Lecture – 27

Linear Quadratic Regulator (LQR) Design – I

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LQR Design: Problem Objective

- To drive the state X of a linear (rather linearized) system $\boxed{\dot{X} = AX + BU}$ to the origin by minimizing the following quadratic performance index (cost function)

$$J = \frac{1}{2}\left(X_f^T S_f X_f\right) + \frac{1}{2}\int_{t_0}^{t_f}\left(X^T Q X + U^T R U\right) dt$$

where

$$S_f, Q \geq 0 \ \ \text{(psdf)}, \quad R > 0 \ \ \text{(pdf)}$$

LQR Design: Guideline for Selection of Weighting Matrices

$S_f \geq 0$ (psdf), $Q \geq 0$ (psdf), $R > 0$ (pdf)

These are usually chosen as diagonal matrices, with

s_{f_i} = maximum expected/acceptable value of $\left(1/x_{i_f}^2\right)$

q_i = maximum expected/acceptable value of $\left(1/x_i^2\right)$

r_i = maximum expected/acceptable value of $\left(1/u_i^2\right)$

LQR Design: Some Facts to Remember

- The pair $\{A, B\}$ needs to be controllable and the pair $\{A, \sqrt{Q}\}$ needs to be detectable

- $S_f \geq 0$ (psdf), $Q \geq 0$ (psdf), $R > 0$ (pdf)
  (these are usually chosen as diagonal matrices)

- By default, it is assumed that $t_f \to \infty$

- Constrained problems (state and control inequality constraints) are not considered here. Those will be considered later.

LQR Design: Problem Statement

- Performance Index (to minimize):

$$J = \underbrace{\frac{1}{2}\left(X_f^T S_f X_f\right)}_{\varphi(X_f)} + \int_{t_0}^{t_f} \underbrace{\frac{1}{2}\left(X^T Q X + U^T R U\right)}_{L(X,U)} dt$$

- Path Constraint: $\dot{X} = AX + BU$

- Boundary Conditions: $X(0) = X_0$: Specified

  t_f : Fixed, $X(t_f)$: Free

LQR Design:
Necessary Conditions of Optimality

- Terminal penalty: $\varphi\left(X_f\right)=\frac{1}{2}\left(X_f^T S_f X_f\right)$
- Hamiltonian: $H=\frac{1}{2}\left(X^T Q X+U^T R U\right)+\lambda^T\left(AX+BU\right)$
- State Equation: $\dot{X}=AX+BU$
- Costate Equation: $\dot{\lambda}=-\left(\partial H/\partial X\right)=-\left(QX+A^T\lambda\right)$
- Optimal Control Eq.: $\left(\partial H/\partial U\right)=0 \quad \Rightarrow \quad U=-R^{-1}B^T\lambda$
- Boundary Condition: $\lambda_f=\left(\partial\varphi/\partial X_f\right)=S_f X_f$

LQR Design: Derivation of Riccati Equation

Guess: $$\lambda(t) = P(t)\,X(t)$$

Justification:

From functional analysis theory of normed linear space, $\lambda(t)$ lies in the "dual space" of $X(t)$, which is the space consisting of all continuous linear functionals of $X(t)$.

Reference: Optimization by Vector Space Methods
D. G. Luenberger, John Wiley & Sons, 1969.

LQR Design: Derivation of Riccati Equation

Guess $\boxed{\lambda(t) = P(t)\,X(t)}$

$$\begin{aligned}
\dot{\lambda} &= \dot{P}X + P\dot{X} \\
&= \dot{P}X + P(AX + BU) \\
&= \dot{P}X + P\left(AX - BR^{-1}B^T\lambda\right) \\
&= \dot{P}X + P\left(AX - BR^{-1}B^T PX\right)
\end{aligned}$$

$$-\left(QX + A^T PX\right) = \left(\dot{P} + PA - PBR^{-1}B^T P\right)X$$

$$\left(\dot{P} + PA + A^T P - PBR^{-1}B^T P + Q\right)X = 0$$

LQR Design: Derivation of Riccati Equation

- Riccati equation

$$\dot{P} + PA + A^T P - PBR^{-1}B^T P + Q = 0$$

- Boundary condition

$$P(t_f)X_f = S_f X_f \quad (X_f \text{ is free})$$

$$P(t_f) = S_f$$

LQR Design: Solution Procedure

- Use the boundary condition $P(t_f) = S_f$ and integrate the Riccati Equation backwards from t_f to t_0
- Store the solution history for the Riccati matrix
- Compute the optimal control online

$$U = -\left(R^{-1} B^T P\right) X = -K\, X$$

LQR Design: Infinite Time Regulator Problem

Theorem (By Kalman)

As $t_f \to \infty,$ for constant Q and R matrices, $\dot{P} \to 0 \quad \forall t$

Algebraic Riccati Equation (ARE)

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

Note:

- ARE is still a nonlinear equation for the Riccati matrix. It is not straightforward to solve. However, efficient numerical methods are now available.
- A positive definite solution for the Riccati matrix is needed to obtain a stabilizing controller.

Example – 1:

Stabilization of Inverted Pendulum

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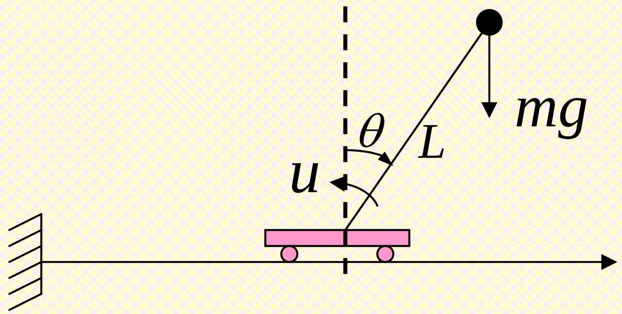
A Motivating Example:
Stabilization of Inverted Pendulum

System dynamics:

$$\ddot{\theta} = \omega_n^2\,\theta - u, \quad \omega_n^2 = g / L$$

(Linearized about vertical equilibrium point)



System dynamics (state space form):

Define: $x_1 = \theta,\ x_2 = \dot{\theta}$

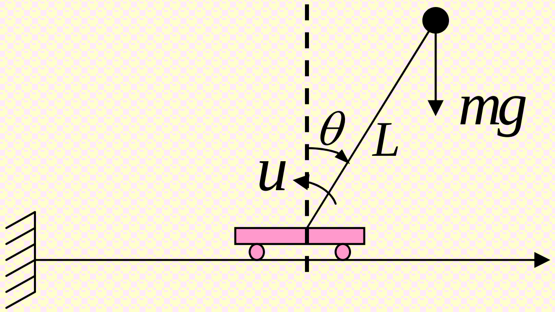
$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} 0 & 1 \\ \omega_n^2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B} u$$

Performance Index (to minimize):

$$J = \frac{1}{2}\int_0^\infty \left(\theta^2 + \frac{1}{c^2} u^2 \right) dt$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \frac{1}{c^2}$$

A Motivating Example:
Stabilization of Inverted Pendulum



ARE:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ (a symmetric matrix)

$$\begin{bmatrix} p_2\omega_n^2 & p_1 \\ p_3\omega_n^2 & p_2 \end{bmatrix} + \begin{bmatrix} p_2\omega_n^2 & p_3\omega_n^2 \\ p_1 & p_2 \end{bmatrix} - \begin{bmatrix} c^2 p_2^2 & c^2 p_2 p_3 \\ c^2 p_2 p_3 & c^2 p_3^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

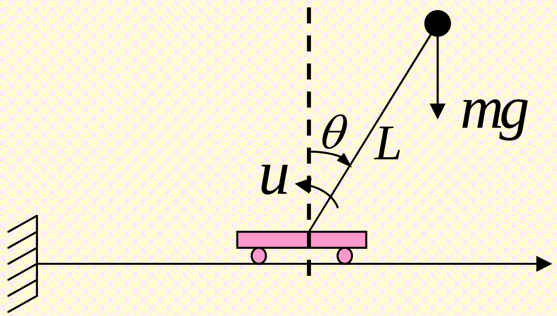
Equations:

$$2p_2\omega_n^2 - c^2 p_2^2 + 1 = 0 \quad \Rightarrow \quad p_2 = \frac{1}{c^2}\left[\omega_n^2 \pm \sqrt{\omega_n^4 + c^2}\right]$$

$$p_1 + p_3\omega_n^2 - c^2 p_2 p_3 = 0 \quad \text{(repeated)}$$

$$2p_2 - c^2 p_3^2 = 0 \quad \Rightarrow \quad p_3 = \pm\frac{1}{c}\sqrt{2p_2}$$

A Motivating Example:
Stabilization of Inverted Pendulum



However, p_3 is a diagonal term, which needs to be real and positive.

Hence, p_2 needs to be positive. Therefore

$$p_2 = \frac{1}{c^2}\left[\omega_n^2 + \sqrt{\omega_n^4 + c^2}\right], \quad p_3 = \frac{1}{c}\sqrt{2p_2}$$

Moreover,

$$p_1 + p_3\omega_n^2 - c^2 p_2 p_3 = 0$$

$$p_1 = c^2 p_2 p_3 - p_3\omega_n^2 \quad \text{(not needed in this problem)}$$

Gain Matrix:

$$K = R^{-1}B^T P = \begin{bmatrix} -c^2 p_2 & -c^2 p_3 \end{bmatrix}$$

Control:

$$u = -K\,X = c^2\left(p_2\theta + p_3\dot{\theta}\right)$$

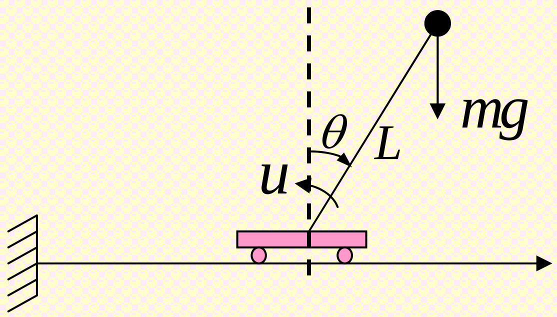
A Motivating Example:
Stabilization of Inverted Pendulum

Analysis

Open-Loop System:

$$\left|\lambda I - A\right| = \begin{vmatrix} \lambda & -1 \\ -\omega_n^2 & \lambda \end{vmatrix} = \lambda^2 - \omega_n^2 = 0$$

$$\lambda = \pm\,\omega_n \quad \left(\text{right half pole: unstable system}\right)$$



Closed-Loop System:

$$A_{CL} = A - BK = \begin{bmatrix} 0 & 1 \\ \omega_n^2 - c^2 p_2 & -c^2 p_3 \end{bmatrix}$$

Closed-Loop Poles:

$$\left|\lambda I - A_{CL}\right| = 0$$

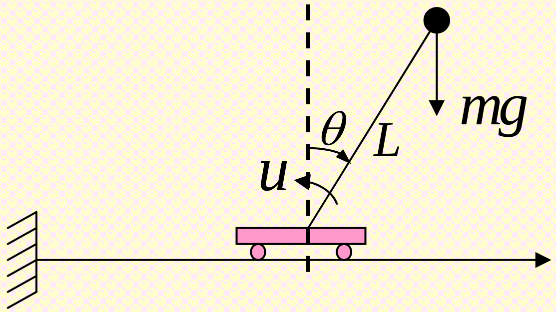
Define:

$$\omega^2 = \sqrt{\omega_n^4 + c^2}$$

$$p_2 = \frac{1}{c^2}\left(\omega_n^2 + \omega^2\right)$$

$$p_3 = \frac{1}{c}\sqrt{2 p_2} = \frac{\sqrt{2}}{c^2}\left(\omega_n^2 + \omega^2\right)^{1/2}$$

A Motivating Example:
Stabilization of Inverted Pendulum



Analysis

Closed-Loop Poles:

$$\lambda^2 + \sqrt{2}\left(\omega_n^2 + \omega^2\right)^{1/2}\lambda + \omega^2 = 0$$

$$\lambda_{1,2} = -\frac{1}{\sqrt{2}}\left(\omega_n^2 + \omega^2\right)^{1/2} \pm j\frac{1}{\sqrt{2}}\left(\omega^2 - \omega_n^2\right)^{1/2}$$

$$\left(\text{Note:} \quad \omega^2 = \sqrt{\omega_n^4 + c^2} \; > \; \omega_n^2\right)$$

Both of the closed-loop poles are strictly in the left-half plane.

Hence, the closed-loop is guaranteed to be "asymptotically stable".

Example – 2:

Finite-time Temperature Control

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Example: Finite Time Temperature Control Problem

System dynamics :

$$\dot{\theta} = -a(\theta - \theta_n) + bu$$

where

a, b : Constants

θ : Temperature

θ_n : Ambient temperature (Constant = 20^0C)

u : Heat input

Problem formulations

Case – 1:

Cost Function:

$$J = \frac{1}{2}\int_0^{t_f} u^2 dt$$

$$\theta(t_f) = \theta_f = 30^0 C$$

(Hard constraint)

Case – 2:

Cost Function:

$$J = \frac{1}{2}\left[s_f \left(\theta_f - 30\right)^2 + \int_0^{t_f} u^2 dt \right]$$

$s_f > 0$: Weightage

i.e. $\theta(t_f) \approx 30^0 C$

(Soft Constraint)

Solution:

Solution:

$$x \triangleq (\theta - \theta_a), \quad \theta(0) = \theta_a$$

$$\dot{x} = -ax + bu, \quad x(0) = (\theta_a - \theta_a) = 0$$

$$H = \frac{1}{2}u^2 + \lambda(-ax + bu)$$

$$\dot{\lambda} = -\left(\frac{\partial H}{\partial x}\right) = a\lambda$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u = -\lambda b$$

Necessary conditions
$\dot{x} = -ax + bu$
$\dot{\lambda} = a\lambda$
$u = -b\lambda$

Solution: Case – 1 (Hard constraint)

$$\lambda = e^{a\left(t-t_f\right)}\lambda_f = e^{-a\left(t_f-t\right)}\lambda_f$$

$$u = -be^{-a\left(t_f-t\right)}\lambda_f$$

$$\dot{x} = -ax - b^2\lambda_f e^{-a\left(t_f-t\right)}$$

Taking laplace transform:

$$\left[sX(s) - \underbrace{x(0)}_{0} \right] = -aX(s) - b^2\lambda_f e^{-at_f}\left(\frac{1}{s-a}\right)$$

Solution: Case – 1 (Hard constraint)

$$X(s) = -b^2 \lambda_f e^{-at_f} \left(\frac{1}{s^2 - a^2} \right)$$

$$= -b^2 \lambda_f e^{-at_f} \frac{1}{2a} \left(\frac{1}{s-a} - \frac{1}{s+a} \right)$$

Hence $x(t) = -b^2 \underbrace{\lambda_f}_{\text{Unknown}} e^{-at_f} \frac{1}{2a} \left(e^{at} - e^{-at} \right)$

However, $x(t_f) = \left(\theta_f - \theta_a \right) = 10^0 C$

Solution: Case – 1 (Hard constraint)

$$x(t_f) = 10 = -b^2 \lambda_f e^{-at_f} \frac{1}{2a}\left(e^{at_f} - e^{-at_f}\right)$$

$$10 = -\left(\frac{b^2 \lambda_f}{2a}\right)\left(1 - e^{-2at_f}\right)$$

$$\lambda_f = \frac{-20a}{b^2\left(1 - e^{-2at_f}\right)}$$

$$x(t) = -b^2\left(\frac{-20\,a}{b^2\left(1 - e^{-2at_f}\right)}\right)e^{-at_f}\frac{1}{2a}\left(e^{at} - e^{-at}\right) = \frac{10\left(e^{at} - e^{-at}\right)}{\left(e^{at_f} - e^{-at_f}\right)}$$

Solution: Case – 1 (Hard constraint)

Note :

$$x(t_f) = \frac{10\cancel{\left(e^{at_f} - e^{-at_f}\right)}}{\cancel{\left(e^{at_f} - e^{-at_f}\right)}} = 10$$

(i.e. The boundary condition is "<u>exactly met</u>".)

Controller :

$$u(t) = -\cancel{b}\,e^{-a\left(t_f - t\right)}\left[\frac{-20a}{b^{\cancel{2}}\left(1 - e^{-2at_f}\right)}\right] = \left[\frac{-20a\,e^{at}}{b\left(e^{at_f} - e^{-at_f}\right)}\right]$$

Solution: Case – 2 (Soft constraint)

$\theta_f \to 30^0 C \quad \Rightarrow x_f \to 10^0 C.$

Hence the cost function is

$$J = \frac{1}{2}\left[s_f \left(x_f - 10 \right)^2 + \int_0^{t_f} u^2 dt \right]$$

$$\lambda_f = s_f \left(x_f - 10 \right) \quad \Rightarrow \quad x_f = \left(\frac{\lambda_f}{s_f} + 10 \right)$$

However, we have

$$x(t) = -\frac{b^2}{2a} \lambda_f e^{-a t_f} \left(e^{at} - e^{-at} \right)$$

Solution: Case – 2 (Soft constraint)

At $t = t_f$, $\quad x\left(t_f\right) = -\dfrac{b^2}{2a}\lambda_f\left(1 - e^{-2at_f}\right) = \dfrac{\lambda_f}{s_f} + 10$

$$i.e. \quad \lambda_f\left[\frac{1}{s_f} + \frac{b^2}{2a}\left(1 - e^{-2at_f}\right)\right] = -10$$

$$i.e.\ \lambda_f = \left[\frac{-20 s_f a}{2a + s_f b^2\left(1 - e^{-2at_f}\right)}\right]$$

Hence $\lambda = e^{-a\left(t_f - t\right)}\lambda_f = e^{-a\left(t_f - t\right)}\left[\dfrac{-20 s_f a}{2a + s_f b^2\left(1 - e^{-2at_f}\right)}\right]$

Solution: Case – 2 (Soft constraint)

$$u(t) = -b\lambda$$

$$= -be^{-a(t_f - t)} \left[\frac{-20 s_f a}{2a + s_f b^2 \left(1 - e^{-2at_f}\right)} \right]$$

$$= -be^{-a(t_f - t)} \left[\frac{10 s_f abe^{at}}{ae^{at_f} + \frac{s_f b^2}{2}\left(e^{at_f} - e^{-at_f}\right)} \right]$$

Correlation between hard and soft constraint results

As $s_f \to \infty$,

$$\lim_{s_f \to \infty} u(t)\Big|_{S.C.} = \lim_{s_f \to \infty} \frac{10abe^{at}}{\left(\dfrac{1}{s_f}\right) ae^{at_f} + \dfrac{b^2}{2}\left(e^{at_f} - e^{-at_f}\right)}$$

$$= \frac{20ae^{at}}{b\left(e^{at_f} - e^{-at_f}\right)} \quad = u(t)\Big|_{H.C.}$$

i.e. The "soft constraint" problem behaves

like the "hard constraint" problem when $s_f \to \infty$.

Thanks for the Attention...!