

$$K = l \left[\frac{1}{4} \left(\frac{5}{2} + \frac{16f^2}{l^2} \right) \sqrt{1 + \frac{16f^2}{l^2}} + \frac{3l}{32f} \log_e \left(\frac{4f}{l} + \sqrt{1 + \frac{16f^2}{l^2}} \right) \right] \tag{6.199}$$

where $\delta'' = d^2\delta/dx^2$.

If elastic elongation and δ'' can be ignored, Eq. (6.198) simplifies to

$$H_L = \frac{\int_0^l M_L \, dx}{\int_0^l y_D \, dx} = \frac{3}{2fl} \int_0^l M_L \, dx \tag{6.200}$$

Thus, for a load uniformly distributed horizontally $w_{L'}$

$$\int_0^l M_L \, dx = \frac{w_L l^3}{12} \tag{6.201}$$

and the increase in the horizontal component of tension due to live load is

$$H_L = \frac{3}{2fl} \frac{w_L l^3}{12} = \frac{w_L l^2}{8f} = \frac{w_L l^2}{8} \frac{8H_D}{w_D l^2} = \frac{w_L}{w_D} H_D \tag{6.202}$$

When a more accurate solution is desired, the value of H_L that is obtained from Eq. (6.202) can be used for an initial trial in solving Eqs. (6.197) and (6.198).

(S. P. Timoshenko and D. H. Young, "Theory of Structures," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; W. T. O'Brien and A. J. Francis, "Cable Movements under Two-Dimensional Loads," *Journal of the Structural Division, ASCE*, vol. 90, no. ST3, *Proceedings Paper* 3929, June 1964, pp. 98–123; W. T. O'Brien, "General Solution of Suspended Cable Problems," *Journal of the Structural Division, ASCE*, vol. 93, no. ST1, *Proceedings Paper* 5085, February 1967, pp. 1–26, www.asce.org; W. T. O'Brien, "Behavior of Loaded Cable Systems," *Journal of the Structural Division, ASCE*, vol. 94, no. ST10, *Proceedings Paper* 6162, October 1968, pp. 2281–2302, www.asce.org; G. R. Buchanan, "Two-Dimensional Cable Analysis," *Journal of the Structural Division, ASCE*, vol. 96, no. ST7, *Proceedings Paper* 7436, July 1970, pp. 1581–1587, www.asce.org.)

6.79 Cable Systems

Analysis of simple cables is described in Art. 6.77. Cables, however, may be assembled into many types of systems. One important reason for such systems is that roofs to be supported are two-or three-dimensional. Consequently, three-dimensional cable arrangements often are advantageous. Another important reason is that cable systems can be designed to offer much higher resistance to vibrations than simple cables do.

Like simple cables, cable systems behave nonlinearly. Thus, accurate analysis is difficult, tedious, and time-consuming. As a result, many designers use approximate methods that appear to have successfully withstood the test of time. Because of the numerous types of cable systems and the complexity of analysis, only general procedures are outlined here.

Cable systems may be stiffened or unstiffened. Stiffened systems are usually used for suspension bridges. Our discussion here deals only with unstiffened systems, that is, systems where loads are carried to supports only by cables. Stiffened cable systems are discussed in Art. 17.15.

Often, unstiffened systems may be classified as a network or as a cable truss, or double-layered plane system.

Networks consist of two or three sets of parallel cables intersecting at an angle (Fig. 6.95). The cables are fastened together at their intersections.

Cable trusses consist of pairs of cables, generally in a vertical plane. One cable of each pair is concave downward, the other concave upward (Fig. 6.96).

Cable Trusses ▪ Both cables of a cable truss are initially tensioned, or prestressed, to a predetermined shape, usually parabolic. The prestress is made large enough that any compression that may be induced in a cable by loads only reduces the tension in the cable; thus, compressive stresses

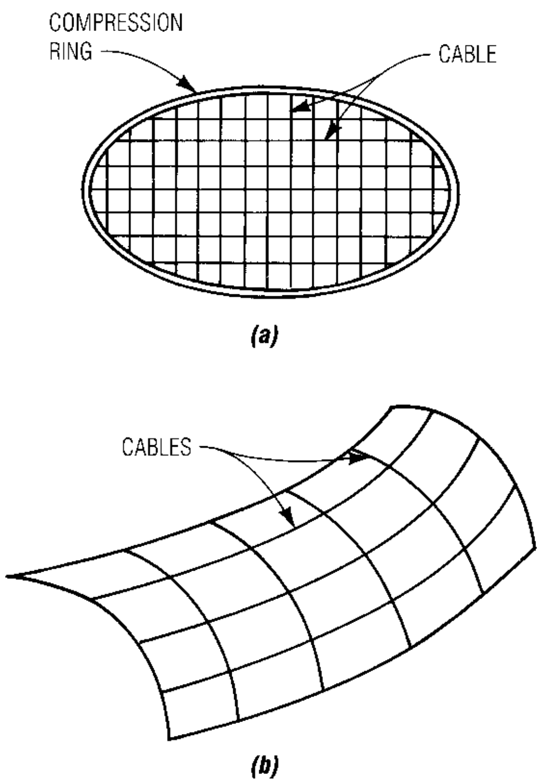


Fig. 6.95 Cable networks: (*a*) Cables forming a dish-shaped surface; (*b*) cables forming a saddle-shaped surface.

cannot occur. The relative vertical position of the cables is maintained by verticals, or spreaders, or by diagonals. Diagonals in the truss plane do not appear to increase significantly the stiffness of a cable truss.

Figure 6.96 shows four different arrangements of the cables, with spreaders, in a cable truss. The intersecting types (Fig. 6.96*b* and *c*) usually are stiffer than the others, for a given size of cables and given sag and rise.

For supporting roofs, cable trusses often are placed radially at regular intervals. Around the perimeter of the roof, the horizontal component of the tension usually is resisted by a circular or elliptical compression ring. To avoid a joint with a jumble of cables at the center, the cables usually are also connected to a tension ring circumscribing the center.

Properly prestressed, such double-layer cable systems offer high resistance to vibrations. Wind or other dynamic forces difficult or impossible to anticipate may cause resonance to occur in a single cable, unless damping is provided. The probability of resonance occurring may be reduced by increasing the dead load on a single cable. But this is not economical because the size of cable and supports usually must be increased as well. Besides, the tactic may not succeed, because future loads may be outside the design range. Damping, however, may be achieved economically with interconnected cables under different tensions, for example, with cable trusses or networks.

The cable that is concave downward (Fig. 6.96) usually is considered the load-carrying cable. If the prestress in that cable exceeds that in the other cable, the natural frequencies of vibration of both cables will always differ for any value of live load. To avoid resonance, the difference between the frequencies of the cables should increase with increase in load. Thus, the two cables will tend to assume different shapes under specific dynamic loads. As a consequence, the resulting flow of energy from one cable to the other will dampen the vibrations of both cables.

Natural frequency, cycles per second, of each cable may be estimated from

$$\omega_n = \frac{n\pi}{l}\sqrt{\frac{Tg}{w}} \tag{6.203}$$

where n = integer, 1 for fundamental mode of vibration, 2 for second mode, . . .

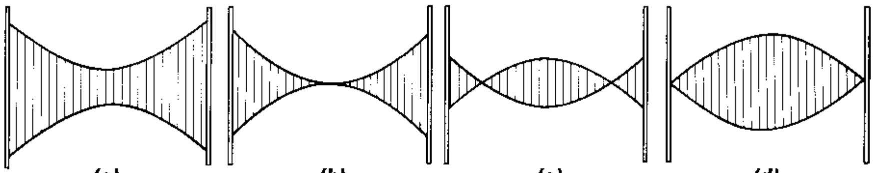


Fig. 6.96 Planar cable systems: (*a*) Completely separated cables; (*b*) cables intersecting at midspan; (*c*) crossing cables; (*d*) cables meeting at supports.

l = span of cable, ft

w = load on cable, kips/ft

g = acceleration due to gravity = 32.2 ft/s^2

T = cable tension, kips

The spreaders of a cable truss impose the condition that under a given load the change in sag of the cables must be equal. But the changes in tension of the two cables may not be equal. If the ratio of sag to span f/l is small (less than about 0.1), Eq. (6.194) indicates that, for a parabolic cable, the change in tension is given approximately by

$$\Delta H = \frac{16}{3}\frac{AEf}{l^2}\Delta f \qquad (6.204)$$

where Δf = change in sag

A = cross-sectional area of cable

E = modulus of elasticity of cable steel

Double cables interconnected with struts may be analyzed as discrete or continuous systems. For a discrete system, the spreaders are treated as individual members. For a continuous system, the spreaders are replaced by a continuous diaphragm that insures that the changes in sag and rise of cables remain equal under changes in load. Similarly, for analysis of a cable network, the cables, when treated as a continuous system, may be replaced by a continuous membrane.

(H. Mollman, "Analysis of Plane Prestressed Cable Structures," *Journal of the Structural Division, ASCE*, vol. 96, no. ST10, *Proceedings Paper* 7598, October 1970, pp. 2059–2082; D. P. Greenberg, "Inelastic Analysis of Suspension Roof Structures," *Journal of the Structural Division, ASCE*, vol. 96, no. ST5, *Proceedings Paper* 7284, May 1970, pp. 905–930; H. Tottenham and P. G. Williams, "Cable Net: Continuous System Analysis," *Journal of the Engineering Mechanics Division, ASCE*, vol. 96, no. EM3, *Proceedings Paper* 7347, June 1970, pp. 277–293, www.asce.org; A. Siev, "A General Analysis of Prestressed Nets," *Publications, International Association for Bridge and Structural Engineering*, vol. 23, pp. 283–292, Zurich, Switzerland, 1963; A. Siev, "Stress Analysis of Prestressed Suspended Roofs," *Journal of the Structural Division, ASCE*, vol. 90, no. ST4, *Proceedings Paper* 4008, August 1964, pp. 103–121; C. H. Thornton and C. Birnstiel, "Three-Dimensional Suspension Structures," *Journal of the Structural Division, ASCE*, vol. 93, no. ST2, *Proceedings Paper* 5196, April 1967, pp. 247–270, www.asce.org.)

Structural Dynamics

Article 6.2 noted that loads can be classified as static or dynamic and that the distinguishing characteristic is the rate of application of load. If a load is applied slowly, it may be considered static. Since dynamic loads may produce stresses and deformations considerably larger than those caused by static loads of the same magnitude, it is important to know reasonably accurately what is meant by slowly.

A useful definition can be given in terms of the natural period of vibration of the structure or member to which the load is applied. If the time in which a load rises from zero to its maximum value is more than double the natural period, the load may be treated as static. Loads applied more rapidly may be dynamic. Structural analysis and design for such loads are considerably different from and more complex than those for static loads.

In general, exact dynamic analysis is possible only for relatively simple structures, and only when both the variation of load and resistance with time are a convenient mathematical function. Therefore, in practice, adoption of approximate methods that permit rapid analysis and design is advisable. And usually, because of uncertainties in loads and structural resistance, computations need not be carried out with more than a few significant figures, to be consistent with known conditions.

6.80 Material Properties Under Dynamic Loading

In general, mechanical properties of structural materials improve with increasing rate of load application. For low-carbon steel, for example, yield strength, ultimate strength, and ductility rise with increasing rate of strain. Modulus of elasticity in the elastic range, however, is unchanged. For concrete, the dynamic ultimate strength in compression may be much greater than the static strength.

Since the improvement depends on the material and the rate of strain, values to use in dynamic analysis and design should be determined by tests approximating the loading conditions anticipated.

Under many repetitions of loading, though, a member or connection between members may fail because of "fatigue" at a stress smaller than the yield point of the material. In general, there is little apparent deformation at the start of a fatigue failure. A crack forms at a point of high stress concentration. As the stress is repeated, the crack slowly spreads, until the member ruptures without measurable yielding. Although the material may be ductile, the fracture looks brittle.

Endurance Limit ▪ Some materials (generally those with a well-defined yield point) have what is known as an **endurance limit**. This is the maximum unit stress that can be repeated, through a definite range, an indefinite number of times without causing structural damage. Generally, when no range is specified, the endurance limit is intended for a cycle in which the stress is varied between tension and compression stresses of equal value.

A range of stress may be resolved into two components: a steady, or mean, stress and an alternating stress. The endurance limit sometimes is defined as the maximum value of the alternating stress that can be superimposed on the steady stress an indefinitely large number of times without causing fracture.

Improvement of Fatigue Strength ▪ Design of members to resist repeated loading cannot be executed with the certainty with which members can be designed to resist static loading. Stress concentrations may be present for a wide variety of reasons, and it is not practicable to calculate their intensities. But sometimes it is possible to improve the fatigue strength of a material or to reduce the magnitude of a stress concentration below the minimum value that will cause fatigue failure.

In general, avoid design details that cause severe stress concentrations or poor stress distribution. Provide gradual changes in section. Eliminate sharp corners and notches. Do not use details that create high localized constraint. Locate unavoidable stress raisers at points where fatigue conditions are the least severe. Place connections at points where stress is low and fatigue conditions are not severe. Provide structures with multiple load paths or redundant members, so that a fatigue crack in any one of the several primary members is not likely to cause collapse of the entire structure.

Fatigue strength of a material may be improved by cold working the material in the region of stress concentration, by thermal processes, or by prestressing it in such a way as to introduce favorable internal stresses. Where fatigue stresses are unusually severe, special materials may have to be selected with high energy absorption and notch toughness.

(C. H. Norris et al., "Structural Design for Dynamic Loads," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; W. H. Munse, "Fatigue of Welded Steel Structures," Welding Research Council, 3 Park Avenue 27th floor, New York, NY 10016.)

6.81 Natural Period of Vibration

A preliminary step in dynamic analysis and design is determination of this period. It can be computed in many ways, including application of the laws of conservation of energy and momentum or Newton's second law of motion, $F = M(dv/dt)$, where F is force, M mass, v velocity, and t time. But in general, an exact solution is possible only for simple structures. Therefore, it is general practice to seek an approximate—but not necessarily inexact—solution by analyzing an idealized representation of the actual member or structure. Setting up this model and interpreting the solution requires judgment of a high order.

Natural period of vibration is the time required for a structure to go through one cycle of free vibration, that is, vibration after the disturbance causing the motion has ceased.

To compute the natural period, the actual structure may be conveniently represented by a system of masses and massless springs, with additional resistances provided to account for energy losses due to friction, hysteresis, and other forms of damping. In simple cases, the masses may be set equal to the actual masses; otherwise, equivalent masses may have to be computed (Art. 6.84). The spring constants are the ratios of forces to deflections.

For example, a single mass on a spring (Fig. 6.97*b*) may represent a simply supported beam with mass that may be considered negligible compared with the load W at midspan (Fig. 6.97*a*). The spring constant k should be set equal to the load that produces a unit deflection at midspan; thus, $k = 48EI/L^3$, where E is the modulus of elasticity, psi; I the moment of inertia, in^4; and L the span, in, of the beam. The idealized mass equals W/g, where W is the weight of the load, lb, and g is the acceleration due to gravity, 386 in/s^2.

Also, a single mass on a spring (Fig. 6.97*d*) may represent the rigid frame in Fig. 6.97*c*. In that case, $k = 2 \times 12EI/h^3$, where I is the moment of inertia, in^4, of each column and h the column height, in. The idealized mass equals the sum of the masses on the girder and the girder mass. (Weight of columns and walls is assumed negligible.)

6.81.1 Degree of a System

The spring and mass in Fig. 6.97*b* and *d* form a one-degree system. The degree of freedom of a system is determined by the least number of coordinates needed to define the positions of its components. In Fig. 6.97, only the coordinate y is needed to locate the mass and determine the state of the spring. In a two-degree system, such as one comprising two masses connected to each other and to the ground by springs and capable of movement in only one direction, two coordinates are required to locate the masses.

One-Degree System ▪ If the mass with weight W, lb, in Fig. 6.97 is isolated, as shown in Fig. 6.97*e*, it will be in dynamic equilibrium under the action of the spring force $-ky$ and the inertia force $(d^2y/dt^2)(W/g)$.

Hence, the equation of motion is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = 0 \qquad (6.205)$$

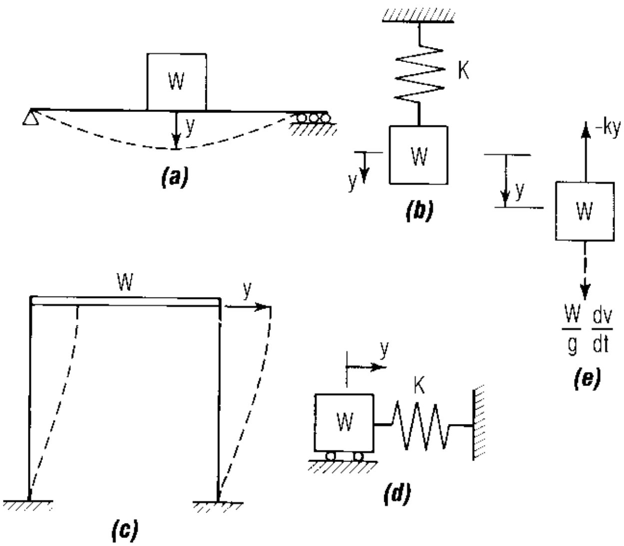


Fig. 6.97 Mass on weightless spring (*b*) or (*d*) may represent the motion of a beam (*a*) or a rigid frame (*c*) in free vibration.

This may be written in the more convenient form

$$\frac{d^2y}{dt^2} + \frac{kg}{W}y = \frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad (6.206)$$

The solution is

$$y = A \sin \omega t + B \cos \omega t \qquad (6.207)$$

where A and B are constants to be determined from initial conditions of the system, and

$$\omega = \sqrt{\frac{kg}{W}} \qquad (6.208)$$

is the natural circular frequency, radians per second.

The motion defined by Eq. (6.207) is harmonic. Its natural period in seconds is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{W}{gk}} \qquad (6.209)$$

Its natural frequency in cycles per second is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} \qquad (6.210)$$

If, at time $t = 0$, the mass has an initial displacement y_0 and velocity v_0, substitution in Eq. (6.207) yields $A = v_0/\omega$ and $B = y_0$. Hence, at any time t, the mass is completely located by

$$y = \frac{v_0}{\omega}\sin \omega t + y_0 \cos \omega t \qquad (6.211)$$

The stress in the spring can be computed from the displacement y, because the spring force equals $-ky$.

Multidegree Systems ▪ In multiple-degree systems, an independent differential equation of motion can be written for each degree of freedom. Thus, in an N-degree system with N masses, weighing $W_1, W_2, \ldots, W_N$, lb, and N^2 springs with constants k_{rj} ($r = 1, 2, \ldots, N$; $j = 1, 2, \ldots, N$), there are N equations of the form

$$\frac{W_r}{g}\frac{d^2y_r}{dt^2} + \sum_{j=1}^{N} k_{rj}y_j = 0 \quad r = 1, 2, \ldots, N \qquad (6.212)$$

Simultaneous solution of these equations reveals that the motion of each mass can be resolved into N harmonic components. They are called the fundamental, second, third, and so on harmonics. Each set of harmonics for all the masses is called a normal mode of vibration.

There are as many normal modes in a system as degrees of freedom. Under certain circumstances, the system could vibrate freely in any one of these modes. During any such vibration, the ratio of displacement of any two of the masses remains constant. Hence, the solutions of Eqs. (6.212) take the form

$$y_r = \sum_{n=1}^{N} a_{rn} \sin \omega_n(t + \tau_n) \qquad (6.213)$$

where a_{rn} and τ_n are constants to be determined from the initial conditions of the system and ω_n is the natural circular frequency for each normal mode.

6.81.2 Natural Periods

To determine ω_n set $y_1 = A_1$ sin ωt; $y_2 = A_2$ sin $\omega_t \ldots$ Then, substitute these and their second derivatives in Eqs. (6.212). After dividing each equation by sin ωt, the following N equations result:

$$\left(k_{11} - \frac{W_1}{g}\omega^2\right)A_1 + K_{12}A_2 + \cdots + k_{1N}A_N = 0$$

$$k_{21}A_1 + \left(k_{22} - \frac{W_2}{g}\omega^2\right)A_2 + \cdots + k_{2N}A_N = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$k_{N1}A_1 + k_{N2}A_2 + \cdots + \left(k_{NN} - \frac{W_N}{g}\omega^2\right)A_N = 0 \qquad (6.214)$$

If there are to be nontrivial solutions for the amplitudes $A_1, A_2, \ldots, A_N$, the determinant of their co-efficients must be zero. Thus,

$$\begin{vmatrix} k_{11} - \frac{W}{g}\omega^2 & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} - \frac{W}{g}\omega^2 & \cdots & k_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ k_{N1} & k_{N2} & \cdots & k_{NN} - \frac{W_N}{g}\omega^2 \end{vmatrix} = 0 \qquad (6.215)$$

Solution of this equation for ω yields one real root for each normal mode. And the natural period for each normal mode can be obtained from Eq. (6.209).

6.81.3 Modal Amplitudes

If ω for a normal mode now is substituted in Eqs. (6.214), the amplitudes $A_1, A_2, \ldots, A_N$ for that mode can be computed in terms of an arbitrary value, usually unity, assigned to one of them. The resulting set of modal amplitudes defines the **characteristic shape** for that mode.

The normal modes are mutually orthogonal; that is,

$$\sum_{r=1}^{N} W_r A_{rn} A_{rm} = 0 \qquad (6.216)$$

where W_r is the rth mass out of a total of N, A represents the characteristic amplitude of a normal mode, and n and m identify any two normal modes. Also, for a total of S springs

$$\sum_{s=1}^{S} k_s y_{sn} y_{sm} = 0 \qquad (6.217)$$

where k_s is the constant for the sth spring and y represents the spring distortion.

6.81.4 Stodola-Vianello Method

When there are many degrees of freedom, the preceding procedure for free vibration becomes very lengthy. In such cases, it may be preferable to solve Eqs. (6.214) by numerical, trial-and-error procedures, such as the Stodola-Vianello method, in which the solution converges first on the highest or lowest mode. Then, the other modes are determined by the same procedure after elimination of one of the equations by use of Eq. (6.216). The procedure requires assumption of a characteristic shape, a set of amplitudes A_{r1}. These are substituted in one of Eqs. (6.214) to obtain a first approximation of ω^2. With this value and with $A_{N1} = 1$, the remaining $(N-1)$ equations are solved to obtain a new set of A_{r1}. Then, the procedure is repeated until assumed and final characteristic amplitudes agree.

6.81.5 Rayleigh Method

Because even the Stodola-Vianello method is lengthy for many degrees of freedom, the Rayleigh approximate method may be used to compute the fundamental mode. The frequency obtained by this method, however, may be a little on the high side.

The Rayleigh method also starts with an assumed set of characteristic amplitudes A_{r1} and depends for its success on the small error in natural frequency produced by a relatively larger error in the shape assumption. Next, relative inertia forces acting at each mass are computed: $F_r = W_r A_{r1}/A_{N1}$, where A_{N1} is the assumed displacement at one of the masses. These forces are applied to the system as a static load and displacements B_{r1} due to them calculated. Then, the natural frequency can be obtained from

$$\omega^2 = \frac{g \sum_{r=1}^{N} F_r B_{r1}}{\sum_{r=1}^{N} W_r B_{r1}^2} \qquad (6.218)$$

where g is the acceleration due to gravity, 386 in/s^2. For greater accuracy, the computation can be repeated with B_{r1} as the assumed characteristic amplitudes.

When the Rayleigh method is applied to beams, the characteristic shape assumed initially may be chosen conveniently as the deflection curve for static loading.

The Rayleigh method may be extended to determination of higher modes by the Schmidt orthogonalization procedure, which adjusts assumed deflection curves to satisfy Eq. (6.216). The procedure is to assume a shape, remove components associated with lower modes, then use the Rayleigh method for the residual deflection curve. The computation will converge on the next higher mode. The method is shorter than the Stodola-Vianello procedure when only a few modes are needed.

For example, suppose the characteristic amplitudes A_{r1} for the fundamental mode have been obtained, and the natural frequency for the second mode is to be computed. Assume a value for the relative deflection of the rth mass A_{r2}. Then the shape with the fundamental mode removed will be defined by the displacements

$$a_{r2} = A_{r2} - c_1 A_{r1} \qquad (6.219)$$

where c_1 is the participation factor for the first mode.

$$c_1 = \frac{\sum_{r=1}^{N} W_r A_{r2} A_{r1}}{\sum_{r=1}^{N} W_r A_{r1}^2} \tag{6.220}$$

Substitute a_{r2} for B_{r1} in Eq. (6.218) to find the second-mode frequency and, from deflections produced by $F_r = W_r a_{r2}$, an improved shape. (For more rapid convergence, A_{r2} should be selected to make c_1 small.) The procedure should be repeated, starting with the new shape.

For the third mode, assume deflections A_{r3} and remove the first two modes:

$$a_{r3} = A_{r3} - c_1 A_{r1} - c_2 A_{r2} \tag{6.221}$$

The participation factors are determined from

$$c_1 = \frac{\sum_{r=1}^{N} W_r A_{r3} A_{r1}}{\sum_{r=1}^{N} W_r A_{r1}^2} \qquad c_2 = \frac{\sum_{r=1}^{N} W_r A_{r3} A_{r2}}{\sum_{r=1}^{N} W_r A_{r2}^2} \tag{6.222}$$

Use a_{r3} to find an improved shape and the third-mode frequency.

6.81.6 Distributed Mass

For some structures with mass distributed throughout, it sometimes is easier to solve the dynamic equations based on distributed mass than the equations based on equivalent lumped masses. A distributed mass has an infinite number of degrees of freedom and normal modes. Every particle in it can be considered a lumped mass on springs connected to other particles. Usually, however, only the fundamental mode is significant, although sometimes the second and third modes must be taken into account.

For example, suppose a beam weighs w lb/lin ft and has a modulus of elasticity E, psi, and moment of inertia I, in^4. Let y be the deflection at a distance x from one end. Then, the equation of motion is

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{w}{g} \frac{\partial^2 y}{\partial t^2} = 0 \tag{6.223}$$

(This equation ignores the effects of shear and rotational inertia.) The deflection y_n for each mode, to satisfy the equation, must be the product of a harmonic function of time $f_n(t)$ and of the characteristic shape $Y_n(x)$, a function of x with undetermined amplitude. The solution is

$$f_n(t) = c_1 \sin \omega_n t + c_2 \cos \omega_n t \tag{6.224}$$

where ω_n is the natural circular frequency and n indicates the mode, and

$$Y_n(x) = A_n \sin \beta_n x + \beta_n \cos \beta_n x + C_n \sin h\beta_n x + D_n \cos h\beta_n x \tag{6.225}$$

where

$$\beta_n = \sqrt[4]{\frac{w\omega_n^2}{EIg}} \tag{6.226}$$

Equations (6.224) to (6.226) apply to spans with any type of end restraints. Figure 6.98 shows the characteristic shape and gives constants for determination of natural circular frequency ω and natural period T for the first four modes of cantilever, simply supported, fixed-end, and fixed-hinged beams. To obtain ω, select the appropriate constant from Fig. 6.98 and multiply it by $\sqrt{EI/wL^4}$. To get T, divide the appropriate constant by $\sqrt{EI/wL^4}$.

6.81.7 Simple Beam

For a simple beam, the boundary (support) conditions for all values of time t are $y = 0$ and bending moment $M = EI\ \partial^2 y/\partial x^2 = 0$. Hence, at $x = 0$ and $x = L$, the span length, $Y_n(x) = 0$ and $d^2 Y_n/dx^2 = 0$. These conditions require that $B_n = C_n = D_n = 0$ and $\beta_n = n\pi/L$, to satisfy Eq. (6.225). Hence, according to Eq. (6.226), the natural circular frequency for a simply supported beam is

$$\omega_n = \frac{n^2 \pi^2}{L^2} \sqrt{\frac{EIg}{w}} \tag{6.227}$$

The characteristic shape is defined by

$$Y_n(x) = \sin \frac{n\pi x}{L} \tag{6.228}$$

The constants c_1 and c_2 in Eq. (6.224) are determined by the initial conditions of the disturbance. Thus, the total deflection, by superposition of modes, is

$$y = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi x}{L} \tag{6.229}$$

where $A_n(t)$ is determined by the load (see Art. 6.83).

TYPE OF SUPPORT	FUNDAMENTAL MODE	SECOND MODE	THIRD MODE	FOURTH MODE
CANTILEVER	L	0.774L	0.5L, 0.132L	0.356L, 0.094L, 0.644L
$\omega\sqrt{wL^4/EI}$ =	0.480	3.031	8.421	16.504
$T\sqrt{EI/wL^4}$ =	13.090	2.073	0.746	0.381
SIMPLE	L	0.5L	L/3, L/3	L/4, L/4, L/2
$\omega\sqrt{wL^4/EI}$ =	1.347	5.389	12.125	21.556
$T\sqrt{EI/wL^4}$ =	4.665	1.166	0.518	0.292
FIXED	L	L/2	0.359L	0.278L, L/2
$\omega\sqrt{wL^4/EI}$ =	3.031	8.421	16.504	27.283
$T\sqrt{EI/wL^4}$ =	2.073	0.746	0.381	0.230
FIXED-HINGED	L	0.56L	0.384L, 0.308L	0.294L, 0.235L, 0.529L
$\omega\sqrt{wL^4/EI}$ =	2.105	6.821	14.231	24.336
$T\sqrt{EI/wL^4}$ =	2.985	0.921	0.442	0.258

Fig. 6.98 Coefficients for computing natural circular frequencies ω and natural periods of vibration T, seconds, for prismatic beams: w = weight of beam, lb/lin ft; L = beam span, ft; E = modulus of elasticity, psi; I = moment of inertia, in^4.

To determine the characteristic shapes and natural periods for beams with variable cross section and mass, use the Rayleigh method. Convert the beam into a lumped-mass system by dividing the span into elements and assuming the mass of each element to be concentrated at its center. Also, compute all quantities, such as deflection and bending moment, at the center of each element. Start with an assumed characteristic shape and apply Eq. (6.218).

Methods are available for dynamic analysis of continuous beams. (R. Clough and J. Penzien, "Dynamics of Structures," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; D. G. Fertis and E. C. Zobel, "Transverse Vibration Theory," The Ronald Press Company). But even for beams with constant cross section, these procedures are very lengthy. Generally, approximate solutions are preferable. (J. M. Biggs, "Introduction to Structural Dynamics," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; N. M. Newmark and E. Rosenblueth, "Fundamentals of Earthquake Engineering," Prentice-Hall, Inc., Englewood Cliffs, N.J., www.prenhall.com.)

6.82 Impact and Sudden Loads

Under impact, there is an abrupt exchange or absorption of energy and drastic change in velocity. Stresses caused in the colliding members may be several times larger than stresses produced by the same weights applied statically.

An approximation of impact stresses in the elastic range can be made by neglecting the inertia of the body struck and the effect of wave propagation and assuming that the kinetic energy is converted completely into strain energy in that body. Consider a prismatic bar subjected to an axial impact load in tension. The energy absorbed per unit of volume when the bar is stressed to the proportional limit is called the **modulus of resilience**. It is given by $f_y^2/2E$, where f_y is the yield stress and E the modulus of elasticity, both in psi. Below the proportional limit, the stress, psi, due to an axial load U, in-lb, is

$$f = \sqrt{\frac{2UE}{AL}} \tag{6.230}$$

where A is the cross-sectional area, in^2, and L the length of bar, in.

This equation indicates that energy absorption of a member may be improved by increasing its length or area. Sharp changes in cross section should be avoided, however, because of associated high stress concentrations. Also, uneven distribution of stress in a member due to changes in section should be avoided. Energy absorption is larger with a uniform stress distribution throughout the length of the member.

If a static axial load W would produce a tensile stress f' in the bar and an elongation e', in, then the axial stress produced when W falls a distance h, in, is

$$f = f' + f'\sqrt{1 + \frac{2h}{e'}} \tag{6.231}$$

if f is within the proportional limit. The elongation due to this impact load is

$$e = e' + e'\sqrt{1 + \frac{2h}{e'}} \tag{6.232}$$

These equations indicate that the stress and deformation due to an energy load may be considerably larger than those produced by the same weight applied gradually.

The same equations hold for a beam with constant cross section struck by a weight at midspan, except that f and f' represent stresses at midspan and e and e', midspan deflections.

According to Eqs. (6.231) and (6.232), a sudden load ($h = 0$) causes twice the stress and twice the deflection as the same load applied gradually.

6.82.1 Impact on Long Members

For very long members, the effect of wave propagation should be taken into account. Impact is not transmitted instantly to all parts of the struck body. At first, remote parts remain undisturbed, while particles struck accelerate rapidly to the velocity of the colliding body. The deformations produced move through the struck body in the form of elastic waves. The waves travel with a constant velocity, ft/s,

$$c = 68.1\sqrt{\frac{E}{\rho}} \tag{6.233}$$

where E = modulus of elasticity, psi

ρ = density of the struck body, lb/ft^3

6.82.2 Impact Waves

If an impact imparts a velocity v, ft/s, to the particles at one end of a prismatic bar, the stress, psi, at that end is

$$f = 0.0147v\sqrt{E\rho} \tag{6.234}$$

if f is in the elastic range. In a compression wave, the velocity of the particles is in the direction of the wave. In a tension wave, the velocity of the particles is in the opposite direction to the wave.

In the plastic range, Eqs. (6.233) and (6.234) hold, but with E as the tangent modulus of elasticity. Hence, c is not a constant and the shape of the stress wave changes as it moves. The elastic portion of the stress wave moves faster than the wave in the plastic range. Where they overlap, the stress and irrecoverable strain are constant.

(The impact theory is based on an assumption difficult to realize in practice—that contact takes place simultaneously over the entire end of the bar.)

At a free end of a bar, a compressive stress wave is reflected as an equal tension wave, and a tension wave as an equal compression wave. The velocity of the particles at the free end equals $2v$.

At a fixed end of a bar, a stress wave is reflected unchanged. The velocity of the particles at the fixed end is zero, but the stress is doubled because of the superposition of the two equal stresses on reflection.

For a bar with a fixed end struck at the other end by a moving mass weighing W_m lb, the initial compressive stress, psi, is, from Eq. (6.234),

$$f_o = 0.0147 v_o \sqrt{E\rho} \tag{6.235}$$

where v_o is the initial velocity of the particles, ft/s, at the impacted end of the bar and E and ρ the modulus of elasticity, psi, and density, lb/ft^3, of the bar. As the velocity of W_m decreases, so does the pressure on the bar. Hence, decreasing compressive stresses follow the wave front. At any time $t < 2L/c$, where L is the length of the bar, in, the stress at the struck end is

$$f = f_o e^{-2\alpha t/\tau} \tag{6.236}$$

where $e = 2.71828$; α is the ratio of W_b, the weight of the bar, to W_m; and $\tau = 2L/c$.

When $t = \tau$, the wave front with stress f_o arrives back at the struck end, assumed still to be in contact with the mass. Since the velocity of the mass cannot change suddenly, the wave will be reflected as from a fixed end. During the second interval, $\tau < t < 2\tau$, the compressive stress is the sum of two waves moving away from the struck end and one moving toward this end.

Maximum stress from impact occurs at the fixed end. For α greater than 0.2, this stress is

$$f = 2f_o(1 + e^{-2\alpha}) \tag{6.237}$$

For smaller values of α, it is given approximately by

$$f = f_o\left(1 + \sqrt{\frac{1}{\alpha}}\right) \tag{6.238}$$

Duration of impact, time it takes for the stress at the struck end to drop to zero, is approximately

$$T = \frac{\pi L}{c\sqrt{\alpha}} \tag{6.239}$$

for small values of α.

When W_m is the weight of a falling body, velocity at impact is $\sqrt{2gh}$, when it falls a distance h, in. Substitution in Eq. (6.235) yields

$$f_o = \sqrt{2EhW_b/AL}$$

since $W_b = \rho AL$ is the weight of the bar. Putting $W_b = \alpha W_m$; $W_m/A = f'$, the stress produced by W_m when applied gradually, and $E = f'L/e'$, where e' is the elongation for the static load, gives

$$f_o = f'\sqrt{2h\alpha/e'}$$

Then, for values of α smaller than 0.2, the maximum stress, from Eq. (6.238), is

$$f = f'\left(\sqrt{\frac{2h\alpha}{e'}} + \sqrt{\frac{2h}{e'}}\right) \tag{6.240}$$

For larger values of α, the stress wave due to gravity acting on W_m during impact should be added to Eq. (6.237). Thus, for α larger than 0.2,

$$f = 2f'(1 - e^{-2\alpha}) + 2f'\sqrt{\frac{2h\alpha}{e'}}(1 + e^{-2\alpha}) \tag{6.241}$$

Equations (6.250) and (6.251) correspond to Eq. (6.231), which was developed without taking wave effects into account. For a sudden load, $h = 0$, Eq. (6.241) gives for the maximum stress $2f'(1 - e^{-2\alpha})$, not quite double the static stress, the result indicated by Eq. (6.231). (See also Art. 6.83.)

(S. Timoshenko and J. N. Goodier, "Theory of Elasticity," S. Timoshenko and D. H. Young, "Engineering Mechanics," and D. D. Barkan, "Dynamics of Bases and Foundations," McGraw-Hill Book Company, New York, books.mcgraw-hill.com.)

6.83 Dynamic Analysis of Simple Structures

Articles 6.81 and 6.82 present a theoretical basis for analysis of structures under dynamic loads. As noted in Art. 6.81, an approximate solution based on an idealized representation of an actual member or structure is advisable for dynamic analysis and design. Generally, the actual structure may be conveniently represented by a system of masses and massless springs, with additional resistances to account for damping. In simple cases, the masses may be set equal to the actual masses; otherwise, equivalent masses may be substituted for the actual masses (Art. 6.85). The spring constants are the ratios of forces to deflections (see Art. 6.81).

Usually, for structural purposes, the data sought are the maximum stresses in the springs and their maximum displacements and the time of occurrence of the maximums. This time generally is computed in terms of the natural period of vibration of the member or structure or in terms of the duration of the load. Maximum displacement may be calculated in terms of the deflection that would result if the load were applied gradually.

The term D by which the static deflection e', spring forces, and stresses are multiplied to obtain the dynamic effects is called the **dynamic load factor**. Thus, the dynamic displacement is

$$y = De' \tag{6.242}$$

and the maximum displacement y_m is determined by the maximum dynamic load factor D_m, which occurs at time t_m.

6.83.1 One-Degree System

Consider the one-degree-of-freedom system in Fig. 6.99*a*. It may represent a weightless beam with a mass weighing W lb applied at midspan and subjected to a varying force $F_o f(t)$, or a rigid frame with a mass weighing W lb at girder level and subjected to this force. The force is represented by an arbitrarily chosen constant force F_o times $f(t)$, a function of time.

If the system is not damped, the equation of motion in the elastic range is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = F_o f(t) \tag{6.243}$$

where k is the spring constant and g the acceleration due to gravity, 386 in/s^2. The solution consists of two parts. The first, called the complementary solution, is obtained by setting $f(t) = 0$. This solution is given by Eq. (6.211). To it must be added the second part, the particular solution, which satisfies Eq. (6.243).

The general solution of Eq. (6.243), arrived at by treating an element of the force-time curve (Fig. 6.99*b*) as an impulse, is

$$y = y_o \cos \omega t + \frac{v_o}{\omega} \sin \omega t + e'\omega \int_0^t f(\tau) \sin \omega(t - \tau)\, d\tau \tag{6.244}$$

where y = displacement of mass from equilibrium position, in

y_o = initial displacement of mass ($t = 0$), in

$\omega = \sqrt{kg/W}$ = natural circular frequency of free vibration

k = spring constant = force producing unit deflection, lb/in

v_o = initial velocity of mass, in/s

$e' = F_o/k$ = displacement under static load, in

A closed solution is possible if the integral can be evaluated.

Assume, for example, the mass is subjected to a suddenly applied force F_o that remains constant (Fig. 6.100*a*). If y_o and v_o are initially zero, the displacement y of the mass at any time t can be obtained from the integral in Eq. (6.244) by setting $f(\tau) = 1$:

$$y = e'\omega \int_0^t \sin \omega(t - \tau)\, d\tau = e'(1 - \cos \omega t) \tag{6.245}$$

The dynamic load factor $D = 1 - \cos \omega t$. It has a maximum value $D_m = 2$ when $t = \pi/\omega$. Figure

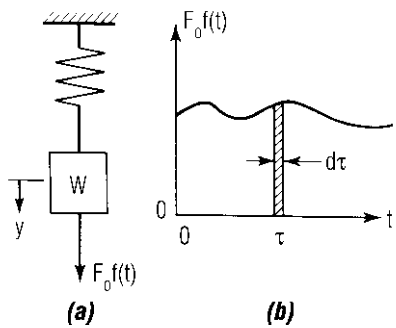


Fig. 6.99 One-degree system acted on by a varying force.

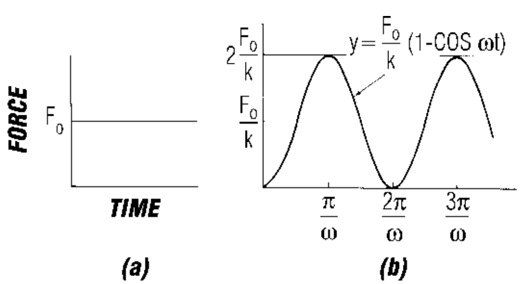


Fig. 6.100 Harmonic vibrations (*b*) result when a constant force (*a*) is applied to an undamped one-degree system like the one in Fig. 6.99*a*.

6.100*b* shows the variation of displacement with time.

6.83.2 Multidegree Systems

A multidegree lumped-mass system may be analyzed by the modal method after the natural frequencies of the normal modes have been determined (Art. 6.80). This method is restricted to linearly elastic systems in which the forces applied to the masses have the same variation with time. For other cases, numerical analysis must be used.

In the modal method, each normal mode is treated as an independent one-degree system. For each degree of the system, there is one normal mode. A natural frequency and a characteristic shape are associated with each mode. In each mode, the ratio of the displacements of any two masses is constant with time. These ratios define the characteristic shape. The modal equation of motion for each mode is

$$\frac{d^2A_n}{dt^2} + \omega_n^2 A_n = \frac{g f(t) \sum_{r=1}^{j} F_r \phi_{rn}}{\sum_{r=1}^{j} W_r \phi_{rn}^2} \tag{6.246}$$

where A_n = displacement in nth mode of arbitrarily selected mass

ω_n = natural frequency of nth mode

$F_r f(t)$ = varying force applied to rth mass

W_r = weight of rth mass

j = number of masses in system

ϕ_{rn} = ratio of displacement in nth mode of rth mass to A_n

g = acceleration due to gravity

We define the modal static deflection as

$$A_n' = \frac{g \sum_{r=1}^{j} F_r \phi_{rn}}{\omega_n^2 \sum_{r=1}^{j} W_r \phi_{rn}^2} \tag{6.247}$$

Then, the response for each mode is given by

$$A_n = D_n A_n' \tag{6.248}$$

where D_n is the dynamic load factor. Since D_n depends only on ω_n and $f(t)$, the variation of force with time, solutions for D_n obtained for one-degree systems also apply to multidegree systems. The total deflection at any point is the sum of the displacements for each mode, $\Sigma A_n \phi_{rn}$, at that point.

6.83.3 Response of Beams

The response of beams to dynamic forces can be determined in a similar way. The modal static deflection is defined by

$$A_n' = \frac{\int_0^L p(x)\phi_n(x)dx}{\omega_n^2 (w/g) \int_0^L \phi_n^2(x)dx} \tag{6.249}$$

where $p(x)$ = load distribution on span [$p(x)f(t)$ is varying force]

$\phi_n(x)$ = characteristic shape of nth mode (see Art. 6.81)

L = span length

w = uniformly distributed weight on span

The response of the beam then is given by Eq. (6.248) and the dynamic deflection is the sum of the modal components, $\Sigma A_n \phi_n(x)$.

Nonlinear Responses ▪ When the structure does not react linearly to loads, the equations of motion can be solved by numerical analysis if resistance is a unique function of displacement. Sometimes, the behavior of the structure can be represented by an idealized resistance displacement diagram that makes possible a solution in closed form. Figure 6.101*a* shows such a diagram.

6.83.4 Elastic-Plastic Response

Resistance is assumed linear ($R = ky$) until a maximum R_m is reached. After that, R remains equal to R_m for increases in y substantially larger than the displacement y_e at the elastic limit. Thus, some portions of the structure deform into the plastic range. Figure 6.101*a*, therefore, may be used for ductile structures only rarely subjected to severe dynamic loads. When this diagram can be used for designing such structures, more economical designs can be produced than for structures limited to the elastic range because of the high energy-absorption capacity of structures in the plastic range.

For a one-degree system, Eq. (6.243) can be used as the equation of motion for the initial sloping part

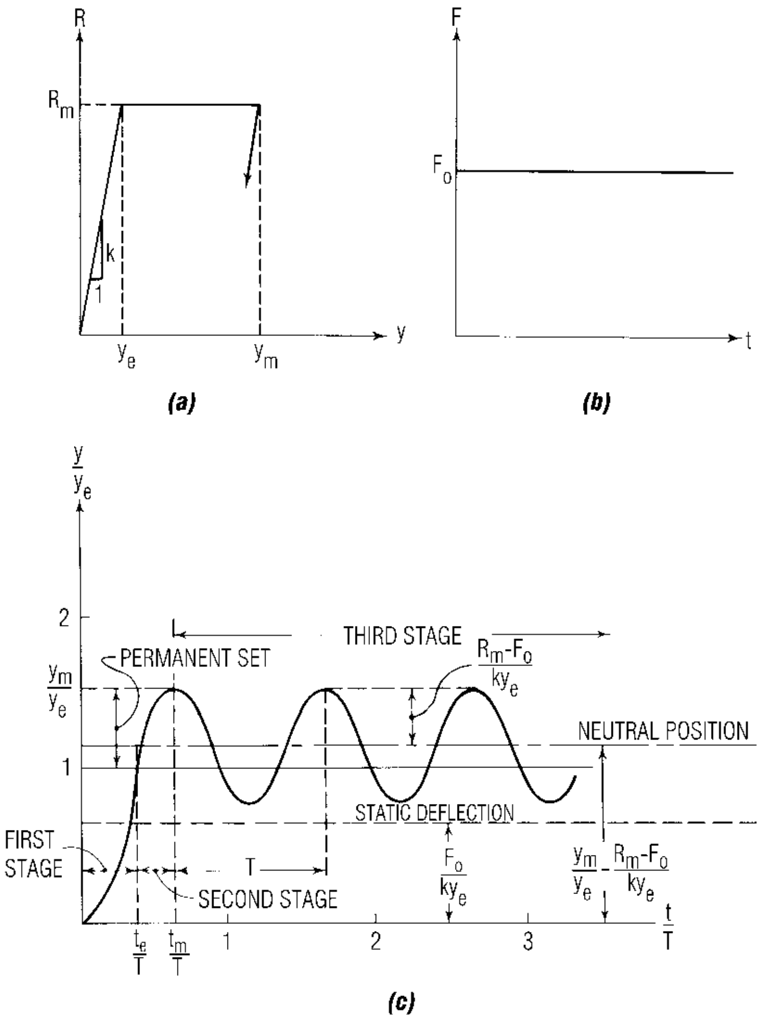


Fig. 6.101 Response in the elastic range of a one-degree system with resistance characteristics plotted in (*a*) to a constant force (*b*) is shown in (*c*).

of the diagram (elastic range). For the second stage, $y_e < y < y_m$ where y_m is the maximum displacement, the equation is

$$\frac{W}{g}\frac{d^2y}{dt^2} + R_m = F_o f(t) \tag{6.250}$$

For the unloading stage, $y < y_m$, the equation is

$$\frac{W}{g}\frac{d^2y}{dt^2} + R_m - k(y_m - y) = F_o f(t) \tag{6.251}$$

Suppose, for example, the one-degree undamped system in Fig. 6.99*a* behaves in accordance with the bilinear resistance function of Fig. 6.101*a* and is subjected to a suddenly applied constant load (Fig. 6.101*b*). With zero initial displacement and velocity, the response in the first stage ($y < y_e$), according to Eq. (6.245), is

$$\begin{aligned} y &= e'(1 - \cos \omega t_1) \\ \frac{dy}{dt} &= e'\omega \sin \omega t_1 \end{aligned} \tag{6.252}$$

Equation (6.245) also indicates that the displacement y_e will be reached at a time t_e such that $\cos \omega t_e = y_e/e'$.

For convenience, let $t_2 = t - t_e$ be the time in the second stage; thus, $t_2 = 0$ at the start of that stage. Since the condition of the system at that time is the same as the condition at the end of the first stage, the initial displacement is y_e and the initial velocity $e'\omega \sin \omega t_e$. The equation of motion is

$$\frac{W}{g}\frac{d^2y}{dt^2} + R_m = F_o \tag{6.253}$$

The solution, taking into account initial conditions after integrating, for $y_e < y < y_m$ is

$$y = \frac{g}{2W}(F_o - R_m)t_2^2 + e'\omega t_2 \sin \omega t_e + y_e \tag{6.254}$$

Maximum displacement occurs at the time

$$t_m = \frac{W\omega e'}{g(R_m - F_o)} \sin \omega t_e \tag{6.255}$$

and can be obtained by substituting t_m in Eq. (6.254).

The third stage, unloading after y_m has been reached, can be determined from Eq. (6.251) and conditions at the end of the second stage. The response, however, is more easily found by noting that the third stage consists of an elastic, harmonic residual vibration. In this stage, the amplitude of vibration is $(R_m - F_o)/k$ since this is the distance between the neutral position and maximum displacement, and in the neutral position the spring force equals F_o. Hence, the response, obtained directly from Eq. (6.245), is $y_m - (R_m - F_o)/k$ for e' because the neutral position, $y = y_m - (R_m - F_o)/k$, occurs when $\omega t_3 = \pi/2$. The solution is

$$y = y_m - \frac{R_m - F_o}{k} + \frac{R_m - F_o}{k} \cos \omega t_3 \tag{6.256}$$

where $t_3 = t - t_e - t_m$.

Response in the three stages is shown in Fig. 6.101*c*. In that diagram, however, to represent a typical case, the coordinates have been made nondimensional by expressing y in terms of y_e and the time in terms of T, the natural period of vibration.

(J. M. Biggs, "Introduction to Structural Dynamics" and R. Clough and J. Penzien, "Dynamics of Structures," McGraw-Hill Book Company, New York, books.mcgraw-hill.com; D. G. Fertis and E. C. Zobel, "Transverse Vibration Theory," The Ronald Press Company, New York; N. M. Newmark and E. Rosenbleuth, "Fundamentals of Earthquake, Engineering," Prentice-Hall, Inc., Englewood Cliffs, N.J., www.prenhall.com.)

6.84 Resonance and Damping

Damping in structures, due to friction and other causes, resists motion imposed by dynamic loads. Generally, the effect is to decrease the amplitude and lengthen the period of vibrations. If damping is large enough, vibration may be eliminated.

When maximum stress and displacement are the prime concern, damping may not be of great significance for short-time loads. These maximums usually occur under such loads at the first peak of response, and damping, unless unusually large, has little effect in a short period of time. But under conditions close to resonance, damping has considerable effect.

Resonance is the condition of a vibrating system under a varying load such that the amplitude of successive vibrations increases. Unless limited by damping or changes in the condition of the system, amplitudes may become very large.

Two forms of damping generally are assumed in structural analysis, viscous and constant (Coulomb). For viscous damping, the damping force is taken proportional to the velocity but opposite in direction. For Coulomb damping, the damping force is assumed constant and opposed in direction to the velocity.

6.84.1 Viscous Damping

For a one-degree system (Arts. 6.81 to 6.83), the equation of motion for a mass weighing W lb and subjected to a force F varying with time but opposed by viscous damping is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = F - c\frac{dy}{dt} \tag{6.257}$$

where y = displacement of mass from equilibrium position, in

k = spring constant, lb/in

t = time, s

c = coefficient of viscous damping

g = acceleration due to gravity = 386 in/s^2

Let us set $\beta = cg/2W$ and consider those cases in which $\beta < \omega$, the natural circular frequency [Eq. (6.208)], to eliminate unusually high damping (overdamping). Then, for initial displacement y_o and velocity v_o, the solution of Eq. (6.257) with $F = 0$ is

$$y = e^{-\beta t}\left(\frac{v_o + \beta y_o}{\omega_d}\sin \omega_d t + y_o \cos \omega_d t\right) \tag{6.258}$$

where $\omega_d = \sqrt{\omega^2 - \beta^2}$ and $e = 2.71828$. Equation (6.258) represents a decaying harmonic motion with β controlling the rate of decay and ω_d the natural frequency of the damped system.

When $\beta = \omega$

$$y = e^{-\omega t}[v_o t + (1 + \omega t)y_o] \tag{6.259}$$

which indicates that the motion is not vibratory. Damping producing this condition is called critical, and the critical coefficient is

$$c_d = \frac{2W\beta}{g} = \frac{2W\omega}{g} = 2\sqrt{\frac{kW}{g}} \tag{6.260}$$

Damping sometimes is expressed as a percent of critical (β as a percent of ω).

For small amounts of viscous damping, the damped natural frequency is approximately equal to the undamped natural frequency minus $\frac{1}{2}\beta^2/\omega$. For example, for 10% critical damping ($\beta = 0.1\omega$), $\omega_d = \omega[1 - \frac{1}{2}(0.1)^2] = 0.995\omega$. Hence, the decrease in natural frequency due to small amount of damping generally can be ignored.

Damping sometimes is measured by **logarithmic decrement**, the logarithm of the ratio of two consecutive peak amplitudes during free vibration.

$$\text{Logarithmic decrement} = \frac{2\pi\beta}{\omega} \tag{6.261}$$

For example, for 10% critical damping, the logarithmic decrement equals 0.2π. Hence, the ratio of a peak to the following peak amplitude is $e^{0.2\pi} = 1.87$.

The complete solution of Eq. (6.257) with initial displacement y_o and velocity v_o is

$$y = e^{-\beta t}\left(\frac{v_o + \beta y_o}{\omega_d}\sin \omega_d t + y_o \cos \omega_d t\right) + e'\frac{\omega^2}{\omega_d}\int_0^t f(\tau)e^{-\beta(1-\tau)} \sin \omega_d(t - \tau)\, d\tau \tag{6.262}$$

where e' is the deflection that the applied force would produce under static loading. Equation (6.262) is identical to Eq. (6.244) when $\beta = 0$.

Unbalanced rotating parts of machines produce pulsating forces that may be represented by functions of the form $F_o \sin \alpha t$. If such a force is applied to an undamped one-degree system, Eq. (6.244) indicates that if the system starts at rest the response will be

$$y = \frac{F_o g}{W}\left(\frac{1/\omega^2}{1 - \alpha^2/\omega^2}\right)\left(\sin \alpha t - \frac{\alpha}{\omega}\sin \omega t\right) \tag{6.263}$$

And since the static deflection would be $F_o/k = F_o g/W\omega^2$, the dynamic load factor is

$$D = \frac{1}{1 - \alpha^2/\omega^2}\left(\sin \alpha t - \frac{\alpha}{\omega}\sin \omega t\right) \tag{6.264}$$

If α is small relative to ω, maximum D is nearly unity; thus, the system is practically statically loaded. If α is very large compared with ω, D is very small; thus, the mass cannot follow the rapid fluctuations in load and remains practically stationary. Therefore, when α differs appreciably from ω, the effects of unbalanced rotating parts are not too serious. But if $\alpha = \omega$, resonance occurs; D increases with time. Hence, to prevent structural damage, measures must be taken to correct the unbalanced parts to change α, or to change the natural frequency of the vibrating mass, or damping must be provided.

The response as given by Eq. (6.263) consists of two parts, the free vibration and the forced part. When damping is present, the free vibration is of the form of Eq. (6.268) and is rapidly damped out. Hence, the free part is called the **transient response**, and the forced part, the **steady-state response**. The maximum value of the dynamic load factor for the steady-state response D_m is called the **dynamic magnification factor**. It is given by

$$D_m = \frac{1}{\sqrt{(1 - \alpha^2/\omega^2)^2 + (2\beta\alpha/\omega^2)^2}} \tag{6.265}$$

With damping, then, the peak values of D_m occur when

$$\alpha = \omega\sqrt{\frac{1 - \beta^2}{\omega^2}}$$

and are approximately equal to $\omega/2\beta$. For example, for 10% critical damping,

$$D_m = \frac{\omega}{0.2\omega} = 5$$

So even small amounts of damping significantly limit the response at resonance.

6.84.2 Coulomb Damping

For a one-degree system with Coulomb damping the equation of motion for free vibration is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = \pm F_f \qquad (6.266)$$

where F_f is the constant friction force and the positive sign applies when the velocity is negative. If initial displacement is y_o and initial velocity is zero, the response in the first half cycle, with negative velocity, is

$$y = \left(y_o - \frac{F_f}{k}\right)\cos\omega t + \frac{F_f}{k} \qquad (6.267)$$

equivalent to a system with a suddenly applied constant force. For the second half cycle, with positive velocity, the response is

$$y = \left(-y_o + 3\frac{F_f}{k}\right)\cos\omega\left(t - \frac{\pi}{\omega}\right) - \frac{F_f}{k}$$

If the solution is continued with the sign of F_f changing in each half cycle, the results will indicate that the amplitude of positive peaks is given by $y_o - 4nF_f/k$, where n is the number of complete cycles, and the response will be completely damped out when $t = ky_oT/4F_f$, where T is the natural period of vibration, or $2\pi/\omega$.

Analysis of the steady-state response with Coulomb damping is complicated by the possibility of frequent cessation of motion.

(S. Timoshenko, D. H. Young, and W. Weaver, "Vibration Problems in Engineering," 4th ed., John Wiley & Sons, Inc., New York, www.wiley.com; D. D. Barkan, "Dynamics of Bases and Foundations," McGraw-Hill Book Company, New York; W. C. Hurty and M. F. Rubinstein, "Dynamics of Structures," Prentice-Hall, Inc., Englewood Cliffs, N.J., www.prenhall.com.)

6.85 Approximate Design for Dynamic Loading

Complex analysis and design methods seldom are justified for structures subjected to dynamic loading because of lack of sufficient information on loading, damping, resistance to deformation, and other factors. In general, it is advisable to represent the actual structure and loading by idealized systems that permit a solution in closed form. (See Arts. 6.80 to 6.83.)

Whenever possible, represent the actual structure by a one-degree system consisting of an equivalent mass with massless spring. For structures with distributed mass, simplify the analysis in the elastic range by computing the response only for one or a few of the normal modes. In the plastic range, treat each stage—elastic, elastic-plastic, and plastic—as completely independent; for example, a fixed-end beam may be treated, when in the elastic-plastic stage, as a simply supported beam.

Choose the parameters of the equivalent system to make the deflection at a critical point, such as the location of the concentrated mass, the same as it would be in the actual structure. Stresses in the actual structure should be computed from the deflection in the equivalent system.

Compute an assumed shape factor ϕ for the system from the shape taken by the actual structure under static application of the loads. For example, for a simple beam in the elastic range with concentrated load at midspan, ϕ may be chosen, for $x < L/2$, as $(Cx/L^3)(3L^2 - 4x^2)$, the shape under static loading, and C may be set equal to 1 to make ϕ equal to 1 when $x = L/2$. For plastic conditions (hinge at midspan), ϕ may be taken as Cx/L, and C set equal to 2, to make $\phi = 1$ when $x = L/2$.

For a structure with concentrated forces, let W_r be the weight of the rth mass, ϕ_r the value of ϕ at the location of that mass, and F_r the dynamic force acting on W_r. Then, the equivalent weight of the idealized system is

$$W_e = \sum_{r=1}^{j} W_r\phi_r^2 \qquad (6.268)$$

where j is the number of masses. The equivalent force is

$$F_e = \sum_{r=1}^{j} F_r\phi_r \qquad (6.269)$$

For a structure with continuous mass, the equivalent weight is

$$W_e = \int w\phi^2 \, dx \tag{6.270}$$

where w is the weight in lb/lin ft. The equivalent force is

$$F_e = \int q\phi \, dx \tag{6.271}$$

for a distributed load q, lb/lin ft.

The resistance of a member or structure is the internal force tending to restore it to its unloaded static position. For most structures, a bilinear resistance function, with slope k up to the elastic limit and zero slope in the plastic range (Fig. 6.101*a*), may be assumed. For a given distribution of dynamic load, maximum resistance of the idealized system may be taken as the total load with that distribution that the structure can support statically. Similarly, stiffness is numerically equal to the total load with the given distribution that would cause a unit deflection at the point where the deflections in the actual structure and idealized system are equal. Hence, the equivalent resistance and stiffness are in the same ratio to the actual as the equivalent forces to the actual forces.

Let k be the actual spring constant, g the acceleration due to gravity, 386 in/s^2, and

$$W' = \frac{W_e}{F_e} \Sigma F \tag{6.272}$$

where ΣF represents the actual total load. Then, the equation of motion of an equivalent one-degree system is

$$\frac{d^2y}{dt^2} + \omega^2 y = g\frac{\Sigma F}{W'} \tag{2.273}$$

and the natural circular frequency is

$$\omega = \sqrt{\frac{kg}{W'}} \tag{6.274}$$

The natural period of vibration equals $2\pi/\omega$. Equations (6.273) and (6.274) have the same form as Eqs. (6.206), (6.208), and (6.243). Consequently, the response can be computed as indicated in Arts. 6.80 to 6.82.

Whenever possible, select a load-time function for ΣF to permit use of a known solution.

For preliminary design of a one-degree system loaded into the plastic range by a suddenly applied force that remains substantially constant up to the time of maximum response, the following approximation may be used for that response:

$$y_m = \frac{y_e}{2(1 - F_o/R_m)} \tag{6.275}$$

where y_e is the displacement at the elastic limit, F_o the average value of the force, and R_m the maximum resistance of the system. This equation indicates that for purely elastic response, R_m must be twice F_o; whereas, if y_m is permitted to be large, R_m may be made nearly equal to F_o, with greater economy of material.

For preliminary design of a one-degree system subjected to a sudden load with duration t_d less than 20% of the natural period of the system, the following approximation can be used for the maximum response:

$$y_m = \frac{1}{2} y_e \left[\left(\frac{F_o}{R_m} \omega t_d \right)^2 + 1 \right] \tag{6.276}$$

where F_o is the maximum value of the load and ω the natural frequency. This equation also indicates that the larger y_m is permitted to be, the smaller R_m need be.

For a beam, the spring force of the equivalent system is not the actual force, or reaction, at the supports. The real reactions should be determined from the dynamic equilibrium of the complete beam. This calculation should include the inertia force, with distribution identical with the assumed deflected shape of the beam. For example, for a simply supported beam with uniform load, the dynamic reaction in the elastic range is $0.39R + 0.11F$, where R is the resistance, which varies with time, and $F = qL$ is the load. For a concentrated load F at midspan, the dynamic reaction is $0.78R - 0.28F$. And for concentrated loads $F/2$ at each third point, it is $0.62R - 0.12F$. (Note that the sum of the coefficients equals 0.50, since the dynamic-reaction equations must hold for static loading, when $R = F$.) These expressions also can be used for fixed-end beams without significant error. If high accuracy is not required, they also can be used for the plastic range.

Structures usually are designed to resist the dynamic forces of earthquakes by use of equivalent static loads. (See Arts. 15.4 and 17.3.)

6.85.1 Basics of Structural Dynamics

The basic element in structural dynamics is the single-degree-of-freedom system. Many of the available vibration criteria utilize a strategy to simplify a complex floor system into this basic element. The single-degree-of-freedom system is represented by a single mass m, spring k, and damper c, as shown in Fig. 6.102. The governing differential equation of motion for this system follows.

Equation of motion for single degree of freedom:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F(t)$$

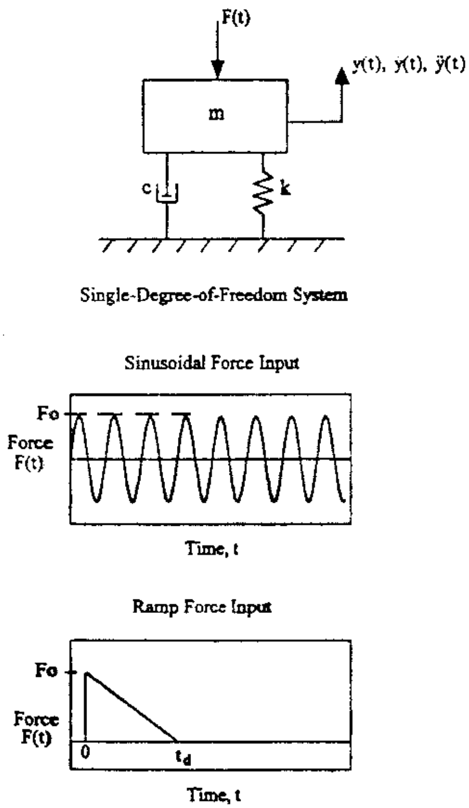


Fig. 6.102 Single-degree-of-freedom system and two common input forces.

When the mass m is subjected to a time-dependent input force $F(t)$, the result is a vibration response which can be described by the displacement $y(t)$, the velocity, $\dot{y}(t)$, and the acceleration, $\ddot{y}(t)$. The equation of motion for a single-degree-of-freedom system can also be formulated in terms of the natural frequency of the free vibration and ratio of critical damping.

$$\ddot{y}(t) + 2\zeta\omega_0\dot{y}(t) + \omega_0^2 y(t) = \frac{F(t)}{m} \tag{6.277}$$

where ω_0 = circular natural frequency, radians/s

$= \sqrt{k/m} = 2\pi f_0$

f_0 = natural frequency, Hz

ζ = ratio of critical damping

$= c/c_{cr}$

c_{cr} = critical damping, the value of damping for which the roots of the characteristic equation are equal

$$c_{cr} = 2\sqrt{km} \tag{6.278}$$

Also shown in Fig. 6.102 are two input forces commonly used to represent different sources of floor excitations. A sinusoidal force input function is often used to predict floor response due to rhythmic excitations. The ramp force input function is often used to assess the floor system response to transient excitations such as walking. The closed-form solutions for the response of a single-degree-of-freedom system subjected to these input forces can be found in most structural dynamics textbooks and will not be presented here.

Continuous systems, such as beams or plate-like structures, contain an infinite number of free vibration modes. Each of these modes can be characterized by its mode shape and its associated natural frequency. Figure 6.103 illustrates the first three modes of vibration for a simply supported beam with a uniform mass distribution. The vibration response at any point on a beam can be approximated by the sum of the individual modal contributions, truncated at some finite mode, at that point in space and time.

The fundamental natural frequency f_1 of a simply supported beam with a uniform mass distribution, as shown in Fig. 6.103, can also be conveniently expressed in terms of the static

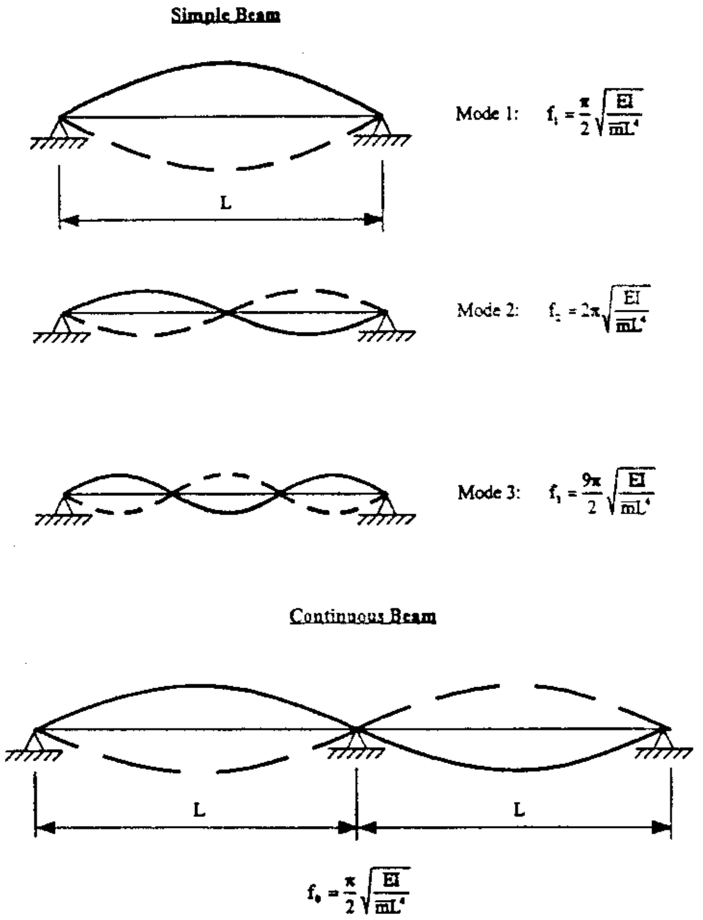


Fig. 6.103 Modes of vibration for beams with uniformly distributed mass.

deflection due to distributed weight. The derivation of this expression is as follows:

$$f_1 = \frac{\pi}{2}\sqrt{\frac{EI}{\bar{m}L^4}} = \frac{\pi}{2}\sqrt{\frac{5g}{384\Delta}} = 0.18\sqrt{\frac{g}{\Delta}} \tag{6.279}$$

$$\Delta = \frac{5wL^4}{384EI} = \frac{5g}{384}\frac{\bar{m}L^4}{EI} \tag{6.280}$$

$$\frac{EI}{\bar{m}L^4} = \frac{5g}{384\Delta} \tag{6.281}$$

where w = uniformly distributed load on a beam $= \bar{m} \cdot g$

$\bar{m}$ = uniformly distributed mass on beam

g = acceleration of gravity = 386.4 in/s^2 or 9800 mm/s^2

L = beam length

E = modulus of elasticity

I = moment of inertia for the beam cross section

The expression for the fundamental natural frequency, in terms of static deflection, is often misused in determining the natural frequency for other beam configurations. In particular, the expression f_1 above cannot be used for continuous beams. There is a common misconception that providing continuity of beams over a support will raise the fundamental frequency of the system. While it is true that continuity reduces the maximum static deflection, the fundamental natural frequency remains the same. This concept is illustrated in Fig. 6.103.

Platelike structures, such as beam and girder systems, also possess an infinite number of natural frequencies and mode shapes. Figure 6.104 illustrates the natural frequencies and mode shapes, for the first four modes, for a one-bay floor system comprised of a slab, joists, and girders. In addition to the mass distribution, the frequencies and mode shapes are affected by the slab, joist (or beam), and girder properties. This concept is explored in the following subsection. Close inspection of Fig. 6.104 and some intuition reveals that an activity like jumping at the center of the floor would cause dynamic amplitudes consisting of the superposition of modes 1, 4 and higher-order modes with a modal amplitude at that point.

One particular phenomenon to carefully consider and, if possible, avoid is that of resonance. Resonance occurs when a component of a harmonic excitation corresponds to one of the natural frequencies of the structure. Vibration amplitudes are greatly amplified in lightly damped structures such as steel floor systems.

6.85.2 Evaluation of Fundamental Natural Frequency for a Floor System

As illustrated in Fig. 6.104, the dynamic behavior of a floor system is very complex. There are, however,

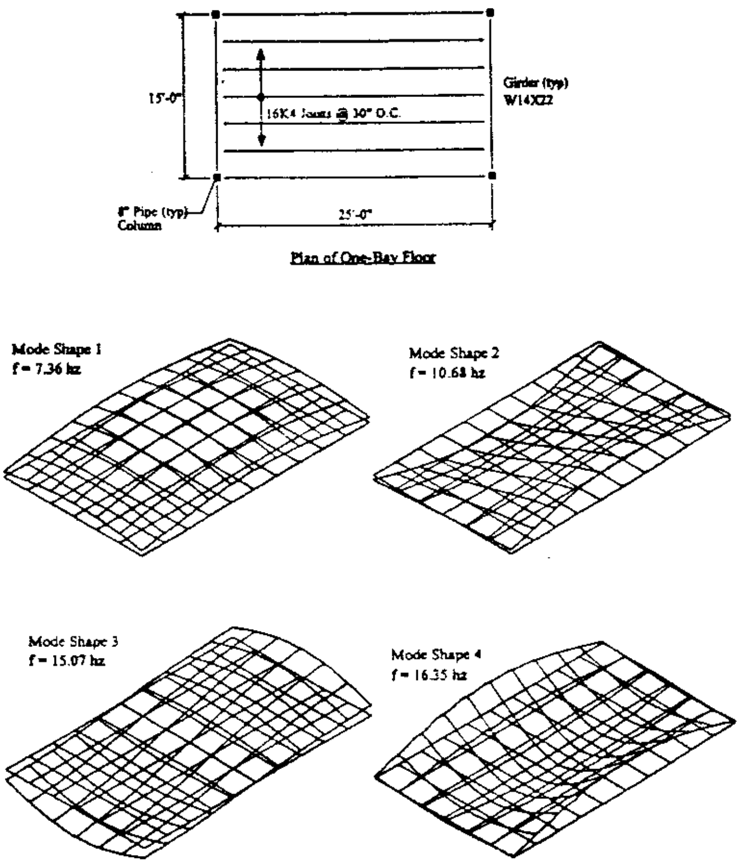


Fig. 6.104 Mode shapes and natural frequencies from a computer analysis.

commonly accepted procedures to determine dynamic characteristics of floor systems. The following discussion provides necessary information and a procedure to estimate the frequency of the first mode of free vibration for a steel floor system. A close approximation of the fundamental natural frequency of a floor system can be achieved by considering the frequencies of the major components of the floor system independently and then combining them as outlined in the procedure below.

Estimated System Frequency

$$\frac{1}{f_s^2} = \frac{1}{f_b^2} + \frac{1}{f_g^2} + \frac{1}{f_c^2} \tag{6.282}$$

where f_s = first natural frequency of the floor system, Hz

f_b = frequency of the beam or joist member, Hz; see equations below

f_g = frequency of the girder members; the lowest girder frequency should be used if the girder frequencies differ; the girder term in the system expression above can be neglected if the beams or joists are supported by a rigid support such as a wall

f_c = frequency of the column, Hz; except in unusual circumstances, this term is generally neglected; the movement of the columns is usually insignificant relative to the beam and girder motion

Beam or Joist Frequency

$$f_b = K\sqrt{\frac{gEI_t}{wL^4}} \tag{6.283}$$

where K = 1.57 for simply supported beams; 0.56 for cantilevered beams; refer to Murray and Hendrick for overhanging beams

g = acceleration of gravity; 386.4 in/s^2 or 9800 mm/s^2

E = modulus of elasticity for transformed section, 29,000 ksi for steel

I_t = transformed moment of inertia; when the steel deck supporting the concrete rests directly on the beam or joist (connected by welds, screws, mechanical shear connectors, etc.), assume composite action between the steel member and the concrete slab; see Sec. 9.3.4 for more information on the computation of composite member properties

w = floor weight per unit length of beam; value should be the actual expected service load on the beam; overestimating this value can result in a nonconservative prediction of acceptability; 10 percent to 25 percent of the live load used in strength calculations is suggested for design

L = beam or joist span

Girder Frequency

$$f_g = K\sqrt{\frac{gEI_t}{wL^4}} \tag{6.284}$$

where I_t = transformed moment of inertia

w = floor weight per unit length of girder; value should be the actual expected service load on the girder; loads from the beams or joists framing into the girder can usually be treated as continuous regardless of the spacing

Note: All other variables are as defined for the beam or joist frequency above.

Column Frequency

$$f_c = \frac{1}{2\pi}\sqrt{\frac{gAE}{PL}} \tag{6.285}$$

where A = area of the column section

P = load on the column; value should be the actual expected service load

L = length of column

Note: All other variables are as defined previously.