Lecture – 35

Stability Analysis of Nonlinear Systems Using Lyapunov Theory – III

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Outline

- Review of Lyapunov Theorems
- LaSalle's Theorem
- Domain of Attraction

Review of Lyapunov Theorems

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Definitions

System Dynamics

$\dot{X} = f(X) \qquad f : D \rightarrow \mathbb{R}^n$ (a locally Lipschitz map)

D: an open and connected subset of $\mathbb{R}^n$

Equilibrium Point (X_e)

$$\dot{X}_e = f(X_e) = 0$$

Definitions

Stable Equilibrium

X_e is stable, provided for each $\varepsilon > 0, \ \exists \delta(\varepsilon) > 0:$

$$\left\| X(0) - X_e \right\| < \delta(\varepsilon) \quad \Rightarrow \quad \left\| X(t) - X_e \right\| < \varepsilon \quad \forall t \geq t_0$$

Unstable Equilibrium

If the above condition is not satisfied, then the equilibrium point is said to be unstable

Definitions

Convergent Equilibrium

$$\text{If } \exists \delta : \quad \left\| X(0) - X_e \right\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} X(t) = X_e$$

Asymptotically Stable

If an equilibrium point is both stable and convergent, then it is said to be asymptotically stable.

Definitions

Exponentially Stable

$$\exists\,\alpha,\lambda > 0: \quad \left\| X(t) - X_e \right\| \le \alpha \left\| X(0) - X_e \right\| e^{-\lambda t} \quad \forall t > 0$$

$$\text{whenever} \quad \left\| X(0) - X_e \right\| < \delta$$

Convention

The equilibrium point $X_e = 0$

(without loss of generality)

Lyapunov Stability Theorems

Theorem – 1 (Stability)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X), \; f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function

such that:

$(i) \quad V(0) = 0$

$(ii) \; V(X) > 0, \; in \; D - \{0\}$

$(iii) \; \dot{V}(X) \leq 0, \; in \; D - \{0\}$

Then $X = 0$ is "stable".

Lyapunov Stability Theorems

Theorem – 2 (Asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X), \; f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function

such that:

$(i) \quad V(0) = 0$

$(ii) \; V(X) > 0, \; in \; D - \{0\}$

$(iii) \, \dot{V}(X) < 0, \; in \; D - \{0\}$

Then $X = 0$ is "asymptotically stable".

Lyapunov Stability Theorems

Theorem – 3 (Globally asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X), \;\; f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function

such that:

$(i) \;\; V(0) = 0$

$(ii) \;\; V(X) > 0, \;\; in \; D - \{0\}$

$(iii) \; V(X)$ is "radially unbounded"

$(iv) \;\; \dot{V}(X) < 0, \;\; in \; D - \{0\}$

Then $X = 0$ is "globally asymptotically stable".

Lyapunov Stability Theorems

Theorem – 3 (Exponentially stable)

Suppose all conditions for asymptotic stability are satisfied.

In addition to it, suppose $\exists$ constants k_1, k_2, k_3, p :

$(i)\quad k_1 \|X\|^p \le V(X) \le k_2 \|X\|^p$

$(ii)\quad \dot{V}(X) \le -k_3 \|X\|^p$

Then the origin $X = 0$ is "exponentially stable".

Moreover, if these conditions hold globally, then the origin $X = 0$ is "globally exponentially stable".

Analysis of Linear Time Invariant System

System dynamics: $\dot{X} = AX, \quad A \in \mathbb{R}^{n \times n}$

Lyapunov function: $V(X) = X^T P X, \qquad P > 0 \ (\text{pdf})$

Derivative analysis:
$$\dot{V} = \dot{X}^T P X + X^T P \dot{X}$$
$$= X^T A^T P X + X^T P A X$$
$$= X^T \left(A^T P + PA \right) X$$

Analysis of Linear Time Invariant System

For stability, we aim for $\dot{V} = -X^T Q X \quad (Q > 0)$

By comparing $\quad X^T \left(A^T P + PA \right) X = -X^T Q X$

For a non-trivial solution

$$PA + A^T P + Q = 0$$

(Lyapunov Equation)

Analysis of Linear Time Invariant Systems

- Choose an arbitrary symmetric positive definite matrix $Q \quad (Q = I)$
- Solve for the matrix P form the *Lyapunov equation* and verify whether it is positive definite
- Result: If P is positive definite, then $\dot{V}(X) < 0$ and hence the origin is "asymptotically stable".

Lyapunov's Indirect Theorem

Let the linearized system about $X = 0$ be $\Delta\dot{X} = A(\Delta X)$.
The theorem says that if all the eigenvalues $\lambda_i \; (i = 1, \ldots, n)$ of the matrix A satisfy $\mathrm{Re}(\lambda_i) < 0$ (i.e. the linearized system is exponentially stable), then for the nonlinear system the origin is locally exponentially stable.

Instability theorem

Consider the autonomous dynamical system and assume $X=0$ is an equilibrium point. Let $V: D \to \mathbb{R}$ have the following properties:

$$(i)\ V(0)=0$$

$$(ii)\ \exists X_0 \in \mathbb{R}^n, \text{arbitrarily close to } X=0, \text{ such that } V\left(X_0\right)>0$$

$$(iii)\ \dot{V}>0 \quad \forall X \in U, \text{ where the set } U \text{ is defined as follows}$$

$$U=\{X \in D: \|X\| \leq \varepsilon \ \text{ and } \ V\left(X\right)>0\}$$

Under these conditions, $X=0$ is unstable

Construction of Lyapunov Functions

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Variable Gradient Method:

* Select a $\nabla V = \dfrac{\partial V}{\partial X} = g\left(X\right)$ that contains some adjustable parameters

* Then $dV\left(X\right) = \left(\dfrac{\partial V}{\partial X}\right)^T dX$

$$\int_{\tilde{X}=0}^{X} dV\left(\tilde{X}\right) = \int_{\tilde{X}=0}^{X} \left(\frac{\partial V}{\partial \tilde{X}}\right)^T d\tilde{X}$$

$$V\left(X\right) - V\left(0\right) = \int_{\tilde{X}=0}^{X} g\left(\tilde{X}\right) d\tilde{X}$$

> <u>Note:</u>
> To recover a unique V, $\nabla V = g(X)$ must satisfy the "<u>Curl Condition</u>":
> $i.e.\ \dfrac{\partial g_i}{\partial x_j} = \dfrac{\partial g_j}{\partial x_i}$

However, note that the intergal value depends on the initial and final states (not on the path followed). Hence, integration can be conveniently done along each of the co-ordinate axes in turn; i.e.

Variable Gradient Method:

$$V(X) = \int_0^{x_1} g_1(\tilde{x}_1, 0, \ldots\ldots, 0)\, d\tilde{x}_1$$

$$+ \int_0^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\, d\tilde{x}_2$$

$$\vdots$$

$$+ \int_0^{x_n} g_n(x_1, \ldots\ldots, x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n$$

Note: The free parameter of $g(X)$ are constrained to satisfy the symmetric condition, which is satisfied by all gradients of a scalar functions.

Variable Gradient Method:

Theorem: A function $g(X)$ is the gradient of a scalar function $V(X)$ if and only if the matrix $\left[\dfrac{\partial g(X)}{\partial X}\right]$ is symmetric; where

$$\left[\frac{\partial g(X)}{\partial X}\right] \triangleq \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_n}{\partial x_1} & \cdots & \dfrac{\partial g_n}{\partial x_n} \end{bmatrix}$$

Krasovskii's Method

Let us consider the system $\dot{X} = f(X)$

Let $A(X) \triangleq \left[\dfrac{\partial f}{\partial X}\right]$: Jacobian matrix

Theorem :

If the matrix $F(X) \triangleq A(X) + A^T(X)$ is ndf for all $X \in D \quad (0 \in D)$,

then the equilibrium point is locally asymptotically stable and a Lyapunov function for the system is

$$V(X) = f^T(X)\, f(X)$$

Note: If $D = \mathbb{R}^n$ and $V(X)$ is radially unbounded,

then the equilibrium point is globally asymptotically stable.

Krasovskii's Method

$$\dot{V}(X) = f^T \dot{f} + \dot{f}^T f$$

$$= f^T \left[\frac{\partial f}{\partial X}\right]^T \dot{X} + \dot{X}^T \left[\frac{\partial f}{\partial X}\right] f$$

$$= f^T \left(A^T + A\right) f$$

$$= f^T F \; f$$

Hence, if $F(X)$ is negative definite, $\dot{V}(X)$ is ndf.

So, by Lyapunov's theorem, $X = 0$ is asymptotically stable.

Generalized Krasovskii's Theorem

Theorem :

Let $$A(X) \triangleq \left[\frac{\partial f(X)}{\partial X} \right]$$

A sufficent condition for the origin to be asymptotically stable is that

$\exists$ two pdf matrices P and Q: $\forall X \neq 0$, the matrix

$$F(X) = A^T P + PA + Q$$

is negative semi-definite in some neighbourhood D of the origin.

In addition, if $D = \mathbb{R}^n$ and $V(X) \triangleq f^T(X)\, P\, f(X)$ is radially unbounded, then the system is globally asymptotically stable.

Generalized Krasovskii's Theorem

Proof : $V(X) = f^T(X)\, P f(X)$

$$\dot{V}(X) = \left[f^T P\, \dot{f} + \dot{f}^T P\, f \right]$$

$$= f^T P \left(\frac{\partial f}{\partial X}\right)^T \dot{X} + \left[\left(\frac{\partial f}{\partial X}\right)^T \dot{X}\right]^T P f$$

$$= f^T P A^T f + f^T A P\, f$$

$$= f^T \left(P A^T + A P + Q - Q\right) f$$

$$= \underbrace{f^T \left(P A^T + A P + Q\right) f}_{nsdf} - \underbrace{f^T Q f}_{ndf}$$

< 0 (ndf) Hence, the result.

Invariant and Limit Sets

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Invariant Set

A set M is said to be an "invariant set" with respect to the system $\dot{X} = f(X)$ if:

$$\boxed{X(0) \in M} \quad \Rightarrow \quad \boxed{X(t) \in M, \ \forall t > 0}$$

<u>Examples:</u>

(i) An equilibrium point ($M = X_e$)

(ii) Any trajectary of an autonomous system $\left(M = X(t)\right)$

Limit Set

Definition:

Let $X(t)$ be a trajectory of the dynamical system $\dot{X} = f(X)$ .Then the set N is called the limit set (or positive limit set) of $X(t)$ if for any $p \in N$, $\exists$ a sequence of times $\{t_n\} \in [0, \infty]$ such that $X(t_n) \to p$ as $t_n \to \infty$.

Note: Roughly, the limit set N of $X(t)$ is whatever $X(t)$ tends to in the limit.

Limit Set

Example:

(i) An asymptotically stable equilibrium point is the limit set of any solution starting from a close neighbourhood of the equilibrium point.

(ii) A stable limit cycle is the limit set for any solution starting sufficiently close to it

LaSalle's Theorem

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A Useful Theorem (Subset of LaSalle's Theorem)

Theorem : The equilibrium point $X = 0$ of the autonomous system $\dot{X} = f(X)$

is asymptotically stable if:

(i) $V(X) > 0$ (pdf) $\forall X \in D \quad [0 \in D]$

(ii) $\dot{V}(X) \leq 0$ (nsdf) in a bounded region $R \subset D$

(iii) $\dot{V}(X)$ does not vanish along any trajectory in R

other than the null solution $X = 0$

Morever,

If the above conditions hold good for R = $\mathbb{R}^n$ and $V(X)$ is radially unbounded,

then $X = 0$ is globally asymptotically stable.

Example

<u>Example:</u>

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - \alpha x_1 - \left(x_1 + x_2\right)^2 x_2$$

<u>Solution:</u> Let $V\left(X\right) = \alpha x^2{}_1 + x_2{}^2 \ , \ \alpha > 0$

$$\dot{V}\left(X\right) = \left(\frac{\partial V}{\partial X}\right)^T f\left(X\right)$$

$$= \left[2\alpha x_1 \quad 2x_2\right] \begin{bmatrix} x_2 \\ -x_2 - \alpha x_1 - \left(x_1 + x_2\right)^2 x_2 \end{bmatrix}$$

$$= 2\alpha x_1 x_2 - 2x_2^2 - 2\alpha x_1 x_2 - 2\left(x_1 + x_2\right)^2 x_2^2$$

Example

$$\dot{V}(X) = -2x_2{}^2\left[1+(x_1+x_2)^2\right]$$

$$\leq 0 \quad \text{(nsdf)}$$

$$\text{Now} \quad \dot{V}(X) = 0 \quad \forall t$$

$$\Leftrightarrow \quad x_2(t) = 0 \quad \forall t$$

$$\Rightarrow \quad \dot{x}_2 = 0$$

$$-x_2 - \alpha x_1 - (x_1+x_2)^2 x_2 = 0 \quad (\text{However, } x_2 = 0)$$

$$\therefore \quad x_1 = 0 \qquad \text{i.e.} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

Here we have :

(i) $\dot{V}(X)$ does not vanish along any trajectory

other than $X = 0$

(ii) $\dot{V} \leq 0$ in $\mathbb{R}^n$

(iii) $V(X)$ is radially unbounded,

Hence, the origin is **Globally asymptotically stable**.

LaSalle's Theorem

Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable (not necessarily pdf) function

and (i) $M \subset D$ be a compact set, which is

invariant with respect to the solution of $\dot{X} = f(X)$

(ii) $\dot{V} \leq 0$ in M

(iii) $E = \{X : X \in M \text{ and } \dot{V}(X) = 0\}$

i.e. E is the set of all points of $M : \dot{V} = 0$

(iv) N is the largest invariant set in E

Then Every solution starting in M approaches N as $t \rightarrow \infty$.

Lasalle's Theorem

Remarks:

(i) $V(X)$ is required only to be continuously differentiable

It need not be positive definite.

(ii) LaSalle's Theorem applies not only to equilibrium points, but also to more general dynamic behaviours such as limit cycles.

(iii) The earlier theorems (on asymptotic stability) can be derived as a corollary of this theorem.

Stability Analysis of a Limit Cycle Using LaSalle's theorem

Example:

$$\dot{x}_1 = x_2 + x_1\left(\beta^2 - x^2{}_1 - x_2{}^2\right)$$

$$\dot{x}_2 = -x_1 + x_2\left(\beta^2 - x^2{}_1 - x_2{}^2\right), \quad \beta > 0$$

Solution:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Morever,

$$\frac{d}{dt}\left(x^2{}_1 + x_2{}^2 - \beta^2\right)$$

$$= 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1\left[x_2 + x_1\left(\beta^2 - x^2{}_1 - x_2{}^2\right)\right]$$

$$+ 2x_2\left[-x_1 + x_2\left(\beta^2 - x^2{}_1 - x_2{}^2\right)\right]$$

Stability Analysis of a Limit Cycle Using LaSalle's theorem

$$= \ 2\left(x^2_1 + x_2^2\right)\left(\beta^2 - x^2_1 - x_2^2\right)$$

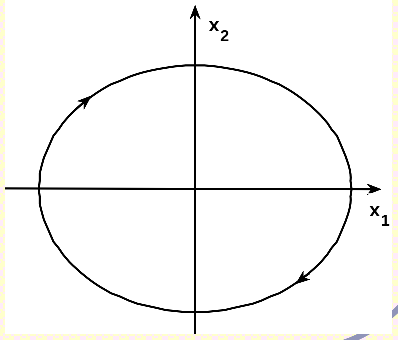
$$= 0 \quad \text{if} \quad x^2_1 + x_2^2 = \beta^2$$

$\therefore$ The set of points defined by $x^2_1 + x_2^2 = \beta^2$ is an invariant set ; i.e any trajectory starting on this circle at t_0 stays on the circle $\forall t \geq t_0$

The trajectories on this invariant set are the solution of :

$$\dot{X} = f(X)\Big|_{\left(x^2_1 + x_2^2 = \beta^2\right)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \Rightarrow \text{ A clock-wise motion}$$



Stability Analysis of a Limit Cycle Using LaSalle's theorem

Let $V(X) = \frac{1}{4}\left(x_1^2 + x_2^2 - \beta^2\right)^2$ [Note: $V(X) \geq 0$ in $\mathbb{R}^2$]

$$\dot{V}(X) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) \\ f_2(X) \end{bmatrix}$$

$$= \left(x_1^2 + x_2^2 - \beta^2\right) \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 + x_1\left(\beta^2 - x_1^2 - x_2^2\right) \\ -x_1 + x_2\left(\beta^2 - x_1^2 - x_2^2\right) \end{bmatrix}$$

$$= \left(x_1^2 + x_2^2 - \beta^2\right)\left(x_1^2 + x_2^2\right)\left(\beta^2 - x_1^2 - x_2^2\right)$$

$$= -\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - \beta^2\right)^2$$

≤ 0 Note: $\dot{V}(X) = -4\left(x_1^2 + x_2^2\right)V(X)$

Stability Analysis of a Limit Cycle Using LaSalle's theorem

Moreover $\dot{V}(X) = 0$

$\Leftrightarrow$ Either $\left(x^2_1 + x_2^{\;2}\right) = 0$ or $x^2_1 + x_2^{\;2} = \beta^2$

i.e Either $\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ or $\underbrace{x^2_1 + x_2^{\;2} = \beta^2}$

origin
Here, $\dot{X} = o$
(i.e it is an equilibrium point)

Circle of radius β
It is an invariant set
(i.e it is a limit cycle)

LaSalle's Theorem :

Step-1: For any $c > \beta$, let us define

$$M = \left\{ X \in \mathbb{R}^2 : V(X) \le c \right\}$$

By construction, M is closed and bounded

In this set, $\dot{V}(X) \le 0$
(and this is true $\forall\ X \in M$)
$\therefore\ M$ is an invariant set

Stability Analysis of a Limit Cycle Using LaSalle's theorem

Step-2 $\left[\text{To find } E = \left\{X \in M : \dot{V}(X) = 0\right\}\right]$

It is already shown that

$$E = (0,0) \cup \left\{X \in \mathbb{R}^2 : x^2{}_1 + x_2{}^2 = \beta^2\right\}$$

Step-3 [To find N: The largest invariant set in E]

Since both the subsets that constitute E are invariant,

$$N = E$$

Hence, By Lasalle's Theorem, every motion starting

in M converges either to the origin or to the limit cycle, $x^2{}_1 + x_2{}^2 = \beta^2$

Stability Analysis (of limit cycle)

Further analysis:

Note that $V(X) = \frac{1}{4}\left(x^2_1 + x_2^{\ 2} - \beta^2\right)^2$ is a measure of

distance of a point $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the limit cycle, since:

$$V(X) = 0 \quad , \text{ if } \quad x^2_1 + x_2^{\ 2} = \beta^2$$

$$\text{Also } \; V(X) = \left(\frac{\beta^4}{4}\right) \text{,if } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Stability Analysis of a Limit Cycle Using LaSalle's theorem

Selecting: (i) $\beta : \beta < \left(\beta^4 / 4\right)$, (i.e. $\beta > \sqrt[3]{4}$)

(ii) $c: \ \beta < c < \left(\beta^4 / 4\right)$

(iii) $M = \left\{ X \in \mathbb{R}^2 : V\left(X\right) \leq c \right\}$ (this excludes origin)

Then applying LaSalle's theorem, it follows that

any trajectory in M will converge to the limit cycle

$\Rightarrow$ The limit cycle is Convergent /Attractive.

Corollary:

Letting $\varepsilon \to 0^+$, this also shows that the origin is unstable!

Domain of Attraction

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Domain of Attraction

Definition: Let $\psi(X,t)$ be trajectories of $\dot{X} = f(X)$ with initial condition X at $t=0$ .Then the Domain of attraction is defined as

$$D_A \triangleq \{X \in D : \psi(X,t) \to X_e \text{ as } t \to \infty\}$$

Philosophy : Around any asymptotically stable equilibrium point, there is a domain of attraction.

Question : Can we estimate a domain of attraction ?

Ans: Yes!

Domain of Attraction

Example:

$$\dot{x}_1 = 3x_2$$

$$\dot{x}_2 = -5x_1 + x_1^3 - 2x_2$$

Eq. point:

$$x_2 = 0$$

$$x_1\left(-5 + x_1^2\right) = 0 \quad \Rightarrow \quad x_1 = 0\ ,\ \pm\sqrt{5}$$

$\therefore$ This system has three eq. points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{5} \\ 0 \end{bmatrix}$

Let us study the stability of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Define $V(X) = a\,x_1^2 - b\,x_1^4 + c\,x_1 x_2 + d\,x_2^2$

Domain of Attraction

where, a, b, c, d need to be choosen "appropriately".

$$\dot{V}(X) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} 3x_2 \\ -5x_1 + x_1^3 - 2x_2 \end{bmatrix}$$

$$= (3c - 4d)\,x_2^2 + (2d - 12b)\,x_1^3 x_2$$

$$+ (6a - 10d - 2c)\,x_1 x_2 + c x_1^4 - 5c\,x_1^2$$

Choose:

$$\left.\begin{array}{l} 2d - 12b = 0 \\ 6a - 10d - 2c = 0 \end{array}\right] \Rightarrow (a = 12, b = 1, c = d = 6) \quad \text{(one choice)}$$

Domain of Attraction

With this choice,

$$V(X) = 3(x_1 + 2x_2)^2 + 9x_1^2 + 3x_2^2 - x_1^4 \quad \text{(locally } pdf\text{)}$$

$$\dot{V}(X) = -6x_2^2 - 30x_1^2 + 6x_1^4 \quad \text{(locally } ndf\text{)}$$

Hence, the system is locally asymptotically stable.

Note: Here, $V(X) > 0$ and $\dot{V}(X) < 0$ as long as $-1.6 < x_1 < 1.6$

We may be tempted to conclude that $D = \left\{ X \in \mathbb{R}^2 : -1.6 < x_1 < 1.6 \right\}$

is a region of attraction .

Surprise : The conclusion is incorrect!

This is because D is NOT an invariant set

Theorem: Domain of Attraction

Theorem:

Let (i) X_e be an equilibrium point of the system $\dot{X} = f(X)$

(ii) $V(X): D \to \mathbb{R}$ be a continuously differentiable function

(iii) $M \subset D$ be a compact set containing X_e such that "M is invariant with respect to the solution of the system"

(iv) $\dot{V}$ is such that

$$\dot{V} < 0 \;\forall X \neq X_e \text{ in } M$$
$$= 0 \text{ if } X = X_e$$

Under these assumption, M is a subset of the domain of attraction, i.e. M is an estimate of domain of attraction.

Proof: In LaSalle's theorem, $E = \left\{ X : X \in M \;\;\&\;\; \dot{V} = 0 \right\} = X_e$. Hence the result !

Example....Contd.

$$V(X) = 12x_1^{\,2} - x_1^{\,4} + 6x_1x_2 + 6x_2^{\,2}$$

$$\begin{bmatrix} \text{Note:} \\ V(0) = 0 \\ \dot{V}(0) = 0 \end{bmatrix}$$

$$\dot{V}(X) = -6x_2^{\,2} - 30x_1^{\,2} + 6x_1^{\,4}$$

We already know that

$V(X) > 0$ and $\dot{V}(X) < 0$ happens in

$$D = \left\{ X \in \mathbb{R}^2 : -1.6 < x_1 < 1.6 \right\}$$

Domain of Attraction

Let us find the minimum of $V(X)$ along the very edge of this set (to restrict this set further).

Then

$$V\big|_{x_1=1.6} = 24.16 + 9.6x_2 + 6x_2^{\;2}$$

$$\frac{\partial}{\partial x_2}\left(V\big|_{x_1=1.6}\right) = 9.6 + 12x_2 = 0$$

$$\Rightarrow x_2 = \frac{-9.6}{12} = -0.8$$

Domain of Attraction

Similarly

$$\frac{\partial}{\partial x_2}\left(V\big|_{x_1=-1.6}\right)=\frac{\partial}{\partial x_2}\left(24.16-9.6x_2+6x_2{}^2\right)$$

$$=-9.6+12x_2=0$$

$$\Rightarrow x_2=0.8$$

Also $$\frac{\partial^2}{\partial x_2}\left(V\big|_{x_1=\pm 1.6}\right)=12>0$$

$\therefore V\left(X\right)$ has local minima when $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1.6 \\ -0.8 \end{bmatrix},\begin{bmatrix} -1.6 \\ 0.8 \end{bmatrix}$

Domain of Attraction

Moreover, $V(1.6, -0.8) = V(-1.6, -0.8) = 20.32$

(i.e. both the minimums are equal)

$\left[\text{Else, we need to choose the minimum of the two minimums.}\right]$

$\therefore M = \left\{ X \in D : V(X) \leq 20.32 - \varepsilon \right\} \subset D$ is an invariant set,

and hence, M is an estimate of the domain of attraction

Note: As long as $\varepsilon > 0$, the local minimums are excluded.

Hence $X(t) \to 0$ as long as it starts in M

An Interesting Result

Lemma

If a real function $V(t)$ satisfies the

in equality $\dot{V}(t) \leq -\alpha V(t) \quad , \quad \alpha \in \mathbb{R}$

Then $V(t) \leq e^{-\alpha t} V(0)$

Proof:

Let $Z(t) = \dot{V} + \alpha V$

then $\dot{V} + \alpha V = Z(t) \qquad \left(\text{Note: } Z(t) \leq 0 \right)$

An Interesting Result

Let us consider $Z(t)$ as an "external input"

to this "linear system"

Then

$$V(t) = e^{-\alpha t} V(0) + \underbrace{\int_0^t \underbrace{e^{-\alpha(t-\tau)}}_{\geq 0} \cdot 1 \cdot \underbrace{Z(\tau)}_{\leq 0} d\tau}_{\leq 0}$$

$$\therefore \quad \boxed{V(t) \leq e^{-\alpha t} V(0)}$$

References

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Thanks for the Attention….!