<u>*Lecture – 3*</u>

Classical Control Overview – I

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Review of Laplace Transforms

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Laplace Transform

Laplace Transform of $f(t)$:

$$F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt$$

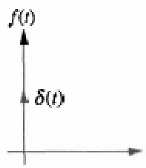
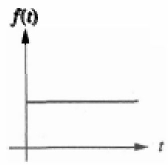
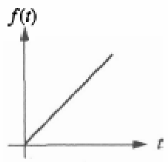
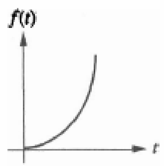
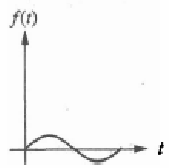
$\left(s = \sigma + j\omega : \text{ a complex variable}\right)$

Inverse Laplace Transform of $F(s)$:

$$L^{-1}\left[F(s)\right] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

$$= f(t)\, u(t) \qquad \text{where } u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Test Signals Commonly Used in Control Systems

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\, dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

Ref: N. S. Nise:
Control Systems Engineering, 4^{th} Ed., Wiley, 2004

Example – 1

$$L\left(t^n\right) = \int_0^\infty e^{-st} t^n dt \qquad \left(\text{by definition}\right)$$

Let $\quad v = st \quad \Rightarrow \quad dv = s\,dt$

$$L\left(t^n\right) = \int_0^\infty e^{-v} \left(\frac{v}{s}\right)^n \frac{dv}{s}$$

$$= \frac{1}{s^{n+1}} \underbrace{\int_0^\infty e^{-v} v^n dv}_{= n!\left(\text{by induction}\right)} = \frac{n!}{s^{n+1}}$$

Example – 2

$$L\left(e^{t}\right)=\int_{0}^{\infty} e^{-st}e^{t}\,dt \qquad \left(\text{by definition}\right)$$

$$=\int_{0}^{\infty} e^{-(s-1)t}\,dt$$

$$=\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_{0}^{\infty}=-\frac{1}{(s-1)}\left[0-1\right]$$

$$=\frac{1}{(s-1)}$$

Laplace Transform

Ref: N. S. Nise:
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Item no.	f(t)	F(s)
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform

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Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

Result : $L\left[e^{-at} f(t)\right] = F(s+a)$

$$L\left[e^{-at} f(t)\right] = \int_0^\infty e^{-st} e^{-at} f(t)\,dt = \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

Let $\hat{s} = s + a$

$$L\left[e^{-at} f(t)\right] = \int_0^\infty e^{-\hat{s}t} f(t)\,dt$$

$$= F(\hat{s}) \qquad \text{(by definition)}$$

$$= F(s+a)$$

Examples

(1) We know: $L(\sin 2t) = \dfrac{2}{s^2 + 2^2}$

Hence $L\left(e^{-3t}\sin 2t\right) = \dfrac{2}{(s+3)^2 + 2^2} = \dfrac{2}{s^2 + 6s + 13}$

(2) We know: $L(\cos 2t) = \dfrac{s}{s^2 + 2^2}$

Hence $L\left(e^{-3t}\cos 2t\right) = \dfrac{s+3}{(s+3)^2 + 2^2} = \dfrac{s+3}{s^2 + 6s + 13}$

Result : $L\left[t^n f(t)\right] = (-1)^n \dfrac{d^n F(s)}{ds^n}$

By definition $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Hence
$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty \left[e^{-st} f(t)\right] dt = \int_0^\infty \frac{d}{ds}\left[e^{-st} f(t)\right] dt$$
$$= \int_0^\infty -te^{-st} f(t)\,dt$$
$$= (-1)\int_0^\infty e^{-st}\left[t\, f(t)\right] dt$$
$$= (-1)\,L\left[t\, f(t)\right]$$

Result : $L\left[t^n f(t)\right] = (-1)^n \dfrac{d^n F(s)}{ds^n}$

Hence $L\left[t\, f(t)\right] = (-1)\dfrac{dF(s)}{ds}$

Similarly $L\left[t^2\, f(t)\right] = (-1)^2 \dfrac{d^2 F(s)}{ds^2}$

$$\vdots$$

$$\vdots$$

In general $L\left[t^n\, f(t)\right] = (-1)^n \dfrac{d^n F(s)}{ds^n}$

Result : $L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$

$$L\left[\frac{df(t)}{dt}\right] = \int_0^\infty e^{-st}\frac{df(t)}{dt}dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\, dt$$

$$= \left[0 - f(0)\right] + s\underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{F(s)}$$

$$= s\,F(s) - \underbrace{f(0)}_{=0\,(\text{Typically})}$$

Hence, multiplication by s is a derivative operator!

Generalization

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = L\left[\frac{d}{dt}\left(\frac{df(t)}{dt}\right)\right] = s\left[sF(s) - f(0)\right] - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

$$L\left[\frac{d^3 f(t)}{dt^3}\right] = s\left[s^2 F(s) - s f(0) - f'(0)\right] - f''(0)$$

$$= s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

Result : $L\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}F(s)$

Let $g(t) = \int_0^t f(\tau)\,d\tau$

Then $g(0) = 0, \quad g'(t) = f(t)$

$$F(s) = L[f(t)] = L[g'(t)] = sL[g(t)] - \underbrace{g(0)}_{=0} = s\,L\left[\int_0^t f(\tau)\,d\tau\right]$$

Hence $L\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}F(s)$

i.e. Division by s is an integral operator!

Transfer Function Representation

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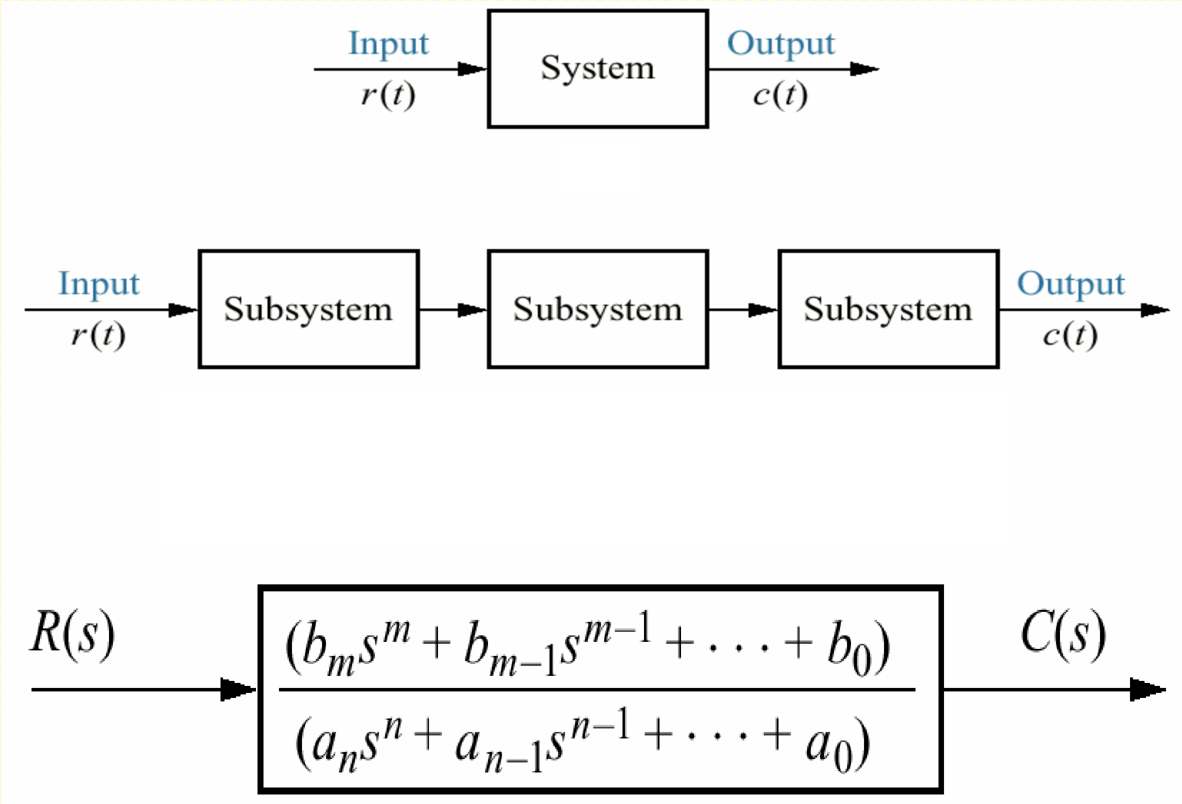
Block Diagram Representation



$$\frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

Transfer Function Representation

Any physical system that can be represented by a linear, time-invariant constant coefficient differential equation can be modeled as a Transfer function

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t)$$

$$= b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

$c(t)$: the output $r(t)$: the input

a_i's and b_i's are constants

Transfer Function Representation

- Taking Laplace Transform

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots$$

$$\cdots + a_0 C(s) + \text{initial condition terms}$$

$$= \quad b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots$$

$$\cdots + b_0 R(s) + \text{initial condition terms}$$

- Assume all initial conditions as zero (linear system)

Then the ratio

$$T(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

is called the TRANSFER FUNCTION

System Block Diagram

$$R(s) \longrightarrow \boxed{\frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}} \longrightarrow C(s)$$

Definitions:

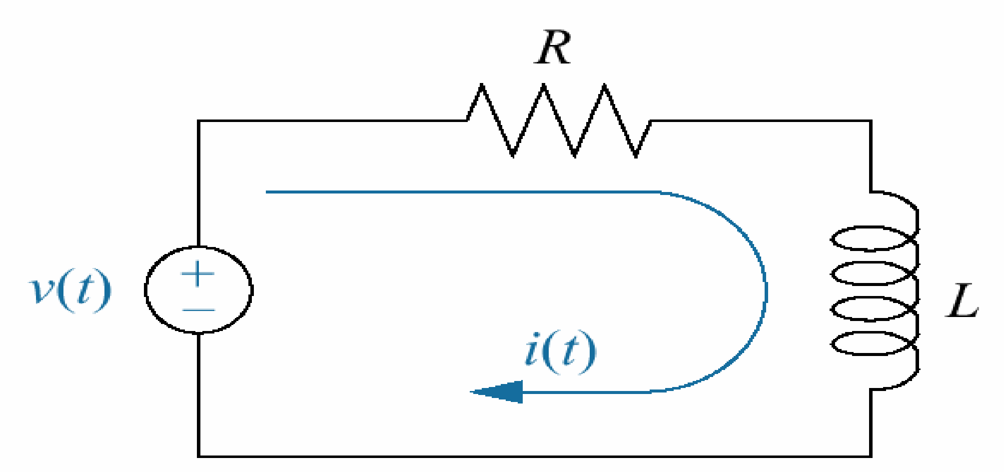
Roots of numerator: ZEROS

Roots of denominator: POLES

$m \leq n$: Proper Transfer Function

$m < n$: Strictly Proper Transfer Function

Example – 1:
Simple First Order System
(R-L Circuit)

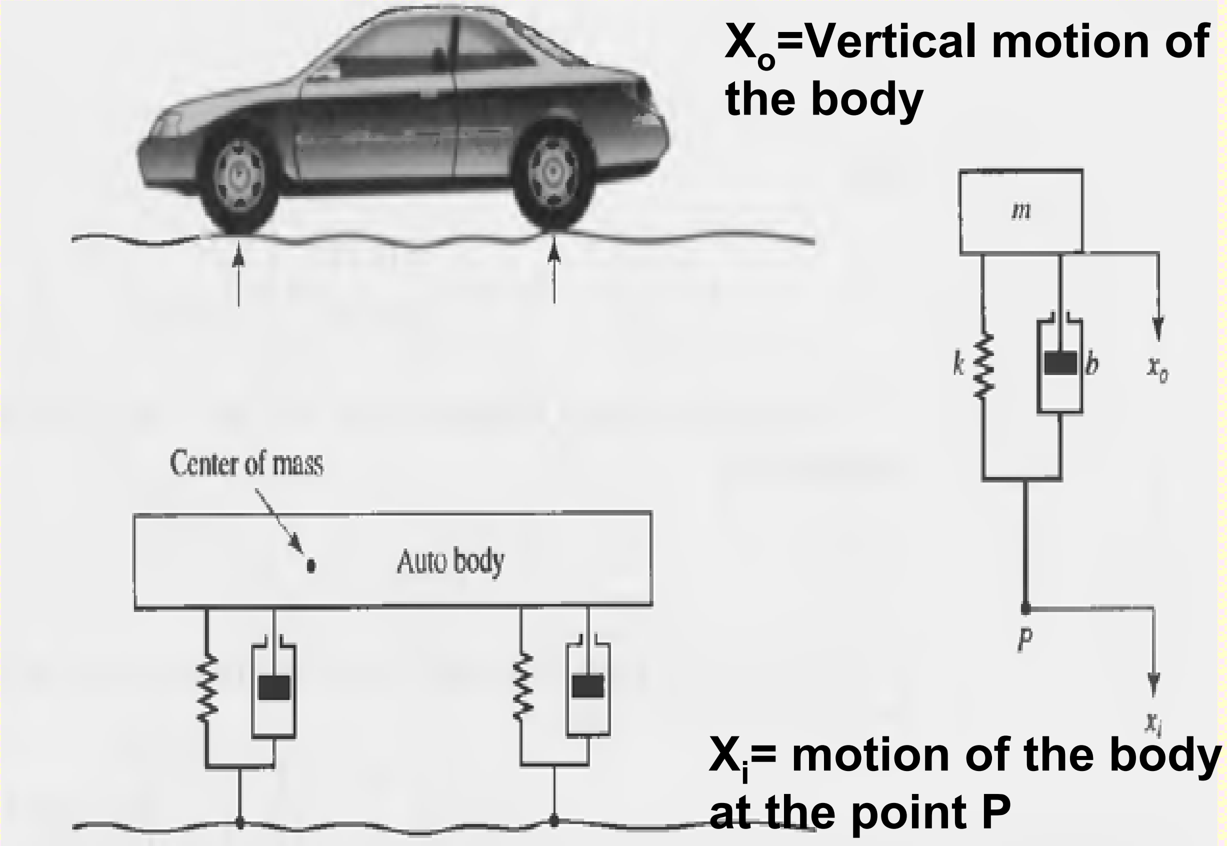


$$v(t) = L\frac{di(t)}{dt} + R\,i(t)$$

laplace transform

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R} \qquad pole = -R/L$$

Example – 2: (Second-order system) Transfer Function Modeling of Car Suspension System



X_o=Vertical motion of the body

X_i= motion of the body at the point P

Car Suspension System

$$m\ddot{x}_0 + b(\dot{x}_0 - \dot{x}_i) + k(x_0 - x_i) = 0$$

$$m\ddot{x}_0 + b\dot{x}_0 + kx_0 = b\dot{x}_i + kx_i$$

Taking Laplace Transform

$$(ms^2 + bs + k)X_0(s) = (bs + k)X_i(s)$$

Hence

$$T(s) = \frac{X_0(s)}{X_i(s)} = \frac{(bs + k)}{(ms^2 + bs + k)}$$

Response of First and Second Order Systems

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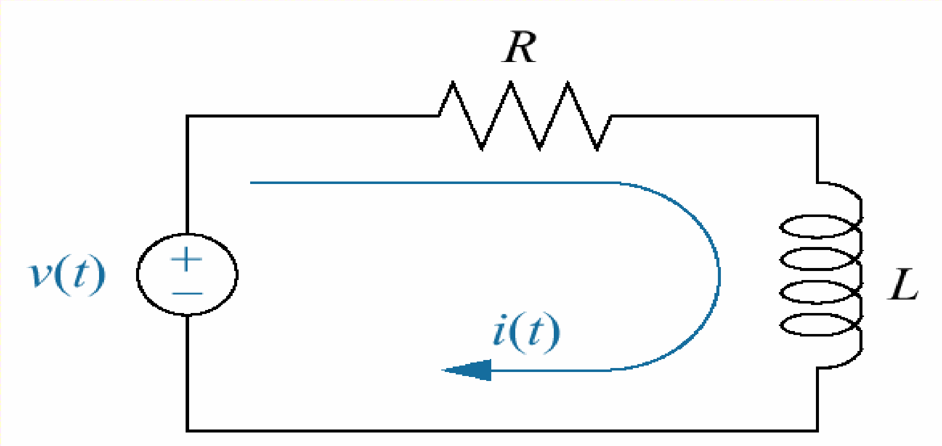
System Response: R-L Circuit

$$v(t) = L\frac{di(t)}{dt} + R\,i(t)$$

laplace transform

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R}$$



Pole location: $-R/L$

Let $L = 1H$ $R = 1\Omega$ and $v(t) = 1V$ (unit step)

System Response: R-L Circuit

$$\frac{I(s)}{V(s)} = \frac{1}{s+2}; \quad pole = -2 \qquad\qquad V(s) = \frac{1}{s}; \quad pole = 0$$

$$I(s) = \frac{1}{s(s+2)}$$

Partial fraction expansion

$$I(s) = \frac{A}{s} + \frac{B}{(s+2)} \qquad A = \left.\frac{1}{s+2}\right|_{s \to 0} \qquad B = \left.\frac{1}{s}\right|_{s \to -2}$$

Taking Inverse Laplace Transform

$$i(t) = \underbrace{\frac{1}{2}}_{\text{forced response}} - \underbrace{\frac{1}{2}e^{-2t}}_{\text{natural response}}$$

System Response: R-L Circuit

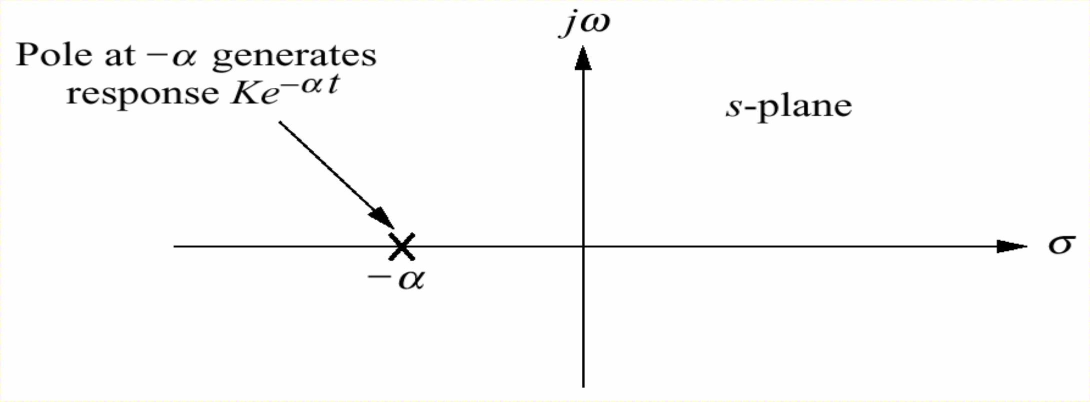
$$\text{Total response} = \underbrace{\text{Forced response}}_{\text{due to input}} + \underbrace{\text{Natural response}}_{\text{due to energy dissipation}}$$

- A Pole of the **input function** generates the form of the ***forced response***
- A Pole of the **system transfer function** generates the form of the ***natural response***
- The zeros and poles together generate the exact ***amplitudes*** for both forced and natural responses

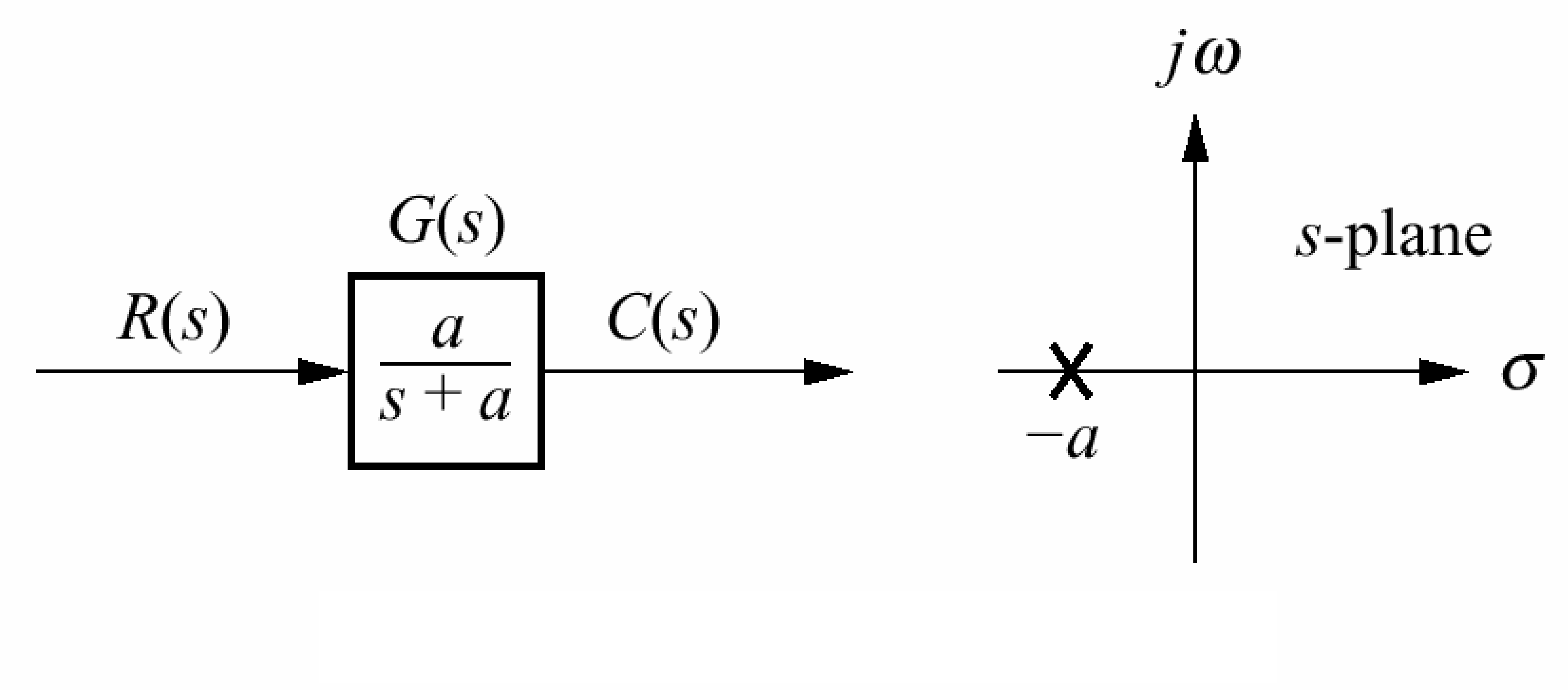
System Response: R-L Circuit

A system is stable if the natural response approaches zero as time approaches infinity.

This demands $e^{-\alpha t}$ *form* in the ***natural response*** that means **all the poles** should lie in **the left half** of the s-plane



First Order Systems



$\tau = (1/a)$: Time constant of the system

Unit Step Response of First-Order System

Output response for a unit step input

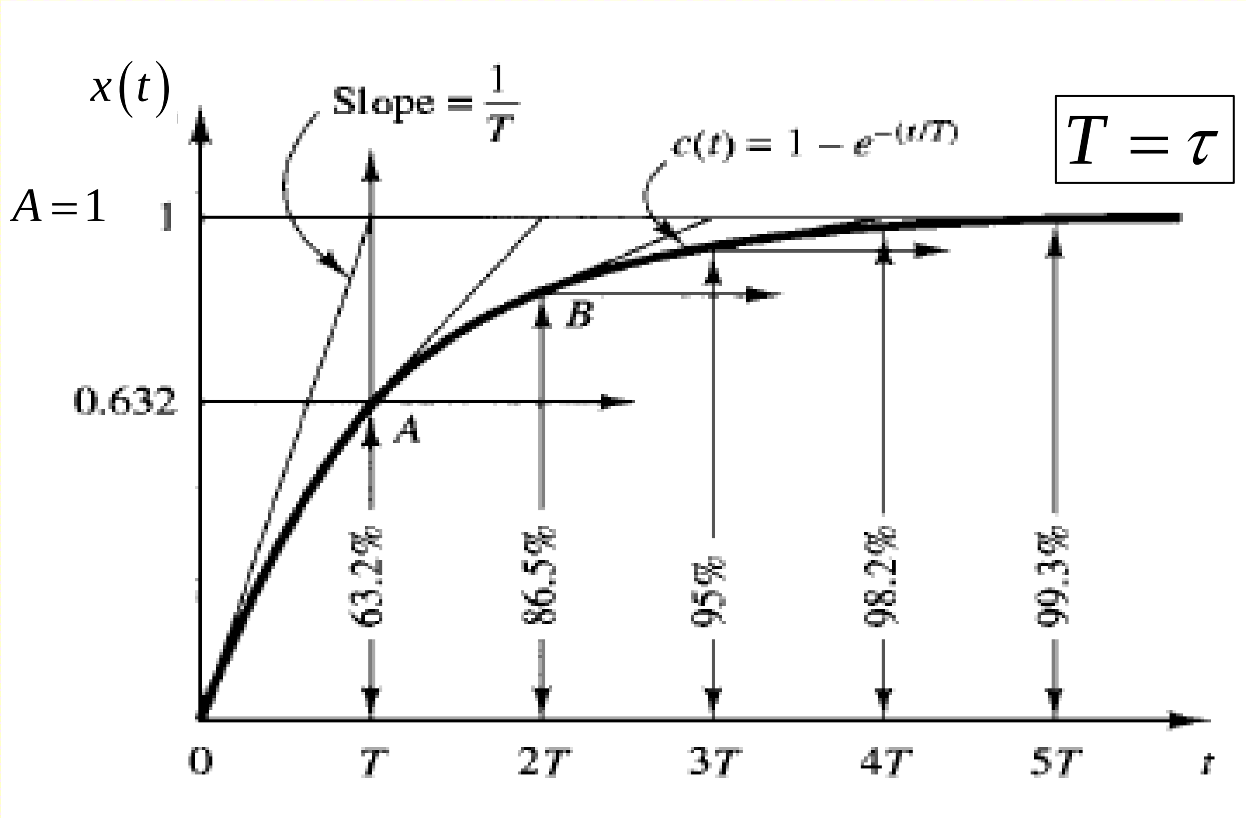
$$c(t) = 1 - e^{-t/\tau}, \qquad for \quad t \geq 0$$

The output will reach its final value as $t \to \infty$.

Initial speed of response:

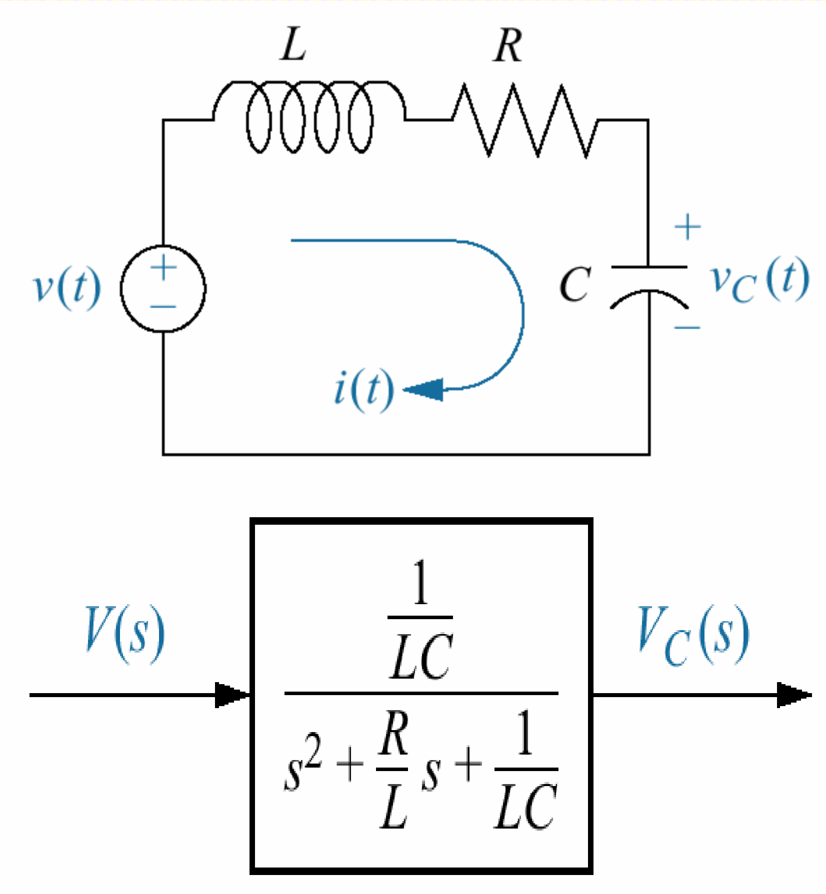
$$\frac{dc}{dt} = \left(\frac{e^{-t/\tau}}{\tau} \right) \Bigg|_{t=0} = \frac{1}{\tau}$$

Unit Step Response of a First-Order System



Settling time: $T_s = 4T = 4\tau$

Second-Order System (R-L-C Circuit)

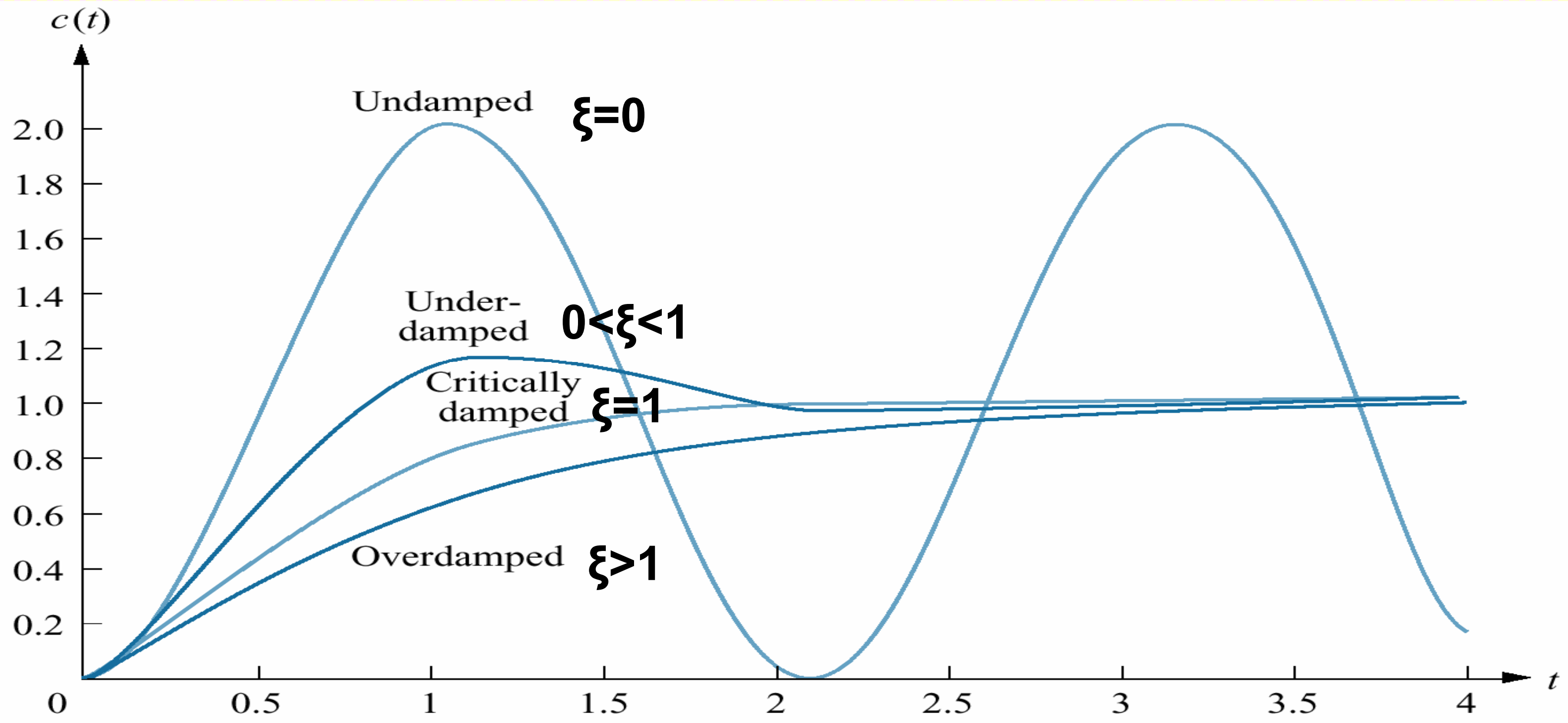


- ζ = damping ratio
- ω_n=un damped natural frequency
- Complex poles

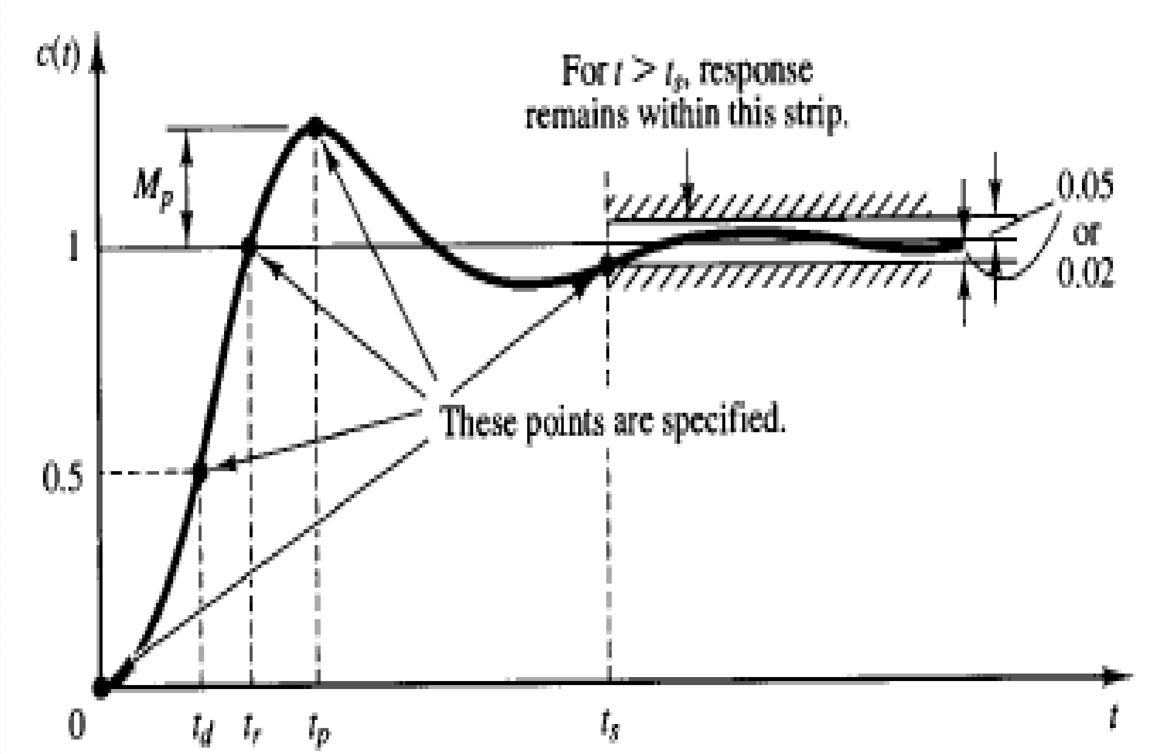
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

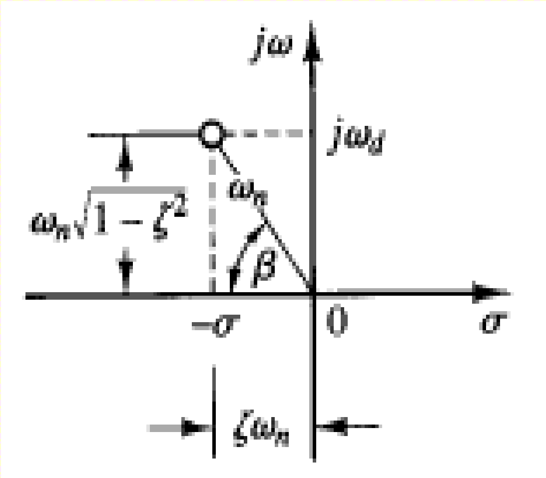
Unit Step Response
Second-Order System

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \phi), \quad \text{where} \;\; \omega_d = \omega_n\sqrt{1-\zeta^2}, \;\; \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$



Transient Response Specifications





Transient response specifications of an Under-damped system

Rise time $T_r = \dfrac{\pi - \beta}{\omega_d}$, Peak time $T_p = \dfrac{\pi}{\omega_d}$

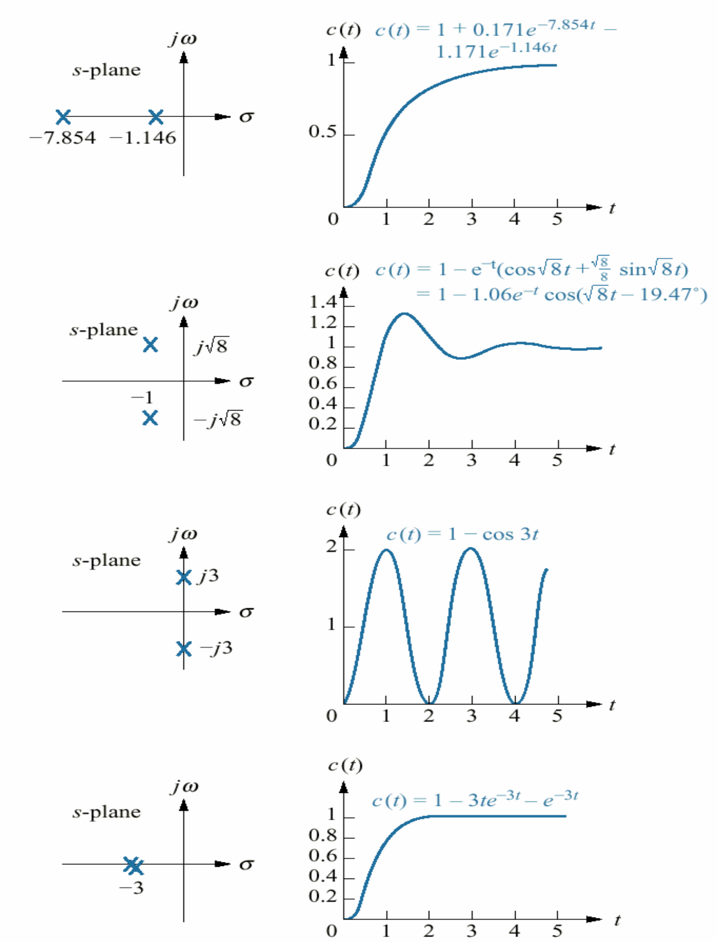
where $\beta = \tan^{-1}\left(\dfrac{\omega_d}{\xi\omega_n}\right)$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Maximum over shoot $M_p = e^{\left(-\pi\xi \big/ \sqrt{1-\zeta^2}\right)}$

Settling time $T_s = \dfrac{4}{\xi\omega_n}$ $(2\%\ criterion)$

$$= \dfrac{3}{\xi\omega_n} \quad (5\%\ criterion)$$

Second-Order Systems: Pole Locations and Step Responses



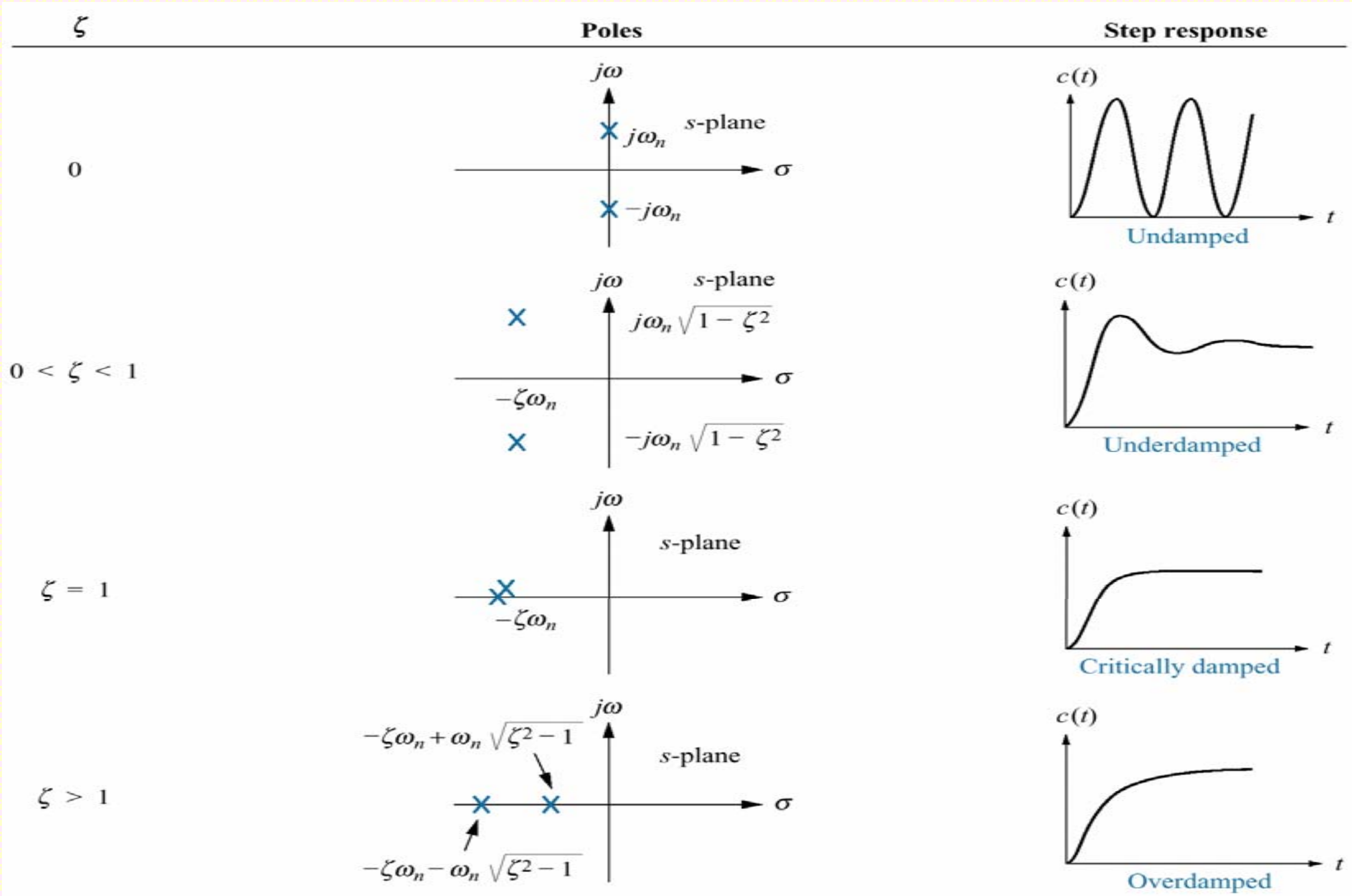
Over damped

Under damped

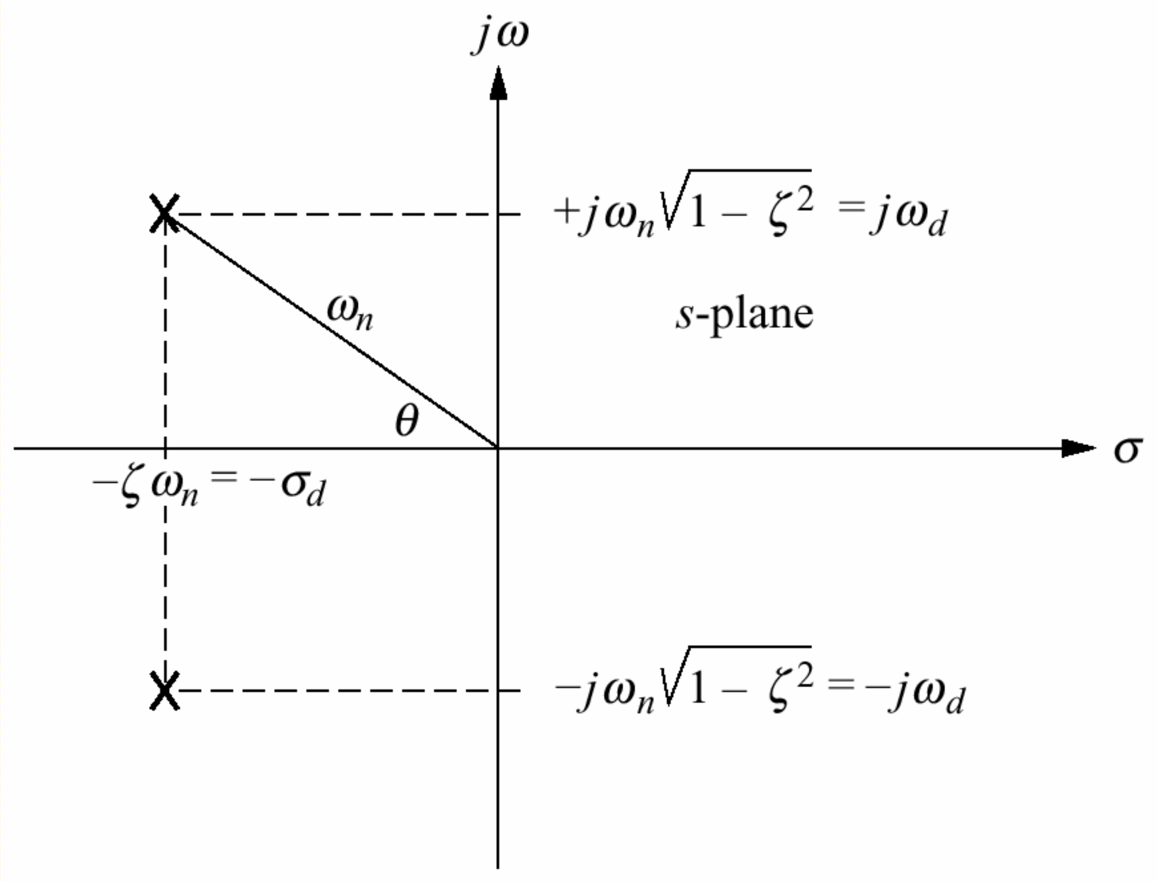
Undamped

Critically damped

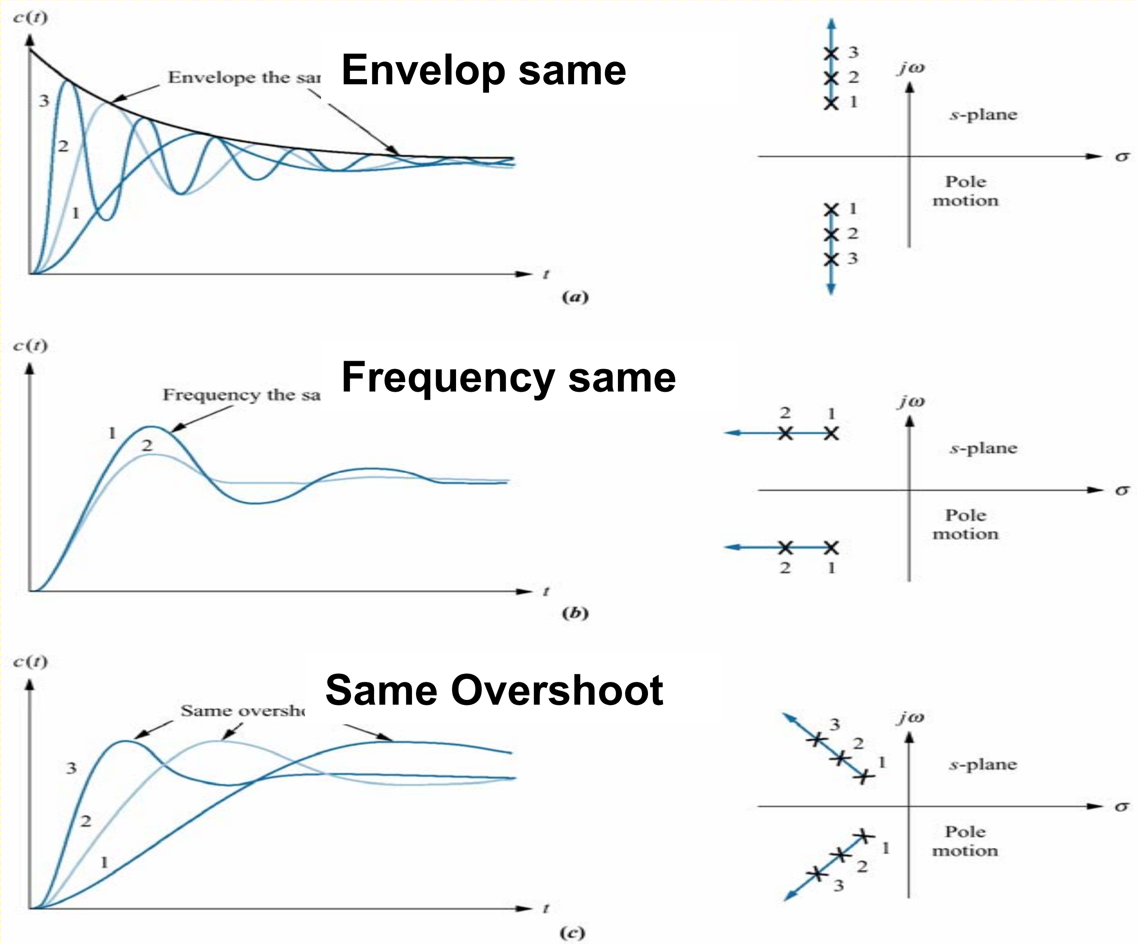
Second-order Response As A Function Of Damping Ratio



Under Damped System Pole Plot



Step Responses of Second Order Under Damped Systems with Pole Movement

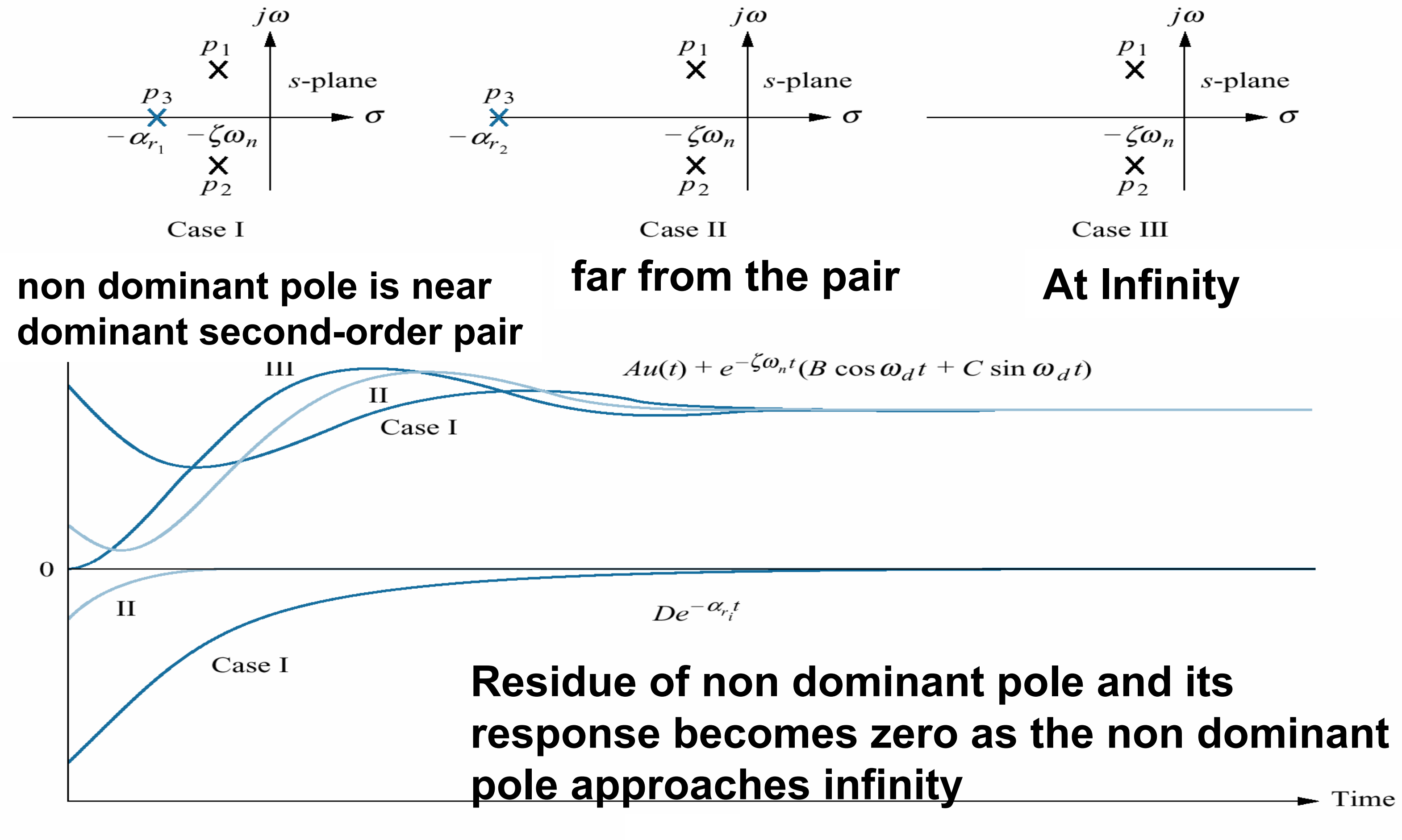


with constant real part

with *constant imaginary* part

with constant damping ratio

Effect of Adding a Pole



non dominant pole is near dominant second-order pair

far from the pair

At Infinity

Residue of non dominant pole and its response becomes zero as the non dominant pole approaches infinity

Effect of Adding a Zero

- Zeros and Poles together dictate the exact response (including magnitude)
- Zeros mainly effect the residues (i.e. the constants in the numerator in the partial fraction expansion)
- Closer the zero is to the dominant poles, the greater is its effect on the transient response

Effect of Adding a Zero: Analysis

Let $C(s)$: Response of a system with unity in the numerator.

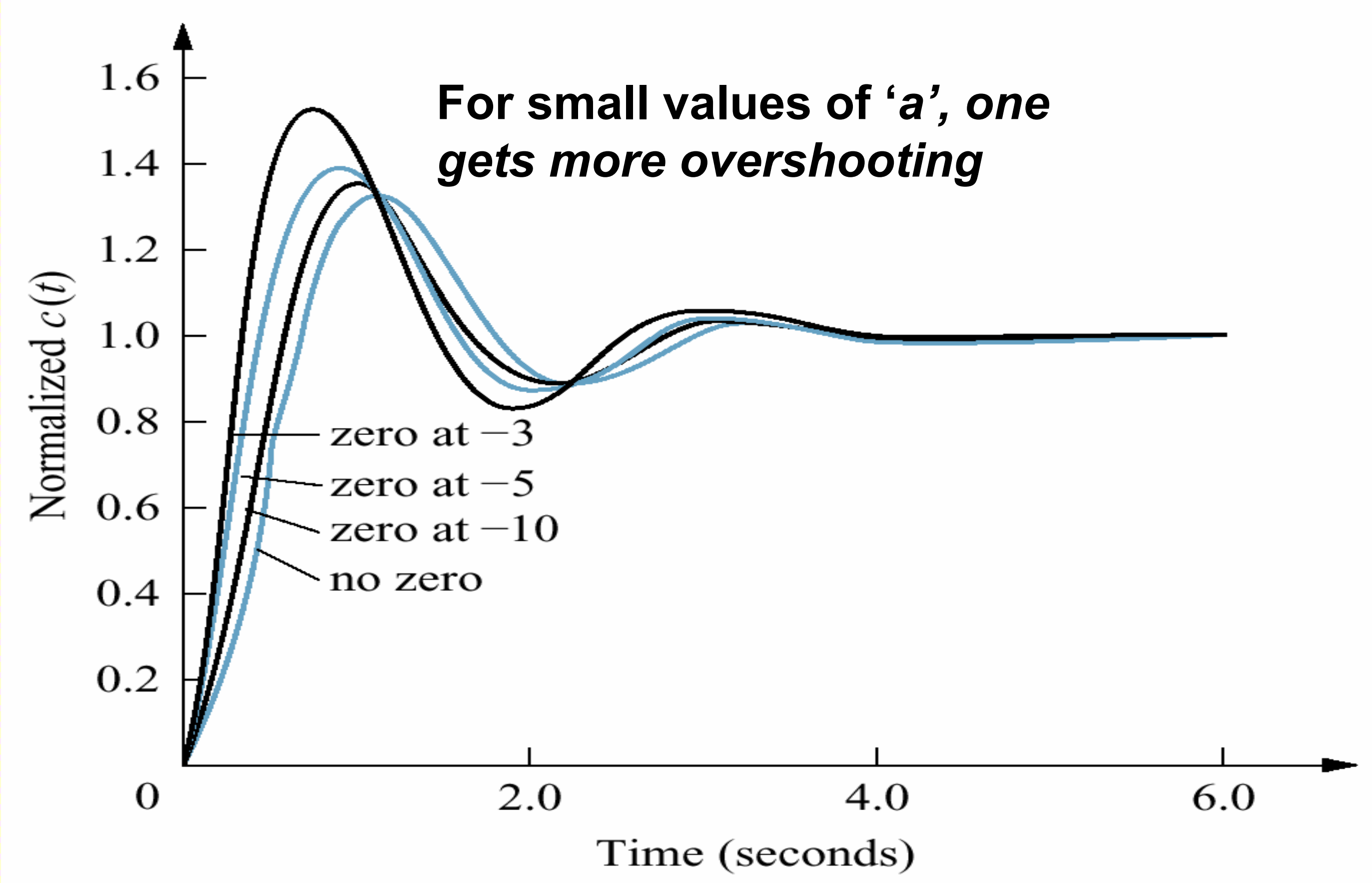
Then by adding a zero, the Laplace transform of the response of the new system will be $(a+s)C(s) = a\,C(s) + s\,C(s)$

$a\,C(s)$: A scaled version of the original response

$s\,C(s)$: The derivative of the original response

Thus, if a is small (in the LH plane), the derivative term is predominant. Hence, more overshooting is expected.

Effect of Adding a Zero for Small Values of *a in the Left-half s-plane*



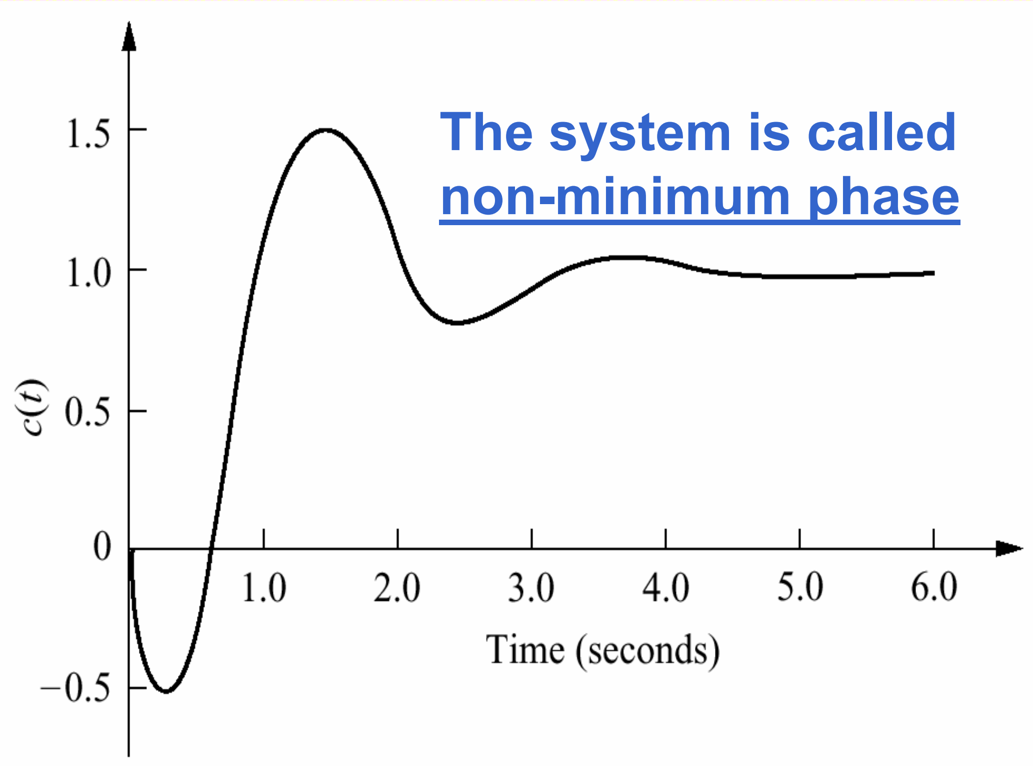
Effect of Adding a Zero in RHS of s-plane

$$(a-s)C(s) = a\,C(s) - s\,C(s), \qquad a > 0$$

In this case the scaled response and derivative terms oppose each other!

Thus, if the derivative term is large, then the system response will initially follow the derivative "in the opposite direction" of the scaled response!

Effect of Adding a Zero in the Right Half *s*-plane



Note: Tail-controlled aerospace vehicles are typical examples for non-minimum phase systems

Thanks for the Attention….!