Lecture – 6

Classical Control Overview – IV

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Lead–Lag Compensator Design

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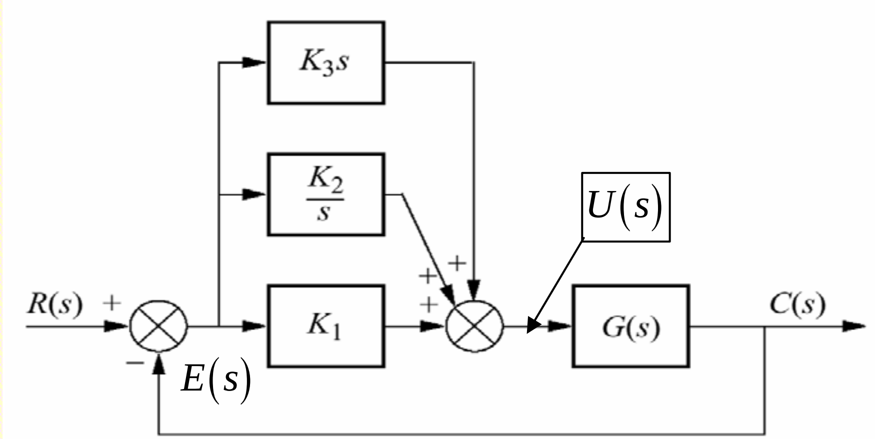
Motivation

- The PID controller involves three components:
  - **Proportional feedback**
  - **Integral feedback**
  - **Derivative feedback**

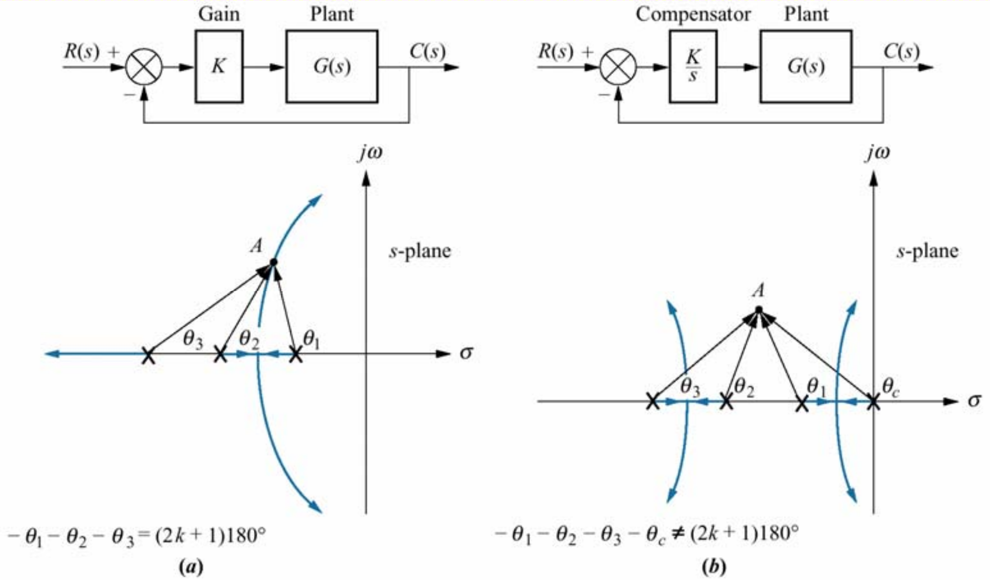


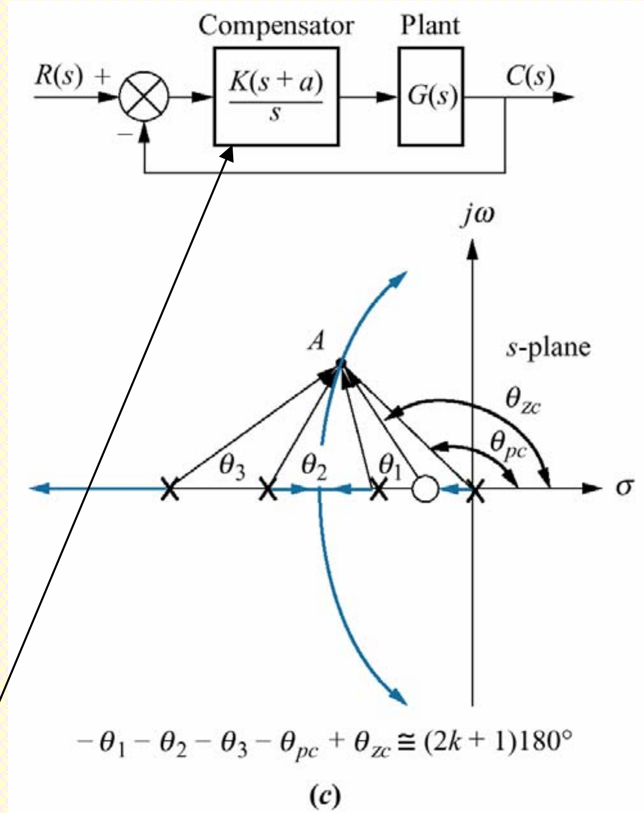
- Problem in PID design:
  - **Requirement of <u>pure integrators</u> and <u>pure differentiators,</u> which are difficult to realize**
  - **Pure integrator pole may travel to the right half plane because of realization inaccuracies.**

Question: Can the difficulties of the PID design be avoided, without compromising much on the basic design philosophy?

Answer: YES! Through Lead-Lag compensator design.

Ideal Integral Compensation for Improving SS Error: PI Controller

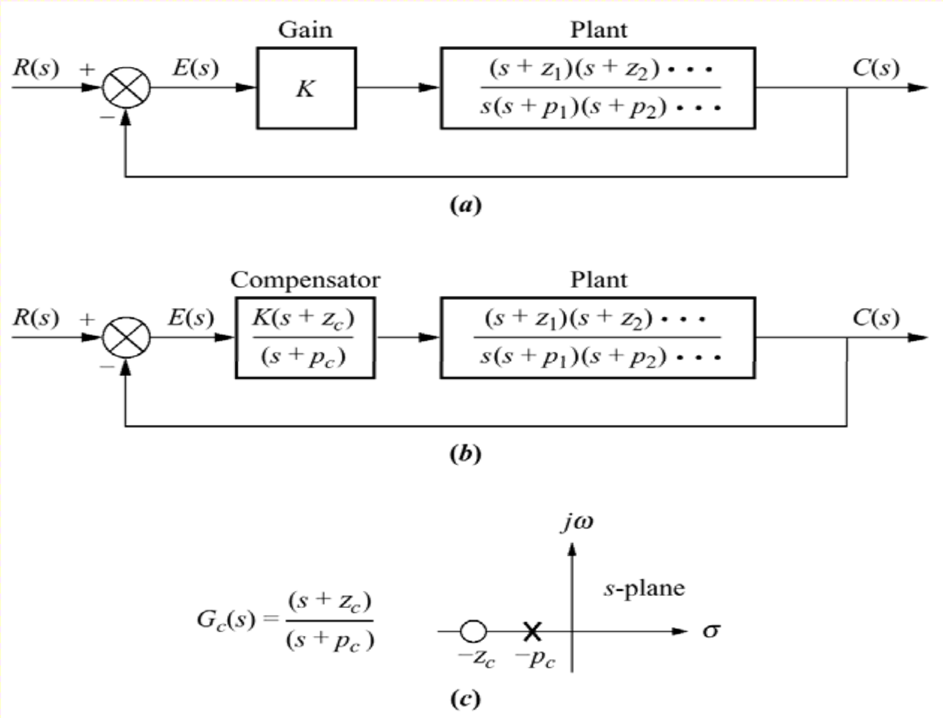




Point A: Desired pole location (already satisfactory transient response). No need to change its location. However, need to improve the steady-state error.

$$\frac{K(s+a)}{s} = K + \frac{Ka}{s} = K + \frac{K_I}{s}$$ (PI)

Lag Compensator
(for Improving Steady State Error)



$$K_{v_0} = \frac{K z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

$$K_{v_1} = \frac{(K z_1 z_2 \cdots)(z_c)}{(p_1 p_2 \cdots)(p_c)}$$

$$= K_{v_0}\left(\frac{z_c}{p_c}\right) > K_{v_0}$$

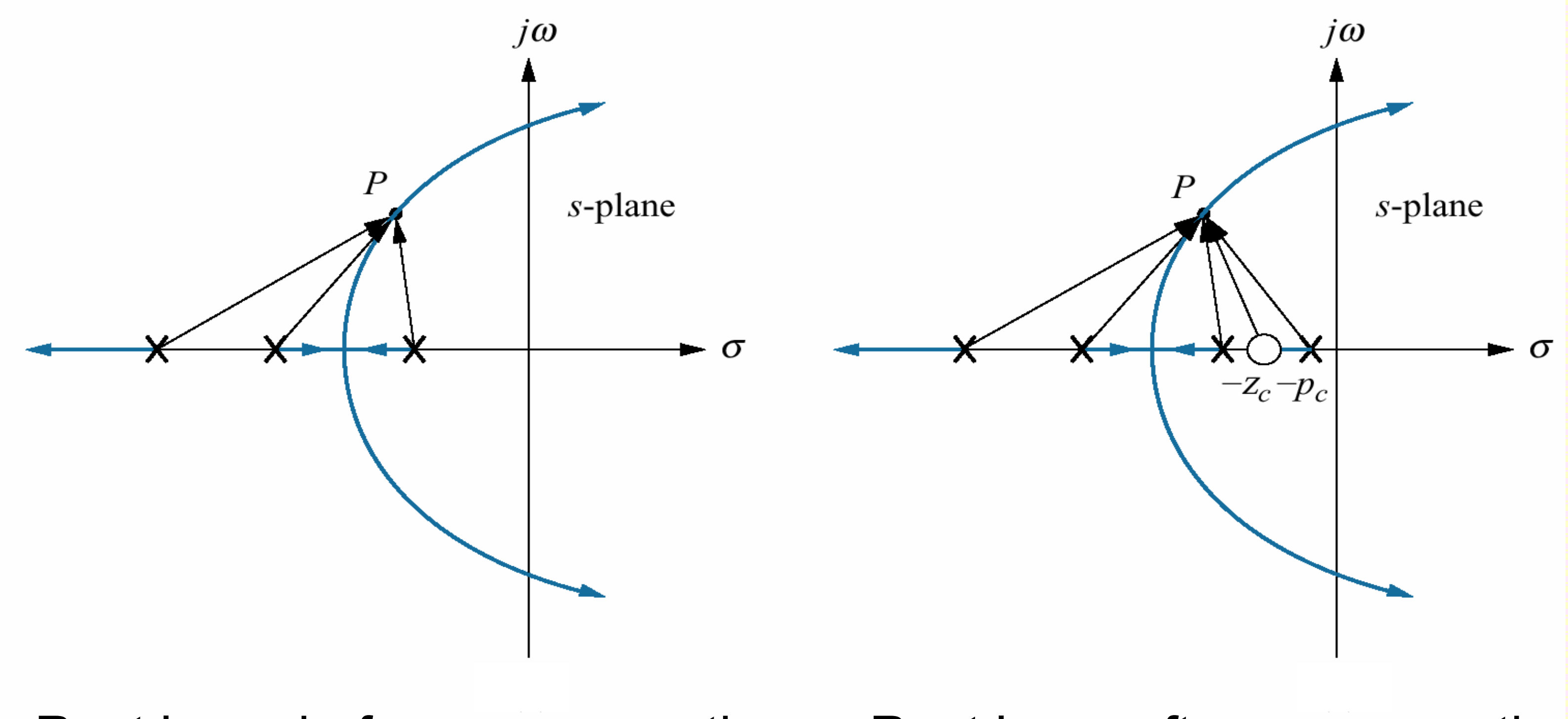
(provided $z_c > p_c$)

Note:

- Pole-zero pair should be close to each other
- $K_{v_1} \gg K_{v_0}$, provided $(z_c / p_c) \gg 1$. Hence put the pair close to origin.

like PI controller

Lag Compensator
(for Improving Steady State Error)



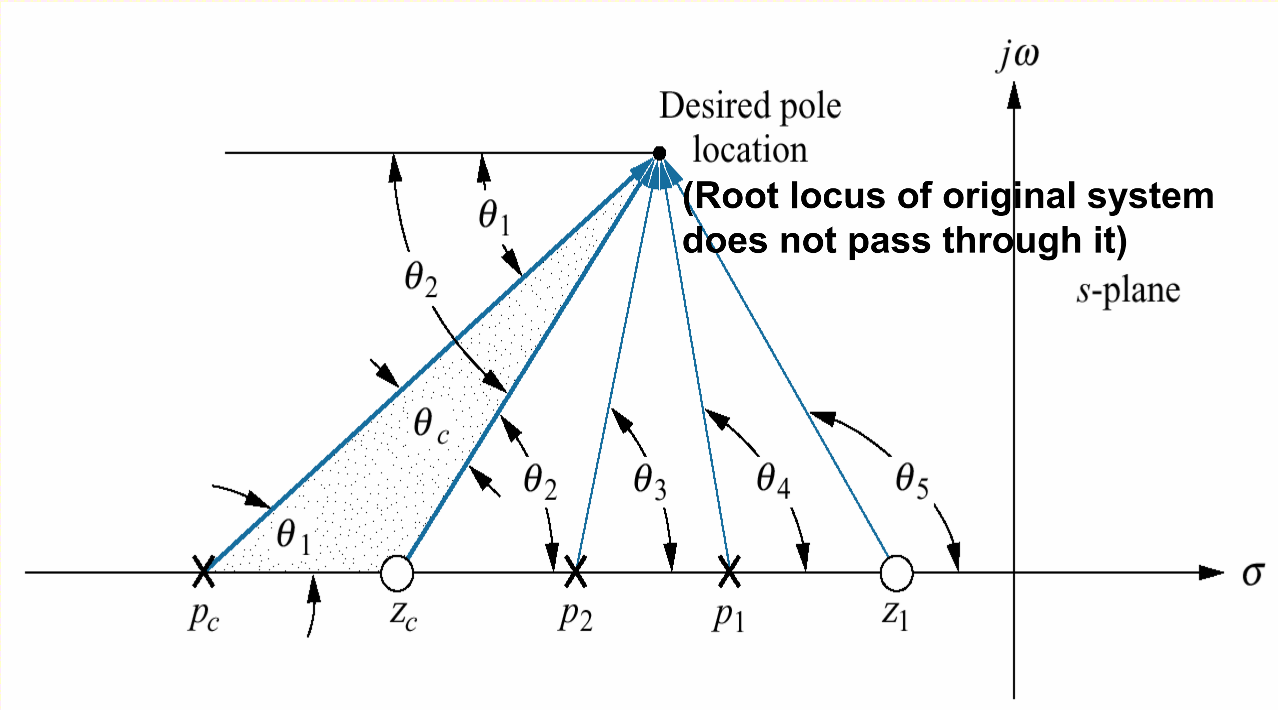
Root locus before compensation Root locus after compensation

No appreciable change in Root locus & closed loop pole location:
No difference in transient response, but improvement in SS error!

Lead Compensator
(for Improving Transient Response)

- Objective:
  - To improve transient performance, avoiding the pure PD realization
- Advantages;
  - Avoids realization difficulties (e.g. avoids requirement of additional power supplies in electrical circuits)
  - Reduces noise amplification due to differentiation
- Drawback:
  - Addition of a pure zero in PD controller tends to reduce the number of branches of Root Locus that travel to RH plane, whereas Led compensators are not capable of doing that.

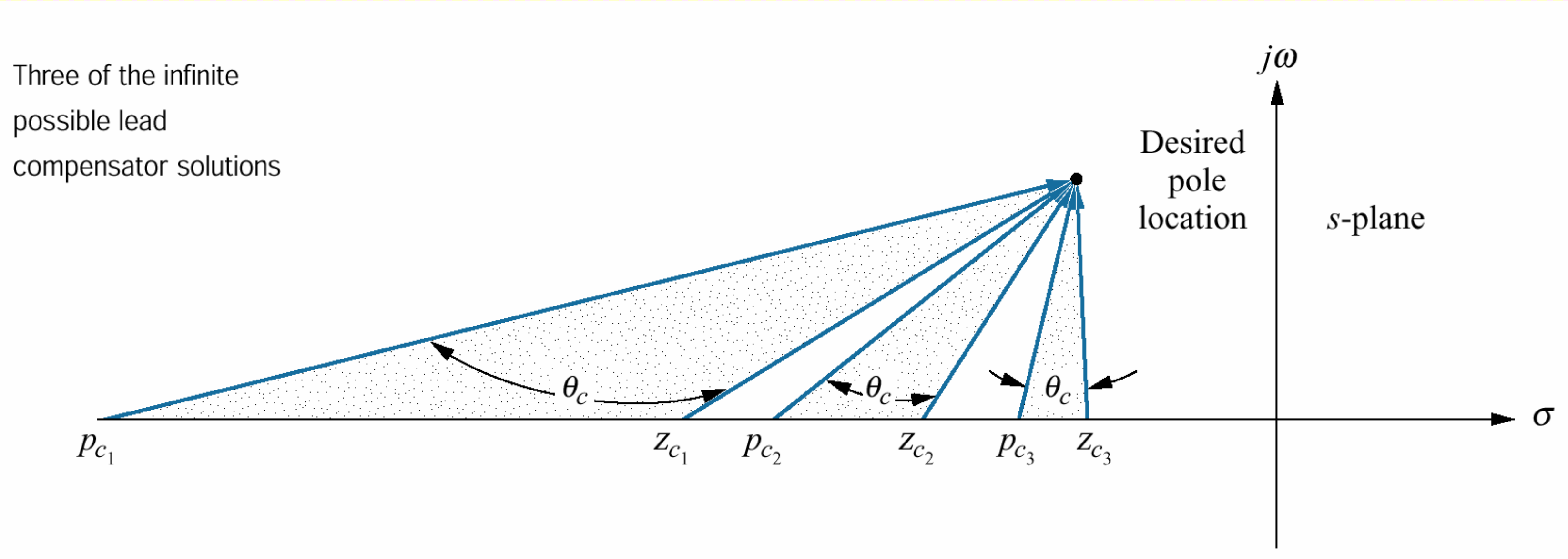
Geometry of Lead Compensator



$$\theta_2 - \theta_1 - \theta_3 - \theta_4 + \theta_5 = (2k+1)180^0$$

where $(\theta_2 - \theta_1) = \theta_c$: Angular contribution of compensator

Many Possibilities of Lead Compensators



Different selections results in:

- Different gain values to reach the desired point
- Different static error constants (that leads to different SS errors); hence different closed loop response in strict sense!

Lead-Lag Compensator Design

Design Steps:

Ref: N. S. Nise: Control Systems Engineering, 4th Ed., Wiley, 2004

- First, evaluate the performance of the uncompensated system.
- If necessary, design a "lead compensator" to improve the transient response.
- Next, design a "lag compensator" to improve the steady state error.
- Simulate the system to be sure that all requirements have been met.
- Redesign the compensators (i.e. retune the compensator gains), if the simulation performance in not satisfactory.

Frequency Response Analysis

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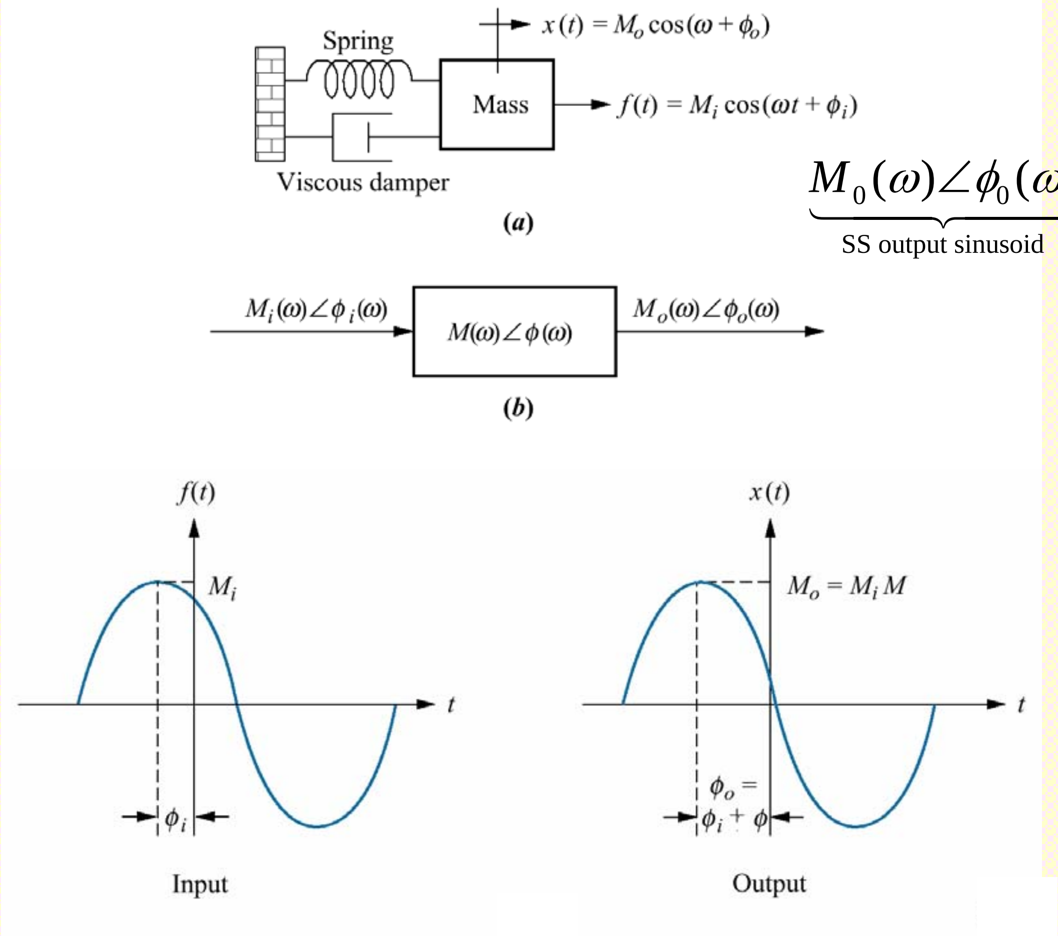
Concept of Frequency Response



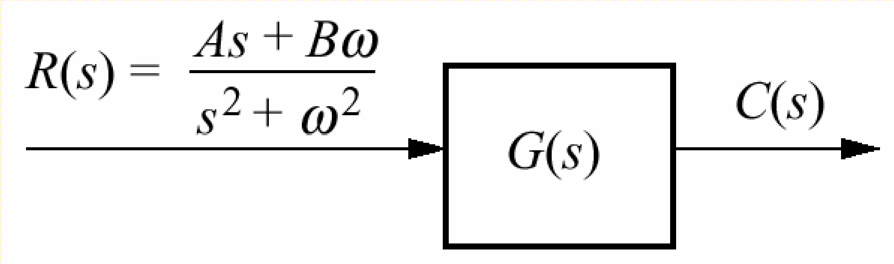
$$\underbrace{M_0(\omega)\angle\phi_0(\omega)}_{\text{SS output sinusoid}} = \underbrace{\left(M_i(\omega)\angle\phi_i(\omega)\right)}_{\text{Frequency Response}}\underbrace{\left(M(\omega)\angle\phi(\omega)\right)}_{\text{Input sinusoid}}$$

$$= M_i(\omega)M(\omega)\angle\left[\phi_i(\omega)+\phi(\omega)\right]$$

$$M(\omega) = \frac{M_0(\omega)}{M_i(\omega)}$$

$$\phi(\omega) = \phi_0(\omega) - \phi_i(\omega)$$

Concept of Frequency Response



$$r(t) = A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\cos\left[\omega t - \tan^{-1}(B/A)\right]$$

$$= M_i \cos\left(\omega t - \phi_i\right), \quad \text{where} \quad M_i = \sqrt{A^2 + B^2}, \; \phi_i = -\tan^{-1}(B/A)$$

After appropriate analysis:

$$c_{ss}(t) = M_i M_G \cos\left(\omega t + \phi_i + \phi_G\right)$$ **where**

$$M_0 \angle \phi_0 = \left(M_i \angle \phi_i\right)\left(M_G \angle \phi_G\right)$$

$$M_G = \left|G\left(j\omega\right)\right|, \quad \phi_G = \angle G\left(j\omega\right)$$

What is frequency response?

- Magnitude and phase relationship between sinusoidal input and the steady state output of a linear system is termed as frequency response.
- Commonly used frequency response analysis:
  - Bode plot
  - Nyquist plot
  - Nichols chart

$$T(s)=\frac{C(s)}{R(s)}, \quad s=j\omega,$$

$$T(j\omega)=\frac{C(j\omega)}{R(j\omega)}=M\angle\phi$$

$$M=|T(j\omega)|, \phi=\angle T(j\omega)$$

Bode Plot Analysis

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Bode Plot Analysis

- Bode plot consists of two simultaneous graphs:
  - Magnitude in dB (20 log |G(jω)|) vs. frequency (in logω)
  - Phase (in degrees) vs. frequency (in logω)
- Steps:

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Then

$$20\log|G(j\omega)| = \begin{bmatrix} 20\log K + 20\log|(s+z_1)| + \cdots + 20\log|(s+z_k)| \\ -20\log|s^m| - 20\log|(s+p_1)| - \cdots - 20\log|(s+p_n)| \end{bmatrix}_{s\to j\omega}$$

$$\angle G(j\omega) = \Big[\big(\angle(s+z_1) + \cdots + \angle(s+z_k)\big) - \big(\angle s^m + \angle(s+p_1) + \cdots + \angle(s+p_k)\big)\Big]_{s\to j\omega}$$

Bode Diagrams: Advantages

- All algebra is through addition and subtraction, and that too mostly through straight line asymptotic approximations
- Low frequency response contains sufficient information about the physical characteristics of most of the practical systems.
- Experimental determination of a transfer function is possible through Bode plot analysis.

Bode Diagrams

- In Bode diagrams, frequency ratios are expressed in terms of:
  - Octave: it is a frequency band from ω_1 *to* $2\omega_1$.
  - Decade: it is a frequency band from ω_1 to $10\omega_1$, where ω_1 is any frequency value.
- The basic factors which occur frequently in an arbitrary transfer function are:
  - Gain K
  - Integral and derivatives: $(j\omega)^{\pm 1}$
  - First order factors: $(1 + j\omega T)^{\pm 1}, \; T = 1/a$
  - Quadratic Factors: $\left(1 + 2\xi (j\omega/\omega_n) + (j\omega/\omega_n)^2\right)^{\pm 1}$

Bode Diagrams

- For Constant Gain K, log-magnitude curve is a horizontal straight line at the magnitude of (20 log K) dB and phase angle is 0 deg.
- Varying the gain K, raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but has no effect on the phase curve
- Logarithmic representation of the frequency-response curve of factor $\left(j(\omega/a)+1\right)$ can be approximated by two straight-line asymptotes
- Frequency at which the two asymptotes meet is called the corner frequency or break frequency.

Example – 1: $G(s) = s + a, \quad G(j\omega) = j\omega + a$

- At low frequencies, $\omega << a, \; G(j\omega) \approx a$
- Magnitude/Phase response:

$$20 \log M = 20 \log a, \quad \angle G(j\omega) = 0^0$$

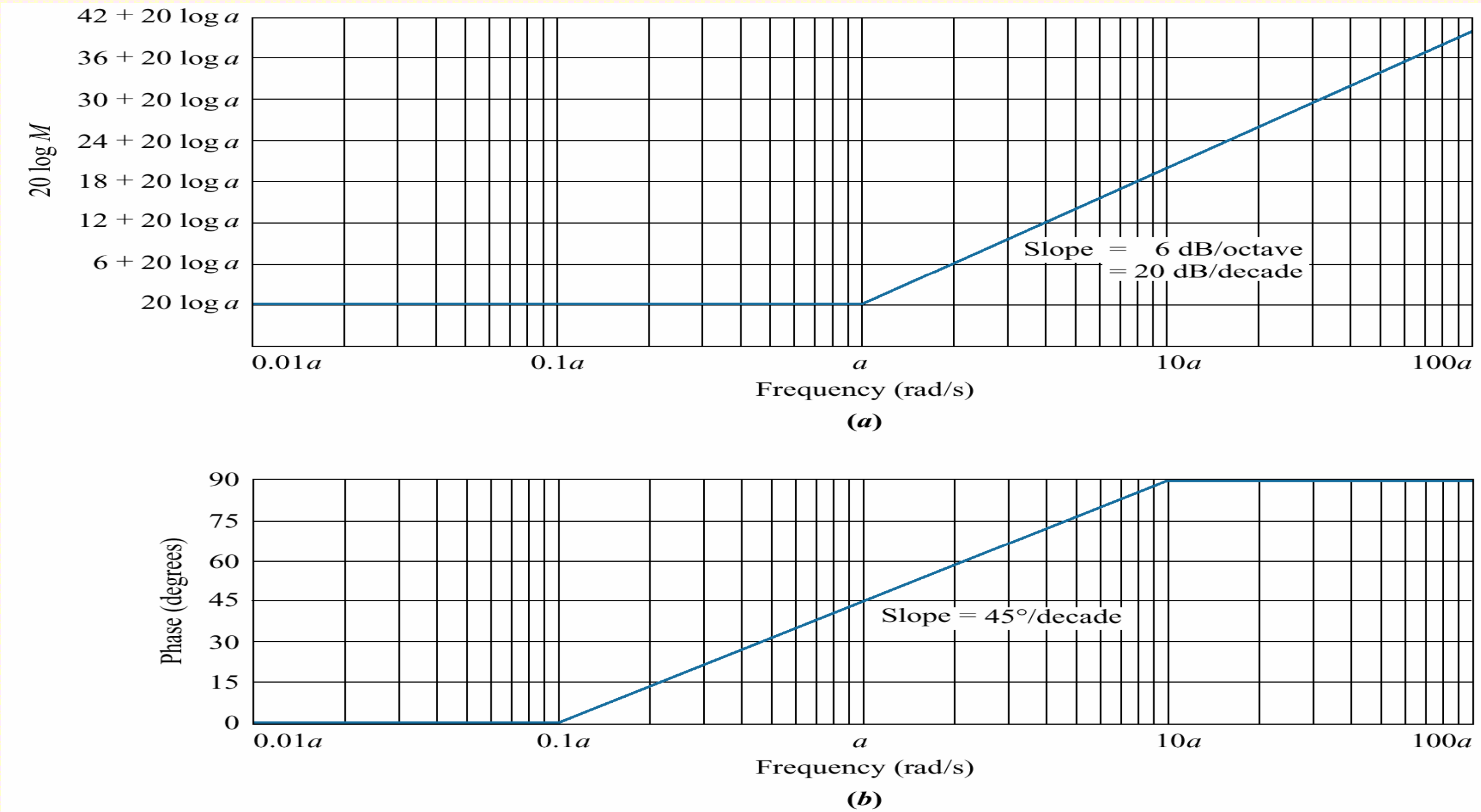
- At high frequencies, $\omega >> a, \; G(j\omega) \approx j\omega$
- Magnitude/Phase response:

$$20 \log M = 20 \log \omega, \quad \angle G(j\omega) = 90^0$$

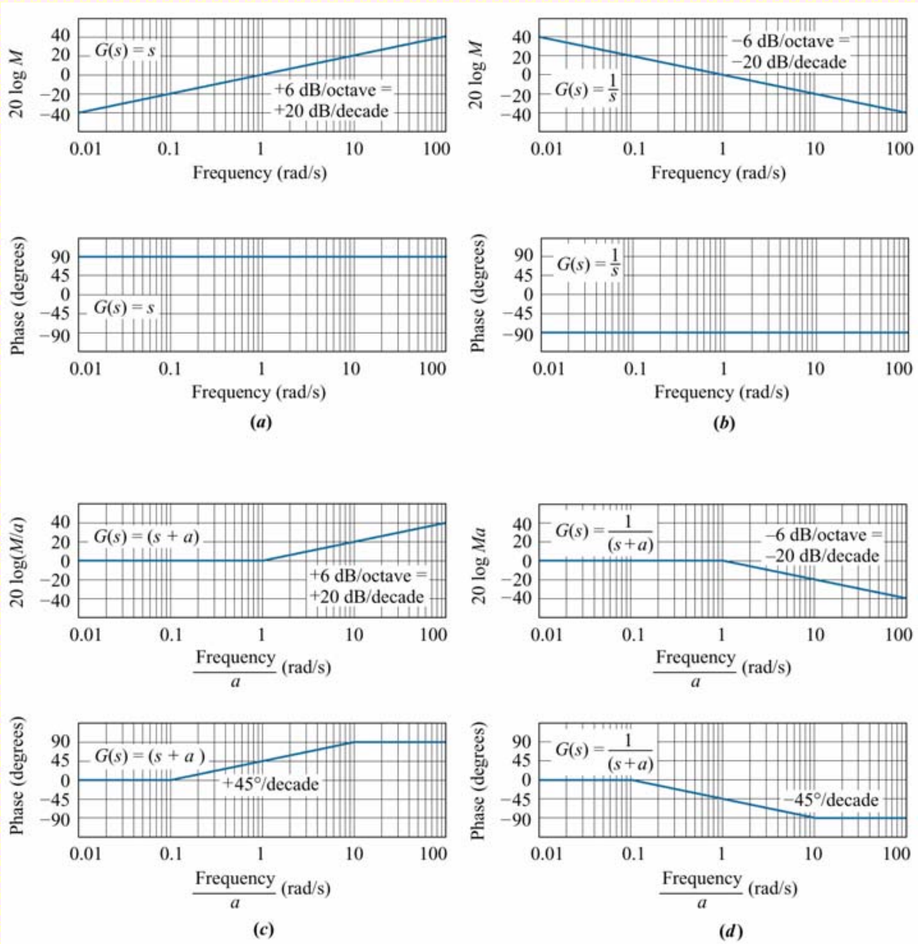
- Corner frequency: $G(j\omega) = j\omega + a = a\left(j\dfrac{\omega}{a} + 1 \right)$

$$\omega_c = a$$

Bode Plot for $(s + a)$



Bode diagrams of some standard first order terms



Ref: N. S. Nise, Control Systems Engineering, 4th Ed. Wiley, 2004.

Bode plots for:

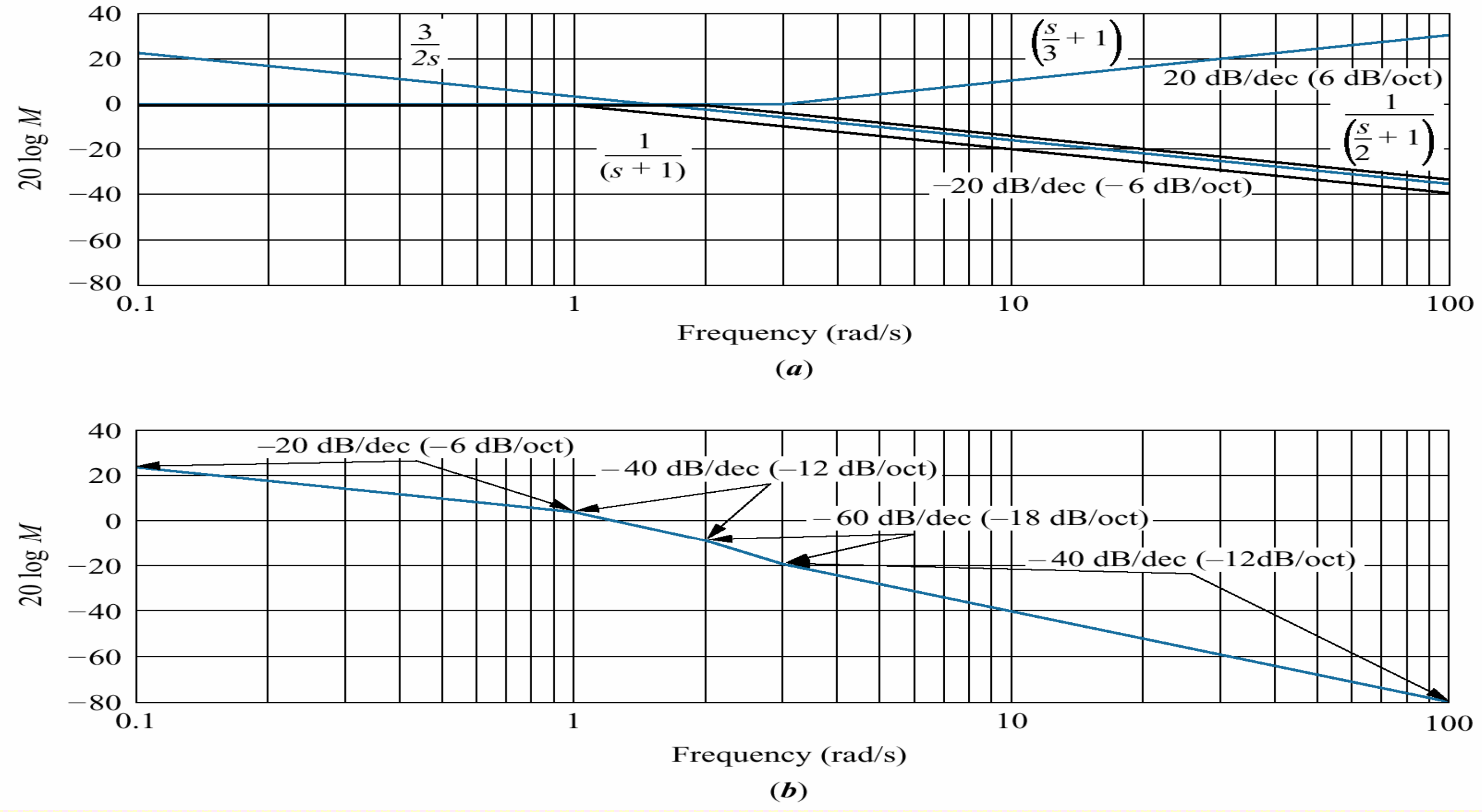
a. $G(s) = s$

b. $G(s) = 1/s$

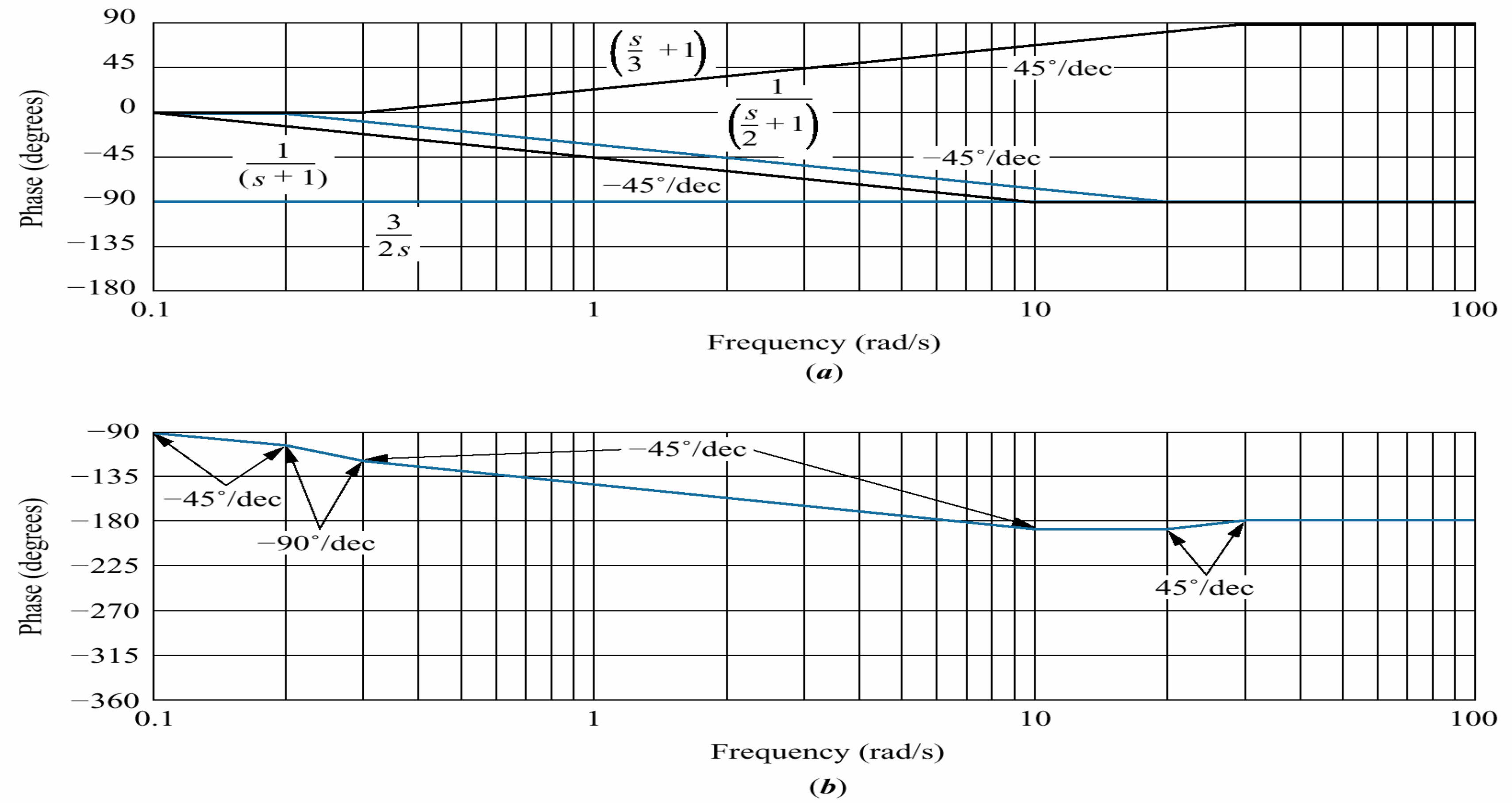
c. $G(s) = (s + a)$

d. $G(s) = 1/(s + a)$

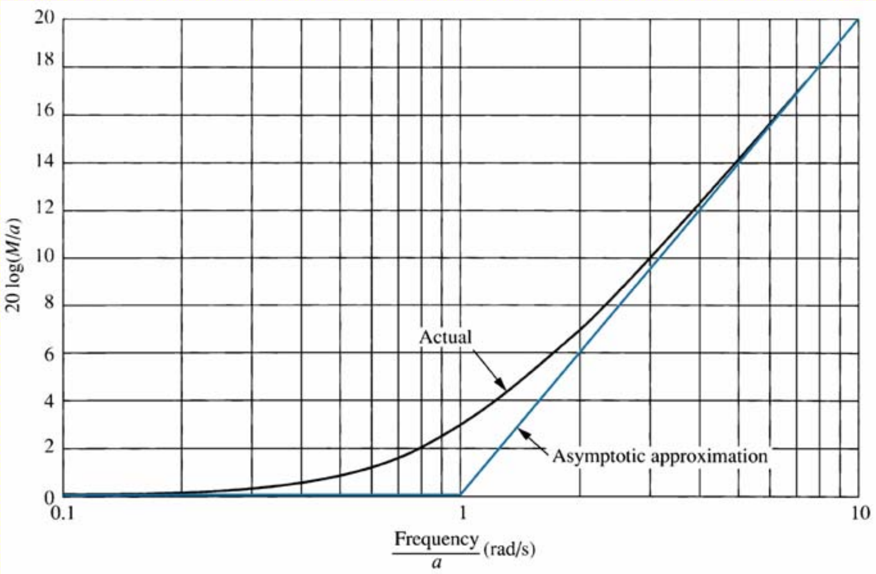
Example – 2: $G(s) = \dfrac{K(s+3)}{s(s+1)(s+2)}$

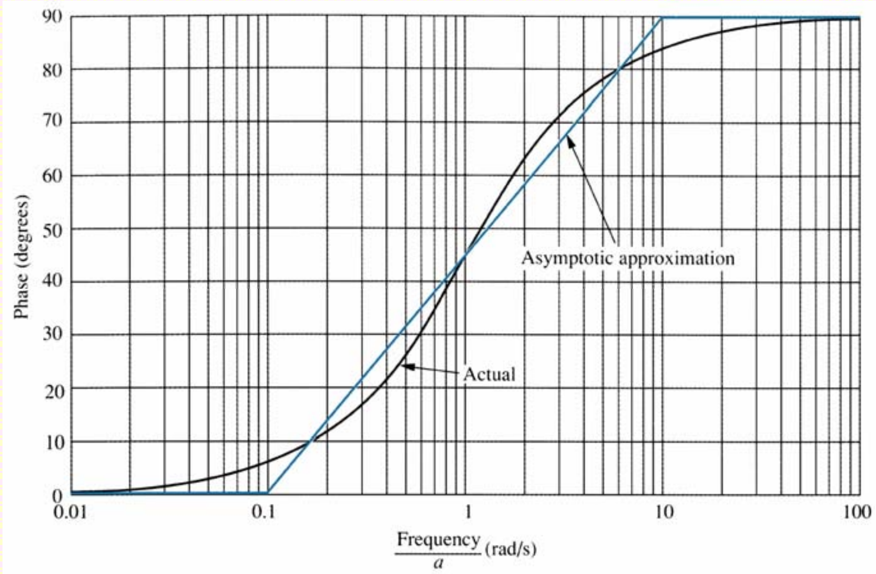


Example – 2: $G(s)=\dfrac{K(s+3)}{s(s+1)(s+2)}$



First Order Terms: Comparison of Actual and Asymptotic Behavior





Magnitude comparison

Phase comparison

Ref: N. S. Nise, Control Systems Engineering, 4th Ed. Wiley, 2004.

Bode plot for second order systems

- System with conjugate zeros when $0<\xi<1$

$$G(s)=s^2+2\xi\omega_n s+\omega_n^2$$

- System with conjugate poles when $0<\xi<1$

$$G(s)=\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}$$

- For complex conjugate poles and zeros, the slope changes by $\pm$ 40dB/decade

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

- For low frequencies ($\omega \ll \omega_n$),
- Log magnitude $20\log\left|\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$ becomes

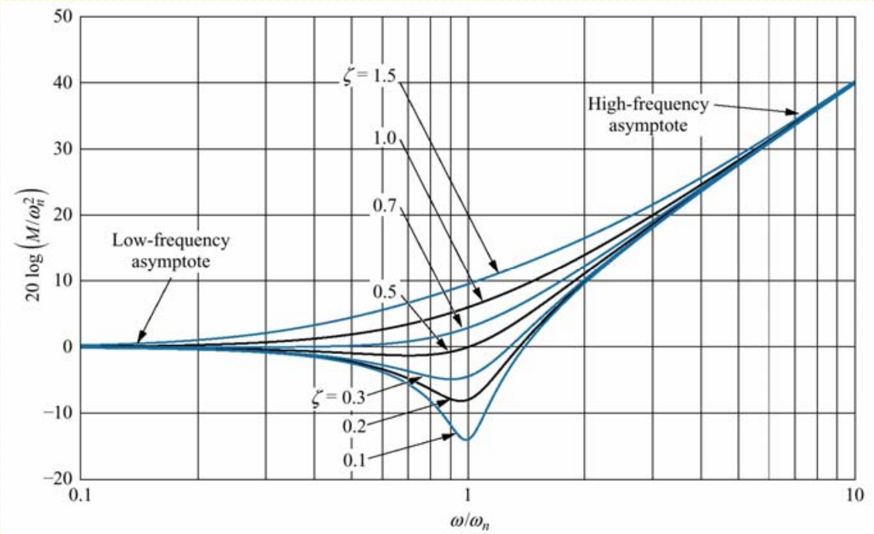
  0dB (20log 1= 0), hence low frequency asymptote is a straight horizontal line
- For high frequencies ($\omega \gg \omega_n$), the log magnitude becomes

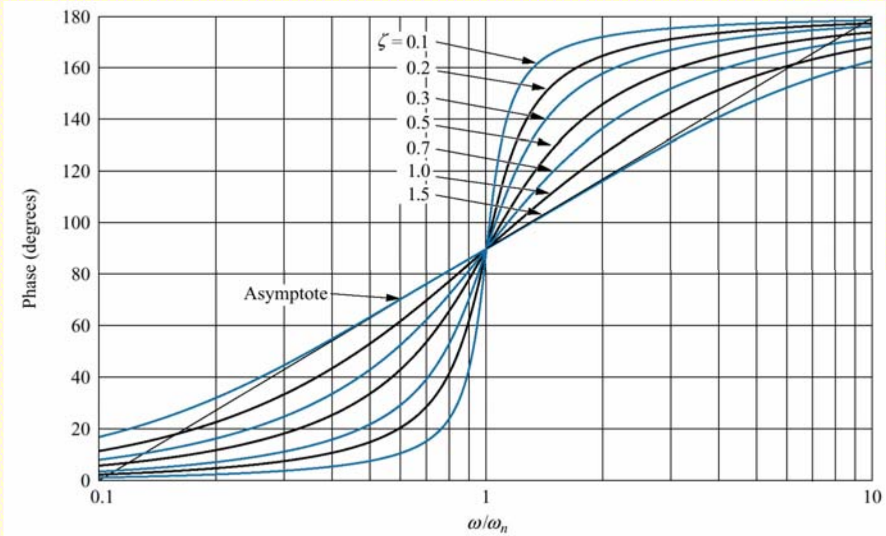
$$-20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}\ \text{dB}$$

- High frequency asymptote is a straight line with slope of -40dB/decade
- The phase angle of the quadratic factor is

$$\phi = \angle\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

Scaled Response for $(s^2 + 2\zeta\omega_n s + \omega_n^2)$

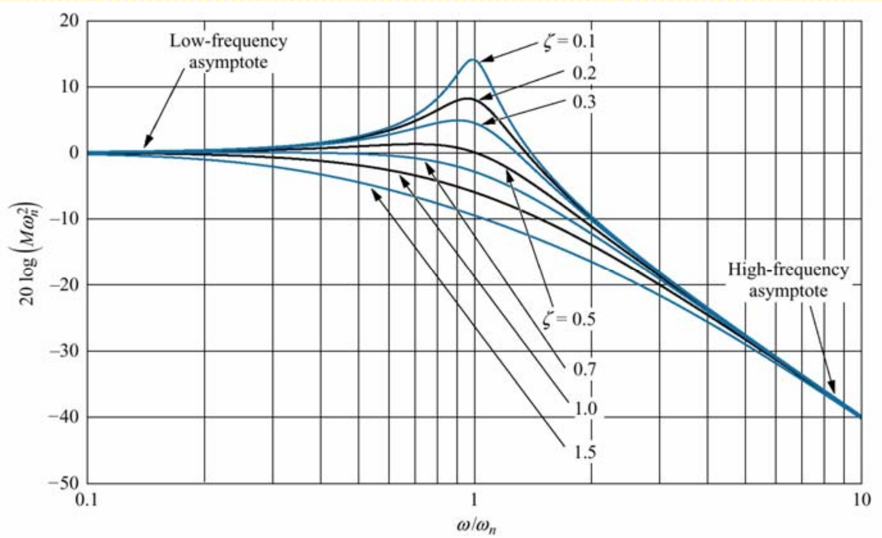


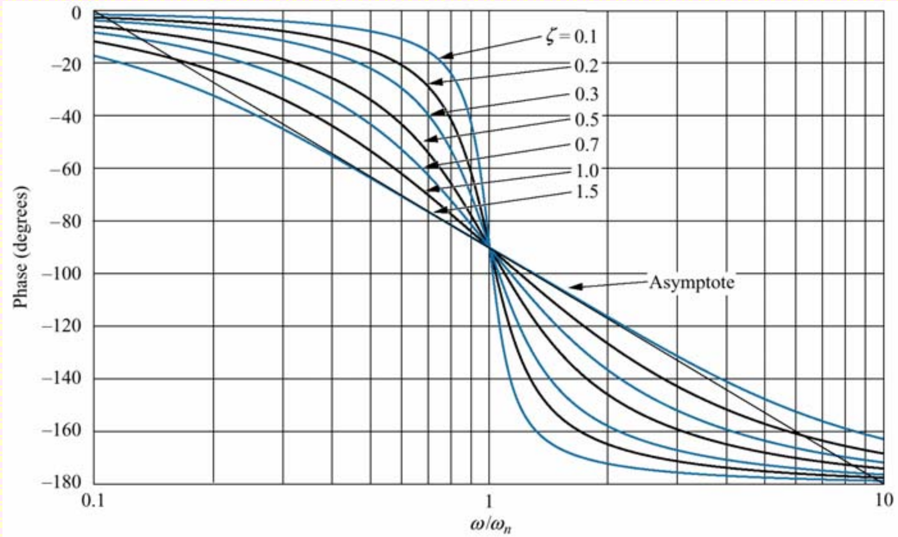


Magnitude Plot

Phase Plot

Scaled Response for $1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$





Magnitude Plot

Phase Plot

Example – 3: $G(s) = \dfrac{(s+3)}{(s+2)(s^2+2s+25)}$

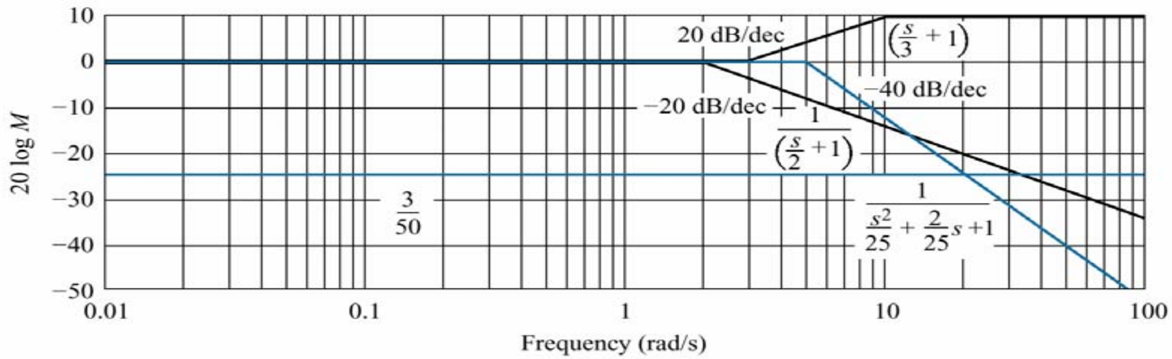
- Second order system is normalized

$$G(s) = \frac{3}{50}\left[\frac{\left(\frac{s}{3}+1\right)}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{25}+\frac{2}{25}s+1\right)}\right]$$

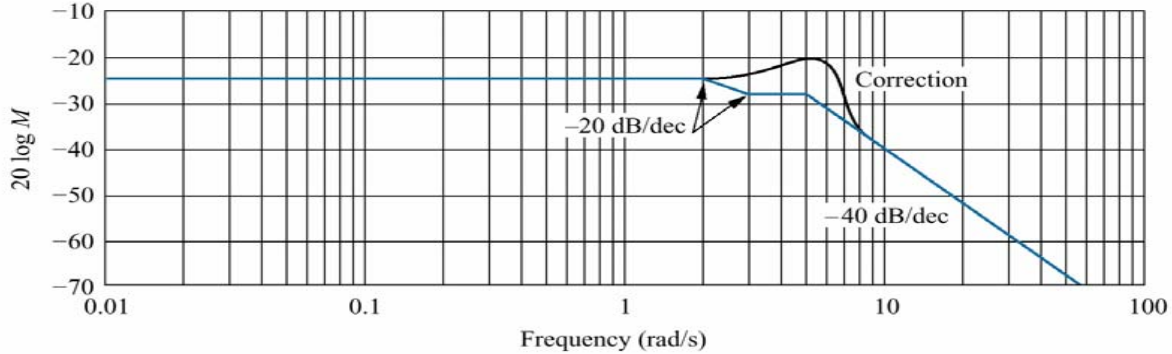
- Bode magnitude plot starts from 20logK = 24.44dB and continues until the next corner frequency at 2 rad/s

Magnitude Plot

Description	Frequency (rad/s)			
	0.01 (Start: Plot)	2 (Start: Pole at −2)	3 (Start: Zero at −3)	5 (Start: $\omega_n = 5$)
Pole at −2	0	−20	−20	−20
Zero at −3	0	0	20	20
$\omega_n = 5$	0	0	0	−40
Total slope (dB/dec)	0	−20	0	−40



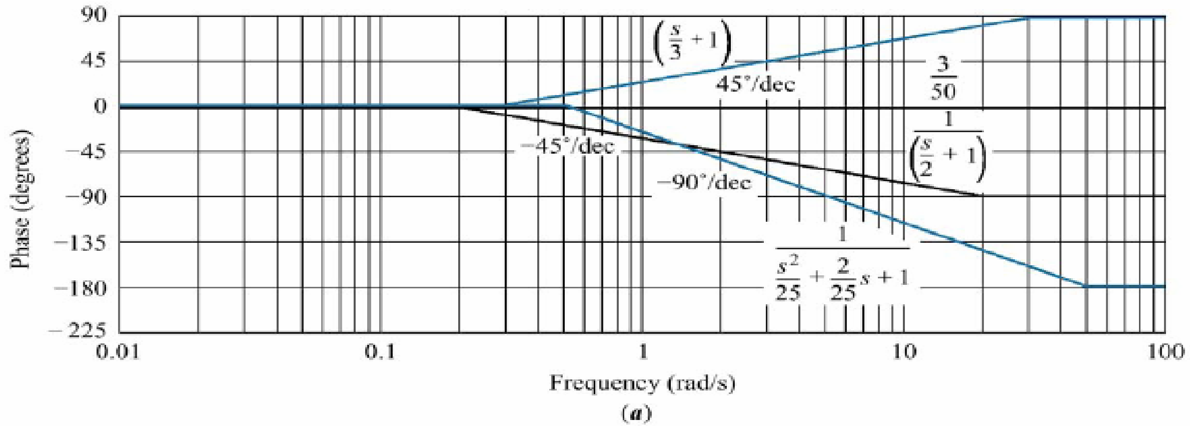
(a)



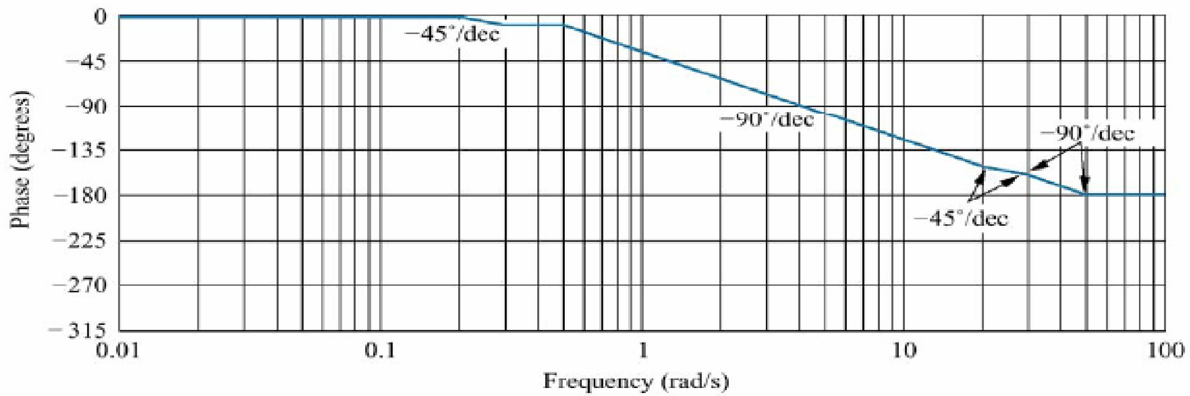
(b)

Phase plot

	Start: pole at −2	Start: zero at −3	Start: ω_n at −5	End: pole at −2	End: zero at −3	End: $\omega_n = 5$
Frequency (rad/s)	0.2	0.3	0.5	20	30	50
Pole at −2	−45	−45	−45	0		
Zero at −3		45	45	45	0	
$\omega_n = 5$			−90	−90	−90	0
Total slope (deg/dec)	−45	0	−90	−45	−90	0



(a)



Nyquist Plot Analysis

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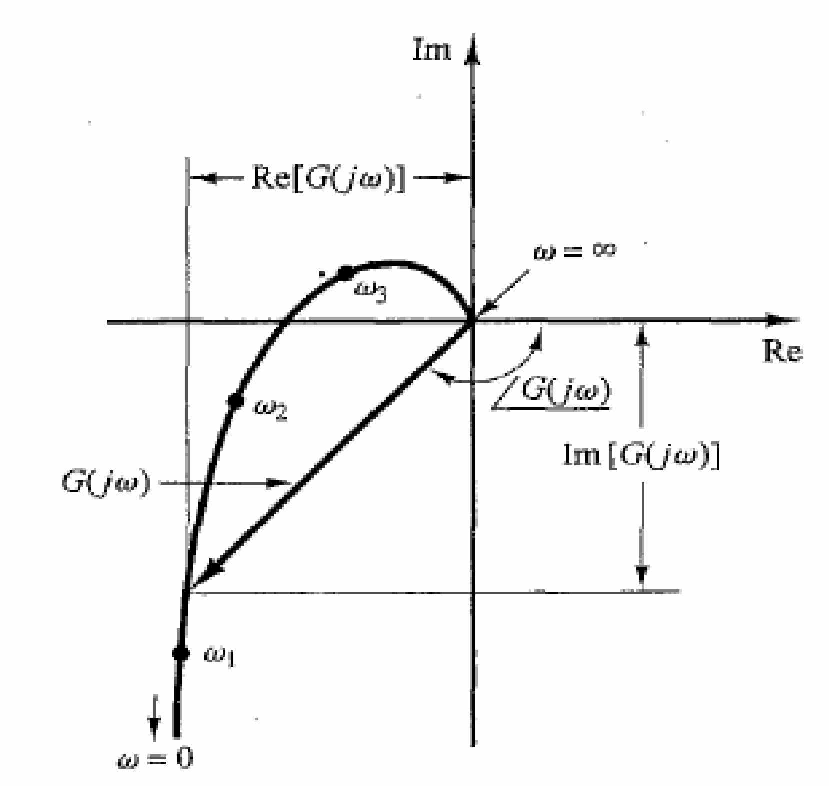
Presenting frequency response characteristics (Polar plots)

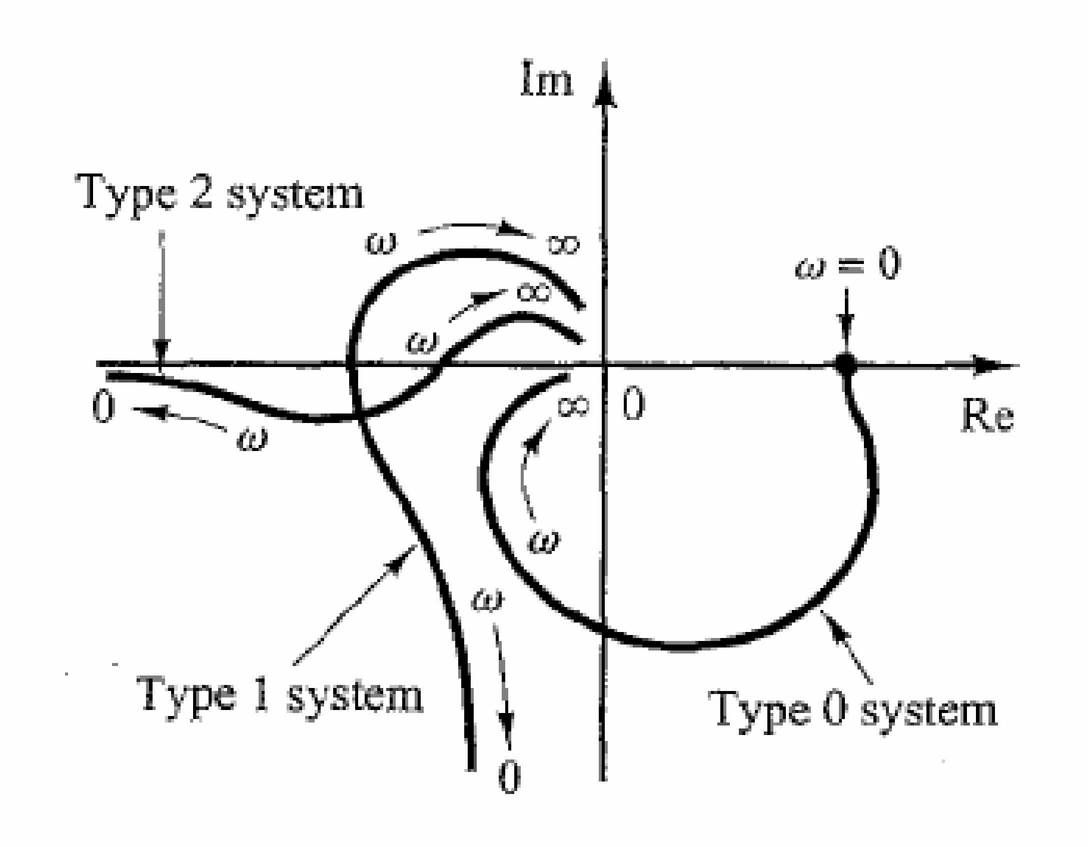


Frequency response of transfer functions can be represented by Polar plots, also called Nyquist plots, as shown in the figure.

$$M = |G(j\omega)|$$

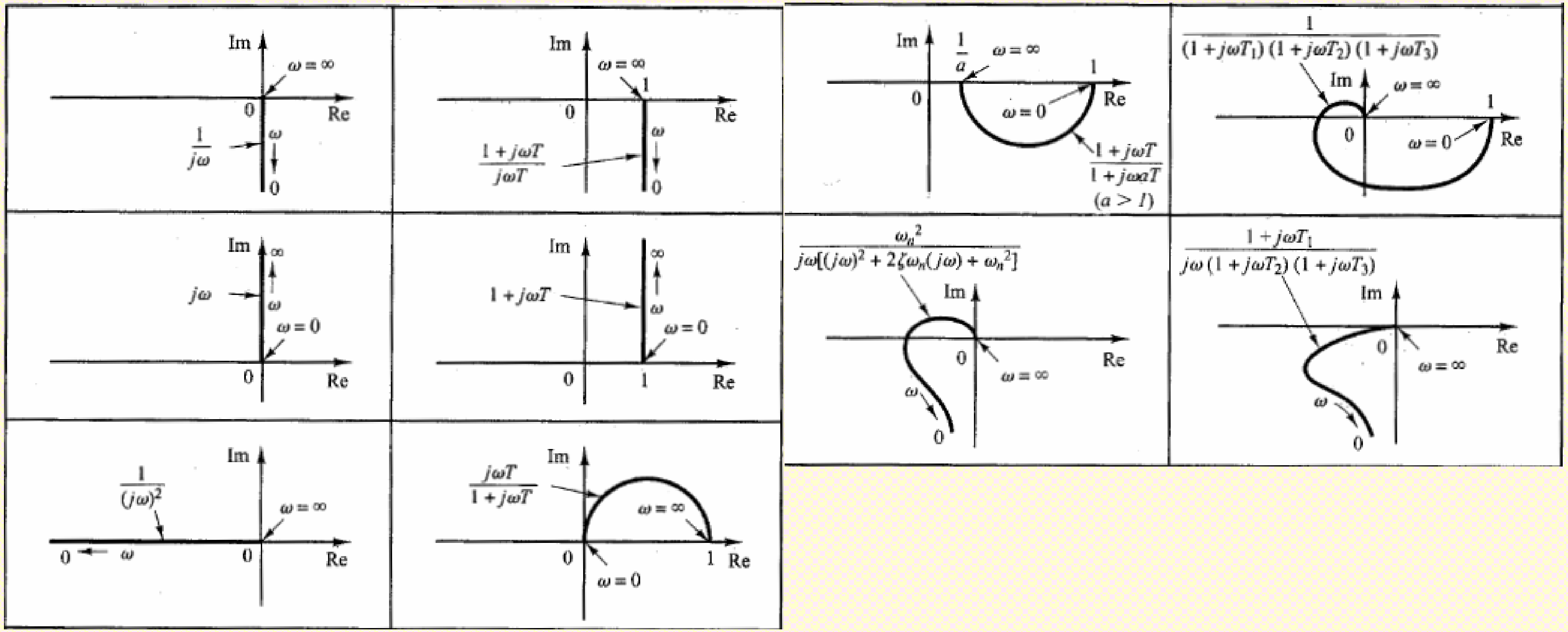
$$\phi = \angle G(j\omega)$$

General nature of Nyquist curves



Ref: K. Ogata: Modern Control Engineering, 3rd Ed., Prentice Hall, 1999.

Nyquist plots for several standard transfer functions

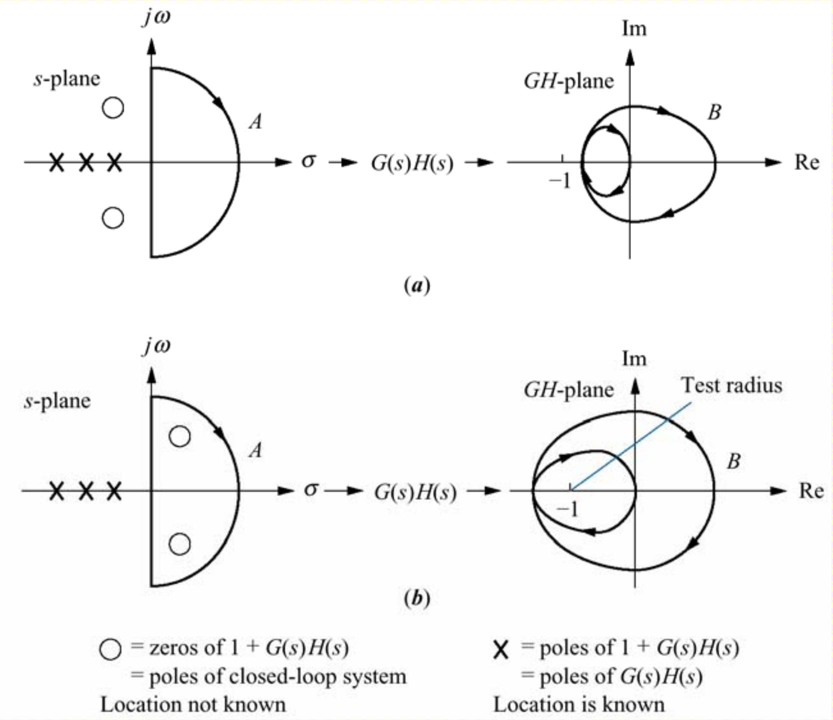


Ref: K. Ogata: Modern Control Engineering, 3rd Ed., Prentice Hall, 1999.

Nyquist stability criterion

Special case: (G(s)H(s) has neither poles nor zeros on the jω axis)

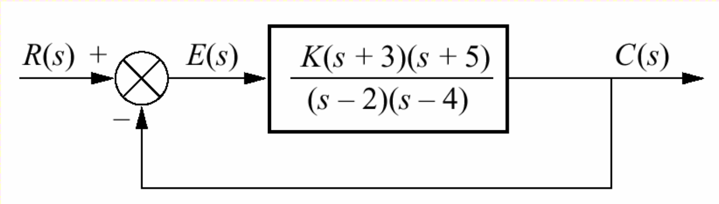
If a contour, that encircles the entire right half –plane is mapped through G(s)H(s). Then number of closed loop poles, Z, in right half of s-plane equals the number of open loop poles P, that are in right half of s-plane minus the number of counter-clockwise revolutions N around -1 of the mapping , i.e. Z = P - N.



a. contour does not enclose closed-loop poles

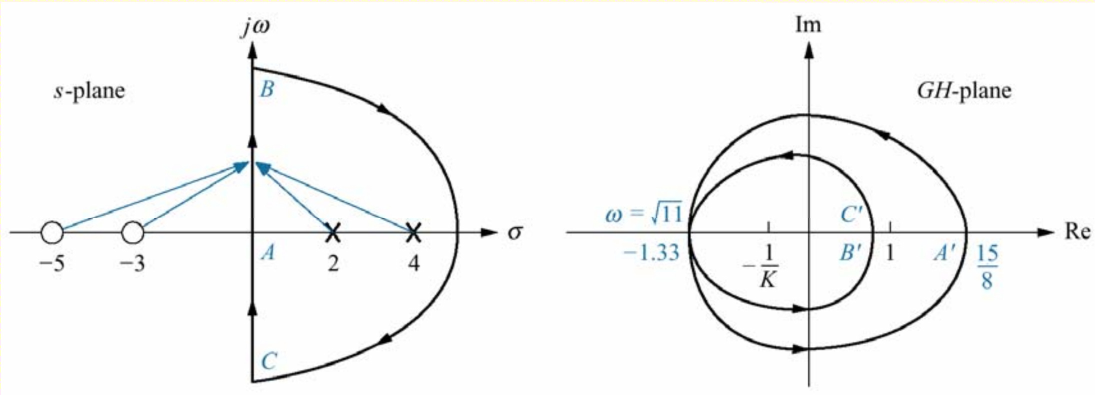
b. contour does enclose closed loop poles

Example – 1



(*a*)

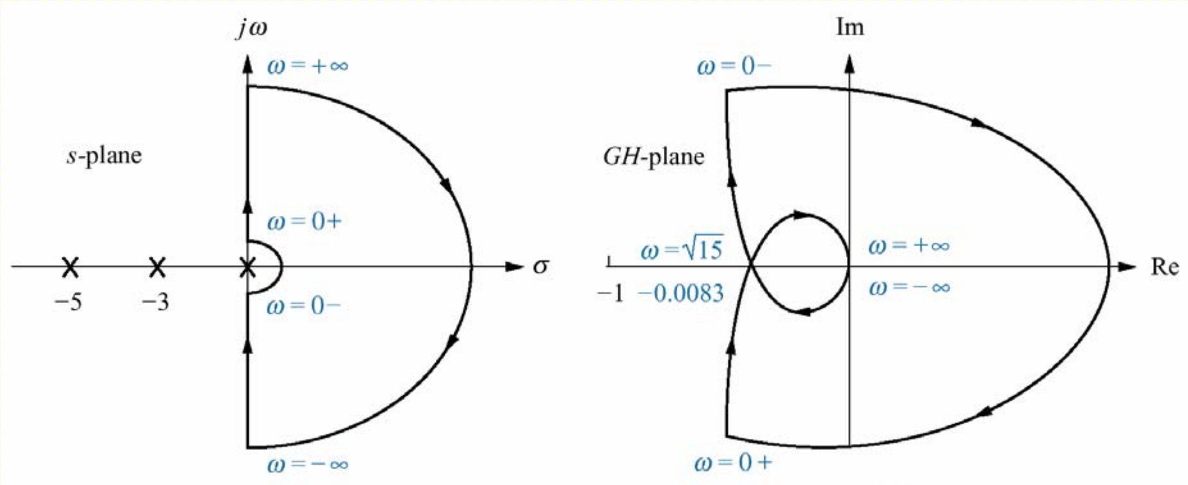
Open loop poles in the right half of s-plane are 2,4, i.e, P = 2
Number of encirclements of (-1), N = 2
Z = P-N = 0, hence the system is stable.



(*b*) (*c*)

Example – 2: $G(s) = \dfrac{K}{s(s+3)(s+5)}$

- There are no open loop poles in the right half of s-plane, i.e, P = 0.
- Number of encirclements N = 0.
- Z = P - N = 0, hence the system is stable.
- Value of K which determines the stability is 120.5. It implies if K < 120.5 then system is stable.
- if K > 120.5, critical point is encircled and N = -1. In that case Z = P - N = 1, and hence the system is unstable



(a) (b)

Robustness Concepts

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Definitions

The closed loop poles can be determined through characteristic equation

$$1 + G(s)H(s) = 0, \quad s = j\omega$$

$$G(s)H(s) = -1 + j0 = 1\angle 180^0$$

<u>Phase cross over frequency</u> (ω_{pc})

Frequency at which the phase angle of the transfer function becomes -180°

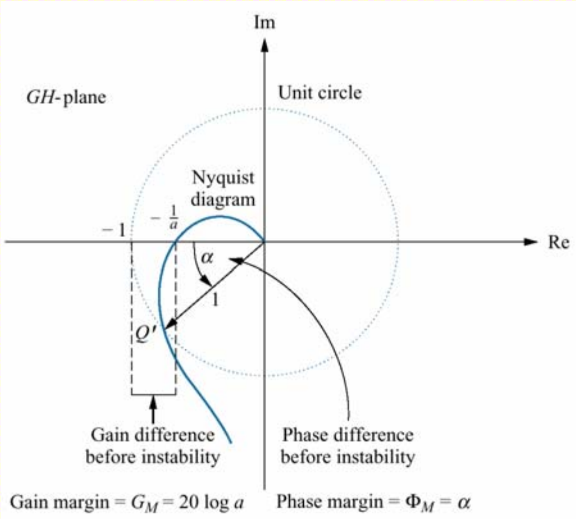
$$\angle G(j\omega)H(j\omega) = -180^0$$

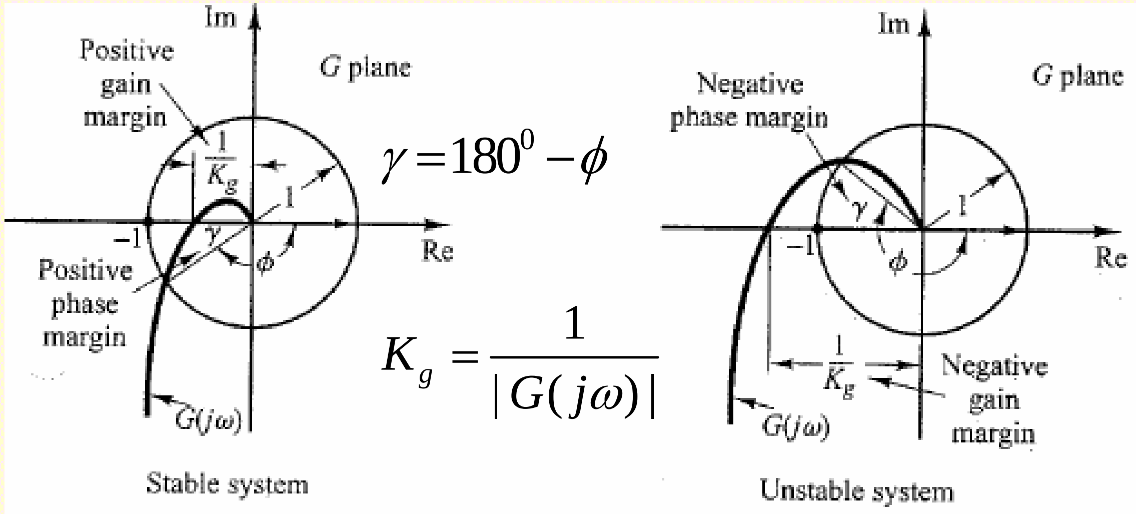
<u>Gain cross over frequency</u> (ω_{gc})

Frequency at which the magnitude of the open loop transfer function, is unity, i.e. $|G(j\omega)H(j\omega)| = 1$.

These frequencies play an important role in determining the stability margins of the system.

Gain and Phase Margins through Nyquist Plots

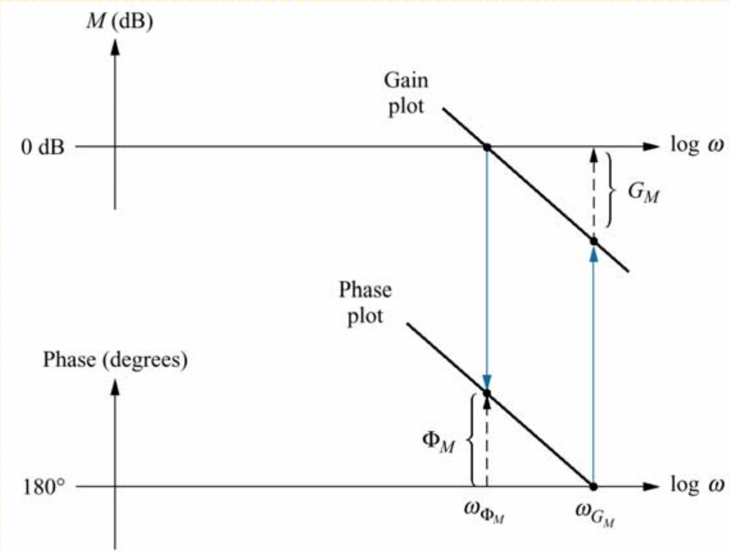


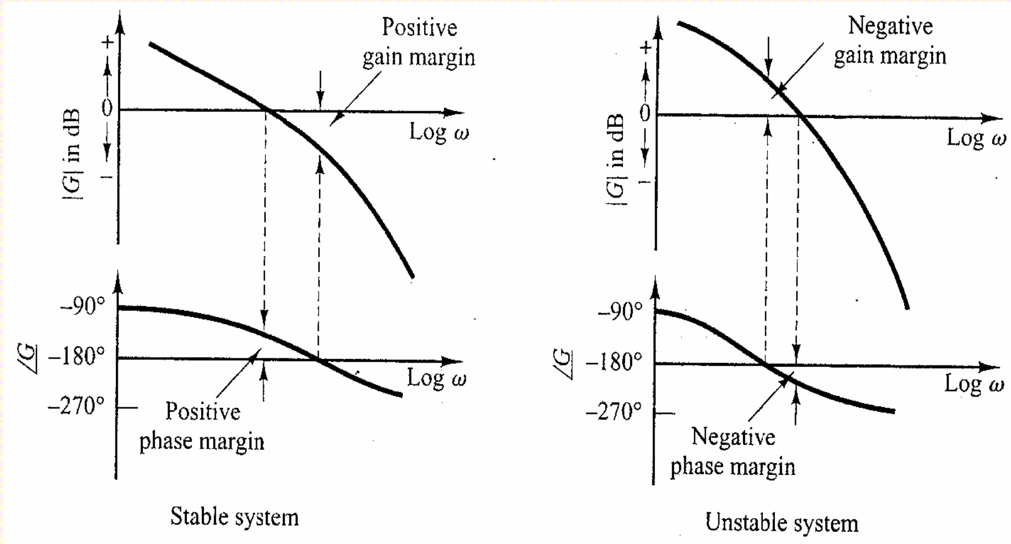


System is said to be

- Stable, if G_M and Φ_M both are positive, i.e. $\omega_{pc} > \omega_{gc}$
- Marginally stable, if G_M and Φ_M both are zero i.e. $\omega_{pc} = \omega_{gc}$
- Unstable, if G_M and Φ_M both are negative i.e. $\omega_{pc} < \omega_{gc}$

Gain and Phase Margins through Bode Plots





- For a stable minimum-phase system, GM/PM indicates how much gain/phase **can be increased** before the system becomes unstable.
- For an unstable system, GM/PM is indicative of how much gain/phase **must be decreased** to make the system stable.

Frequency Response Characteristics

- Consider a closed loop transfer function of second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- The closed loop frequency response

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}} = Me^{j\alpha}$$

$$M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}, \qquad \alpha = -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

- The frequency at which the M reaches its peak value is called resonant frequency (ω_P)

$$\omega_P = \omega_n\sqrt{1 - 2\xi^2}$$

- At ω_P, the slope of the magnitude curve is zero.

Frequency Response Characteristics

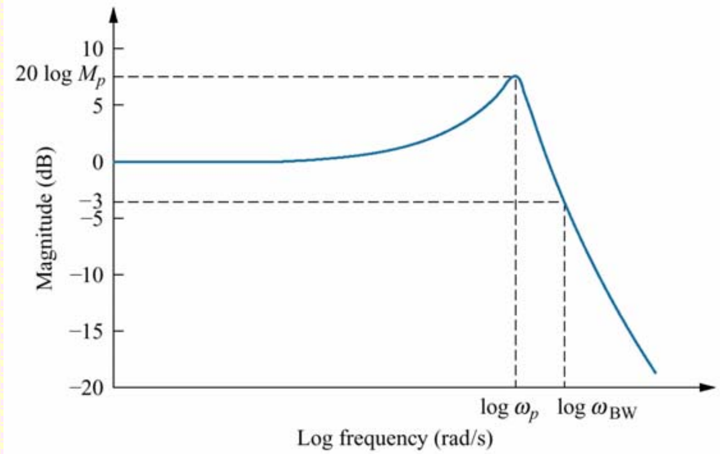
- The maximum value of magnitude is known as the resonant peak (M_p)

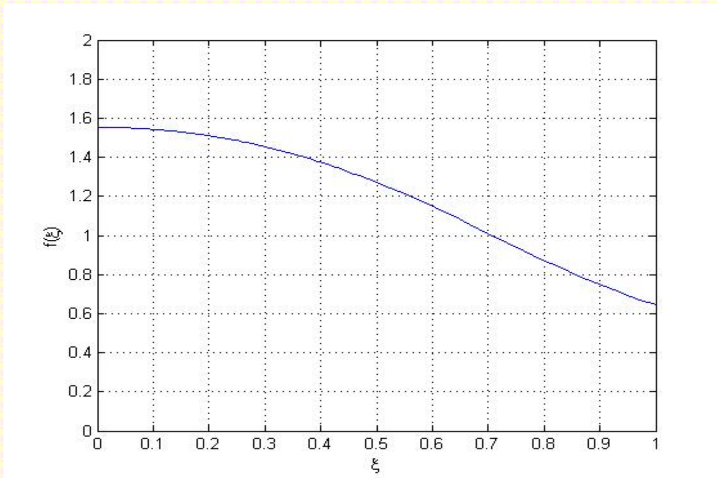
$$M_P = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

- Bandwidth (ω_{BW}) is the frequency at which the magnitude response curve is 3dB down from its value at zero frequency .

$$\omega_{BW} = \omega_n \underbrace{\sqrt{\left(1-2\xi^2\right)+\sqrt{4\xi^4-4\xi^2+2}}}_{f(\xi)}$$

$$T_s \approx \left(4/\xi\right)/\omega_n$$





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Some Points to Remember

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Some Points to Remember

- Block diagrams are not circuit diagrams!
- Stability and robustness are necessary of any good control design. After that, one must look for performance (fast response, optimality etc.)
- High controller gains:
  - Good benefits – Robust stability, Good tracking
  - Bad effects – Control saturation, Noise amplification
- Inter-coupling of variables, nonlinearities, control and state saturation limits, time delays, quantization errors etc. are always present: Classical SISO approach may not be adequate; <u>Advanced techniques are needed.</u>

Thanks for the Attention….!