Lecture – 8

State Space Representation of Dynamical Systems

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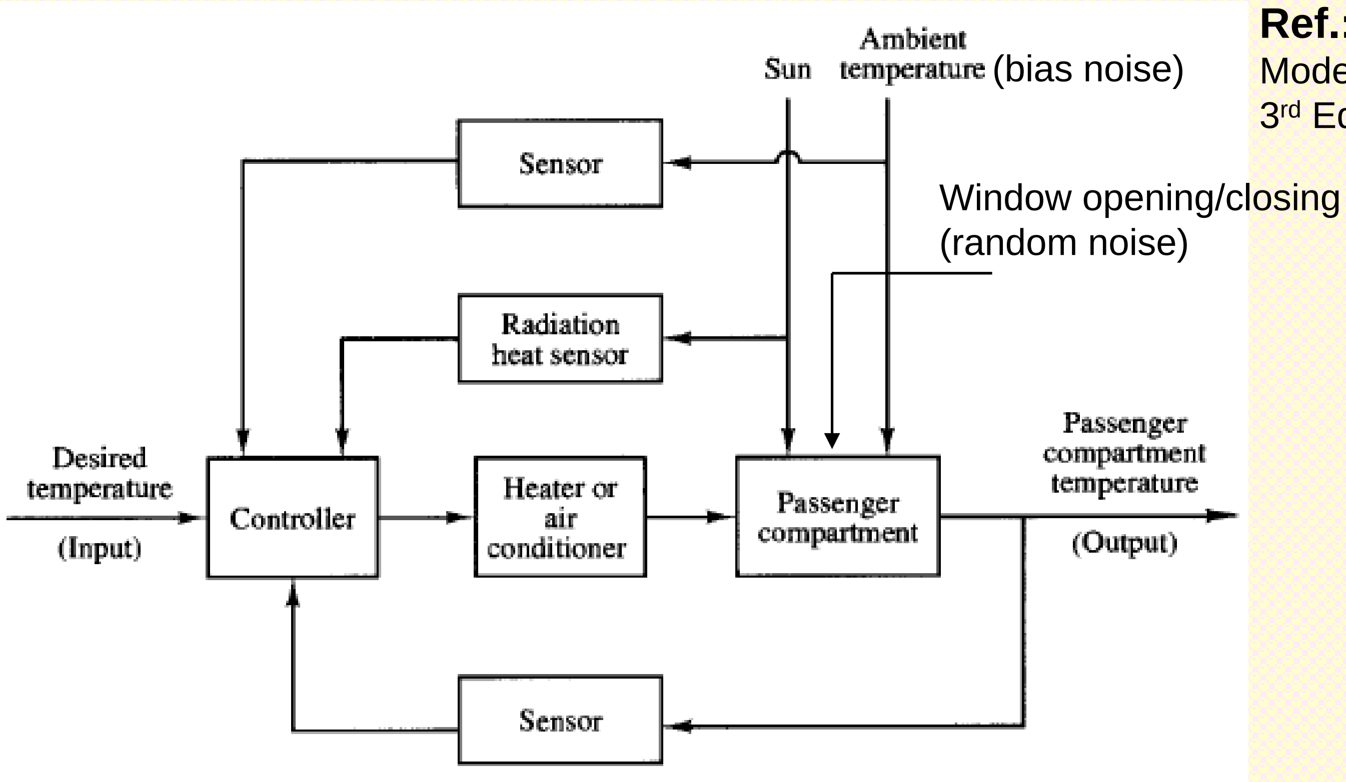
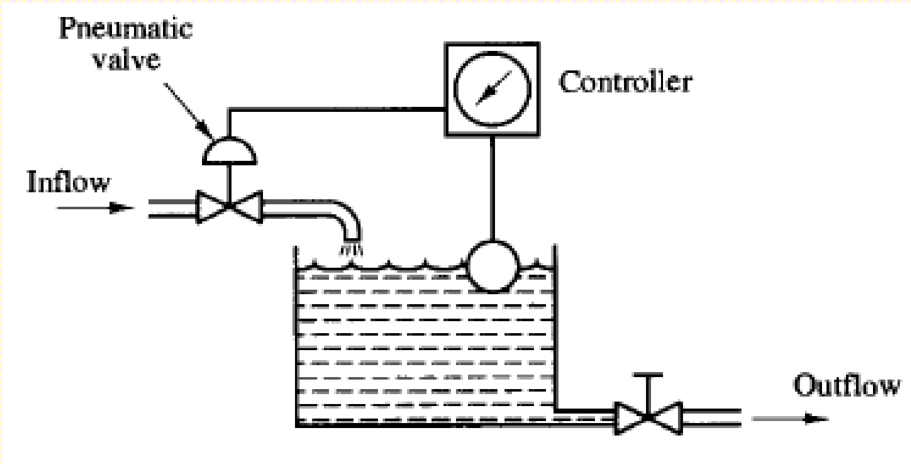
A Practical Control System

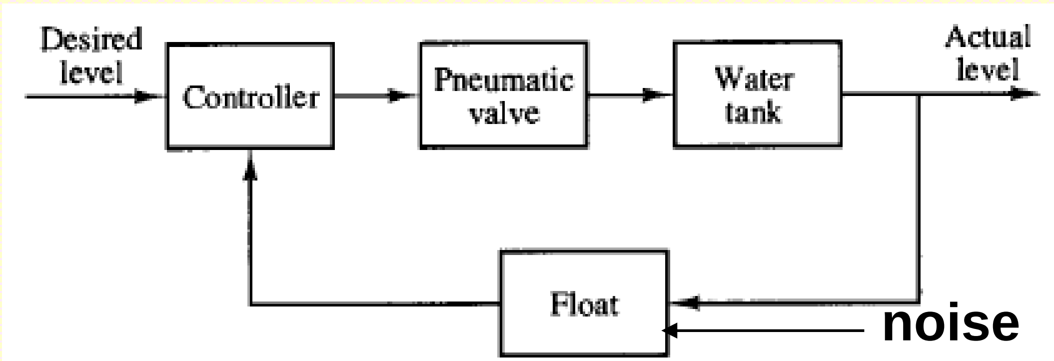


Ref.: K. Ogata,
Modern Control Engineering
3rd Ed., Prentice Hall, 1999.

Temperature control system in a car

Another Practical Control System





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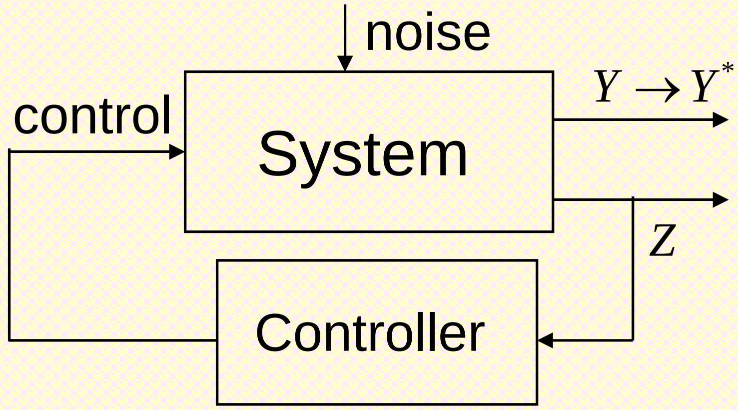
Water level control in an overhead tank

State Space Representation

- **Input variable**:
  - Manipulative (control)
  - Non-manipulative (noise)



- **Output variable**:

  Variables of interest that can be either be measured or calculated

- **State variable:**

  Minimum set of parameters which completely summarize the system's status.

Definitions

State : The state of a dynamic system is the smallest number of variables (called **state variables**) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t = t_0$, completely determine the behaviour of the system for any time $t \geq t_0$.

Note : State variables need not be physically measurable or observable quantities. This gives extra flexibility.

Definitions

State Vector : A n-dimensional vector whos components are n state variables that describe the system completely.

State Space : The n-dimensional space whose co-ordinate axes consist of the x_1 axis, x_2 axis, $\cdots$, x_n axis is called a state space.

Note : For any dynamical system, the state space remains unique, but the state variables are not unique.

Critical Considerations while Selecting State Variables

- Minimum number of variables
  - Minimum number of first-order differential equations needed to describe the system dynamics completely
  - Lesser number of variables: won't be possible to describe the system dynamics
  - Larger number of variables:
    - Computational complexity
    - Loss of either controllability, or observability or both.
- Linear independence. If not, it may result in:
  - Bad: May not be possible to solve for all other system variables
  - Worst: May not be possible to write the complete state equations

State Variable Selection

- Typically, the number of state variables (*i.e.* the order of the system) is equal to the number of independent energy storage elements. However, there are exceptions!

- Is there a restriction on the selection of the state variables ?

  YES! *All state variables should be linearly independent and they must collectively describe the system completely.*

Advantages of State Space Representation

- Systematic analysis and synthesis of higher order systems without truncation of system dynamics
- Convenient tool for MIMO systems
- Uniform platform for representing time-invariant systems, time-varying systems, linear systems as well as nonlinear systems
- Can describe the dynamics in almost all systems (mechanical systems, electrical systems, biological systems, economic systems, social systems etc.)
- **Note:** Transfer function representations are valid for only for linear time invariant (LTI) systems

Generic State Space Representation

$$X \triangleq \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n, \quad U \triangleq \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}^T \in \mathbb{R}^m$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} f_1\left(t,\ x_1 \quad \cdots \quad x_n,\ u_1 \quad \cdots \quad u_m\right) \\ \vdots \\ \vdots \\ f_n\left(t,\ x_1 \quad \cdots \quad x_n,\ u_1 \quad \cdots \quad u_m\right) \end{bmatrix}}_{f(t, X, U)}, \quad t \in \mathbb{R}^+$$

Generic State Space Representation

$$Y \triangleq \begin{bmatrix} y_1 & \cdots & y_p \end{bmatrix}^T \in \mathbb{R}^p$$

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_p \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} h_1(t,\ x_1 & \cdots & x_n,\ u_1 & \cdots & u_m) \\ & & \vdots & & \\ & & \vdots & & \\ h_p(t,\ x_1 & \cdots & x_n,\ u_1 & \cdots & u_m) \end{bmatrix}}_{h(t,X,U)}, \quad t \in \mathbb{R}^+$$

Summary:

$\dot{X} = f(t, X, U)$:	A set of differential equations
$Y = h(t, X, U)$:	A set of algebraic equations

State Space Representation
(noise free systems)

- **Nonlinear System**

$$\dot{X} = f(X, U) \qquad X \in R^n, U \in R^m$$

$$Y = h(X, U) \qquad Y \in R^p$$

- **Linear System**

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

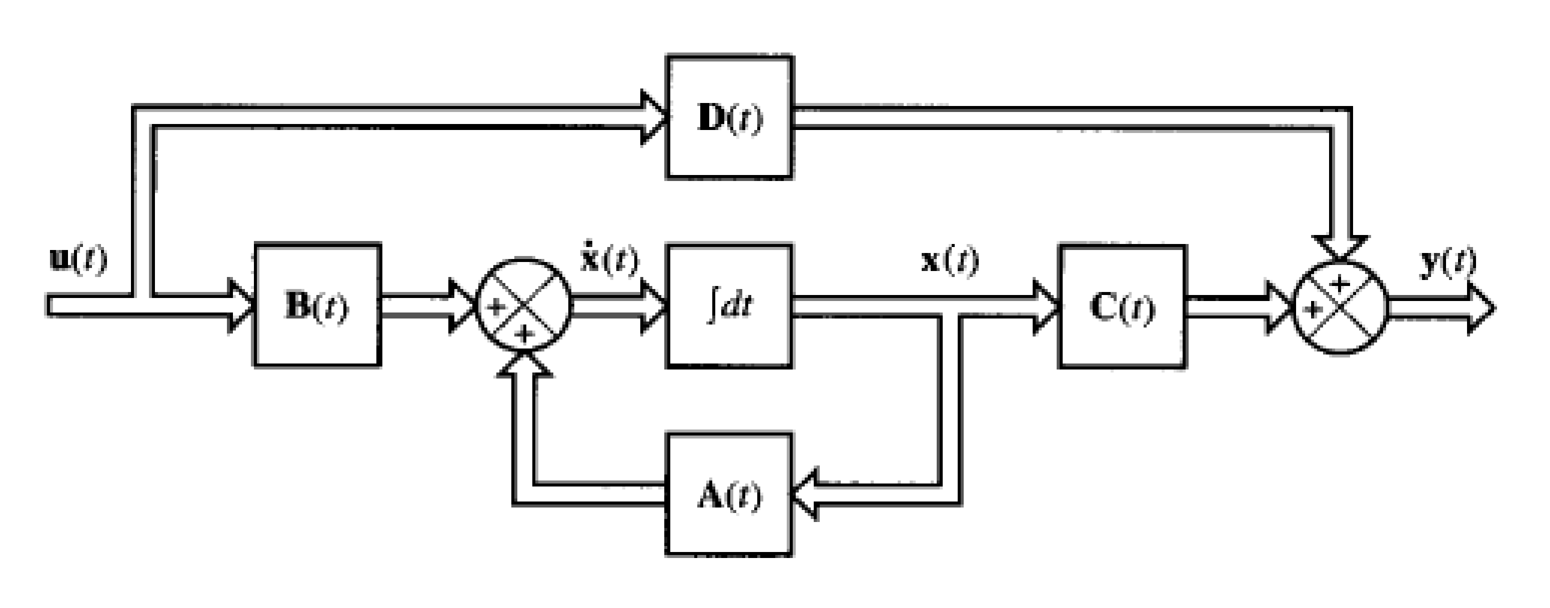
A - System matrix- *n x n*

B - Input matrix- *n x m*

C - Output matrix- *p x n*

D - Feed forward matrix – *p x m*

Block diagram representation of linear systems



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Writing Differential Equations in First Companion Form

(Phase variable form/Controllable canonical form)

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \quad \cdots \quad + a_0 y = u$$

Choose output $y(t)$ and its $(n-1)$ derivatives as state variables

$$\begin{bmatrix} x_1 = y \\ x_2 = \dfrac{dy}{dt} \\ \vdots \\ x_n = \dfrac{d^{n-1}y}{dt^{n-1}} \end{bmatrix} \xrightarrow{\text{differentiating}} \begin{bmatrix} \dot{x}_1 = \dfrac{dy}{dt} \\ \dot{x}_2 = \dfrac{d^2 y}{dt^2} \\ \vdots \\ \dot{x}_n = \dfrac{d^n y}{dt^n} \end{bmatrix}$$

First Companion Form
(Controllable Canonical Form)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & \cdots & \cdots & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Example – 1

- Dynamical system $\ddot{x} + 3\dot{x} + 2x = u$

$$y = x$$

- State variables: $x_1 \triangleq x, \quad x_2 \triangleq \dot{x}$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - 3x_2 + u$$

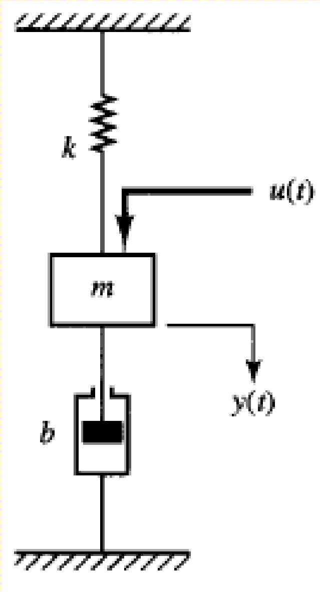
Hence $$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

Example – 2 (spring-mass-damper system)

- System dynamics

$$m\ddot{y} + c\dot{y} + ky = bu$$



- State variables

$$x_1 \triangleq y, \quad x_2 \triangleq \dot{y}$$

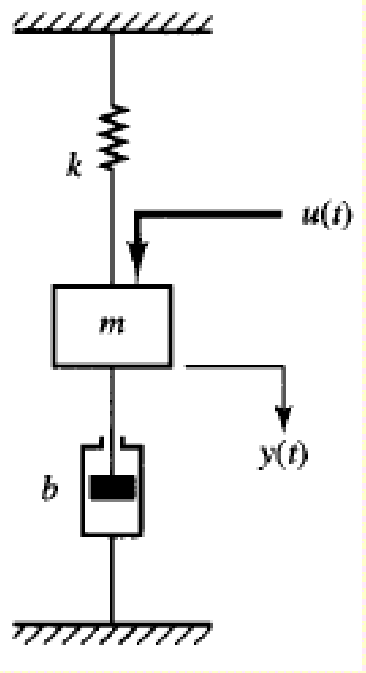
- State dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \frac{1}{m}\left(-ky - c\dot{y}\right) + \frac{b}{m}u \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}\left(-kx_1 - cx_2\right) + \frac{b}{m}u \end{bmatrix}$$

Example – 2 (spring-mass-damper system)

- State dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{m} \end{bmatrix} u$$

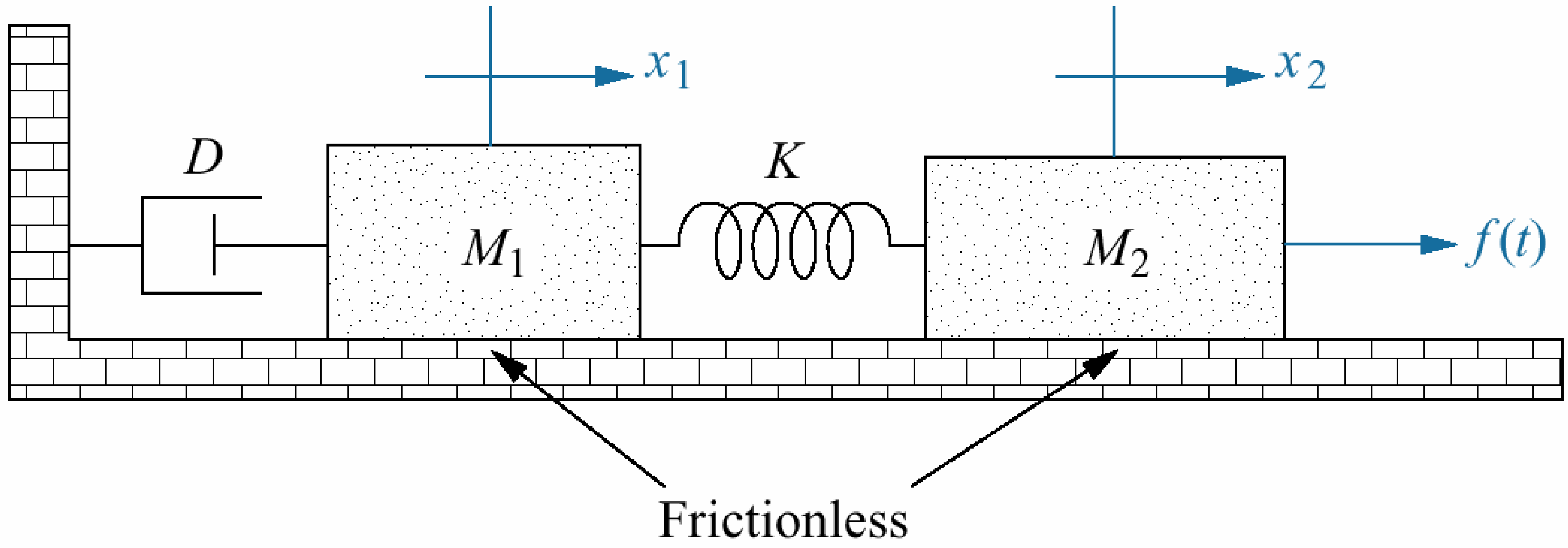


- Output equation

$$y \triangleq x_1$$

$$y = \begin{bmatrix} 1 & \cdots & 0 \end{bmatrix} X + \begin{bmatrix} 0 \end{bmatrix} u$$

Example – 3: Translational Mechanical System



$$M_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + K\left(x_1 - x_2\right) = 0$$

$$M_2 \frac{d^2 x_2}{dt^2} + K\left(x_2 - x_1\right) = f\left(t\right)$$

Ref: N. S. Nise:
Control Systems Engineering,
4th Ed., Wiley, 2004

Example – 3: Translational Mechanical System

- Define $$v_1 \triangleq \frac{dx_1}{dt}, \quad v_2 \triangleq \frac{dx_2}{dt}$$

- System equations
$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$
$$\frac{dv_2}{dt} = \frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t)$$

- State space equations in standard form
$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M_1} & -\frac{D}{M_1} & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & -\frac{K}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} f(t)$$

Example – 4: Nonlinear spring in Example – 3

- Dynamic equations

$$M_1 \frac{d^2 x_1}{dt^2} + D\frac{dx_1}{dt} + K\left(x_1 - x_2\right)^3 = 0$$

$$M_2 \frac{d^2 x_2}{dt^2} + K\left(x_2 - x_1\right)^3 = f\left(t\right)$$

- State space equation

$$\dot{x}_1 = v_1$$

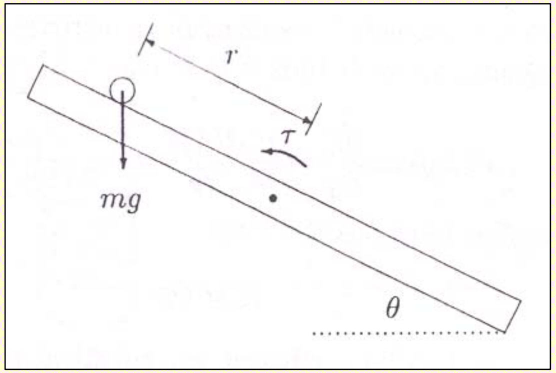
$$\dot{v}_1 = -\frac{K}{M_1}\left(x_1 - x_2\right)^3 - \frac{D}{M_1} v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = -\frac{K}{M_2}\left(x_2 - x_1\right)^3 + \frac{1}{M_2} f\left(t\right)$$

Example – 5
The Ball and Beam System

The beam can rotate by applying a torque at the centre of rotation, and ball can move freely along the beam



Moment of Inertia of beam: J

Mass, moment of inertia and radius of ball: m, J_b, R

Example – 5
The Ball and Beam System

State Space Model

$$x_1 = r,\ x_2 = \dot{r},\ x_3 = \theta,\ x_4 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-mg\sin x_3 + m\,x_1 x_4^2}{m + \dfrac{J_b}{R^2}}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{\tau - mg\,x_1\cos x_3 - 2m\,x_1 x_2 x_4}{m\,x_1^2 + J + J_b}$$

Example – 6: Van-der Pol's Oscillator (Limit cycle behaviour)

- Equation $M\,\ddot{x} + 2c\left(x^2 - 1\right)\dot{x} + k\,x = 0 \qquad \{c, k > 0\}$

- State variables $x_1 \triangleq x, \qquad x_2 \triangleq \dot{x}$

- State Space Equation

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} x_2 \\ -\dfrac{2c}{m}\left(x_1^2 - 1\right)x_2 - \dfrac{k}{m}x_1 \end{bmatrix}}_{F(X)} : \text{ Homogeneous nonlinear system}$$

Example – 7: Spinning Body Dynamics (Satellite dynamics)

Dynamics:

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 + \left(\frac{1}{I_1}\right)\tau_1$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1 + \left(\frac{1}{I_2}\right)\tau_2$$

$$\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2 + \left(\frac{1}{I_3}\right)\tau_3$$

I_1, I_2, I_3 : MI about principal axes

$\omega_1, \omega_2, \omega_3$: Angular velocities about principal axes

τ_1, τ_2, τ_3: Torques about principal axes

Example – 8: Airplane Dynamics, Six Degree-of-Freedom Nonlinear Model

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

$$\dot{U} = VR - WQ - g\sin\Theta + \left(F_{A_X} + F_{T_X}\right)/m$$

$$\dot{V} = WP - UR + g\sin\Phi\cos\Theta + \left(F_{A_Y} + F_{T_Y}\right)/m$$

$$\dot{W} = UQ - VP + g\cos\Phi\cos\Theta + \left(F_{A_Z} + F_{T_Z}\right)/m$$

$$\dot{P} = c_1 QR + c_2 PQ + c_3\left(L_A + L_T\right) + c_4\left(N_A + N_T\right)$$

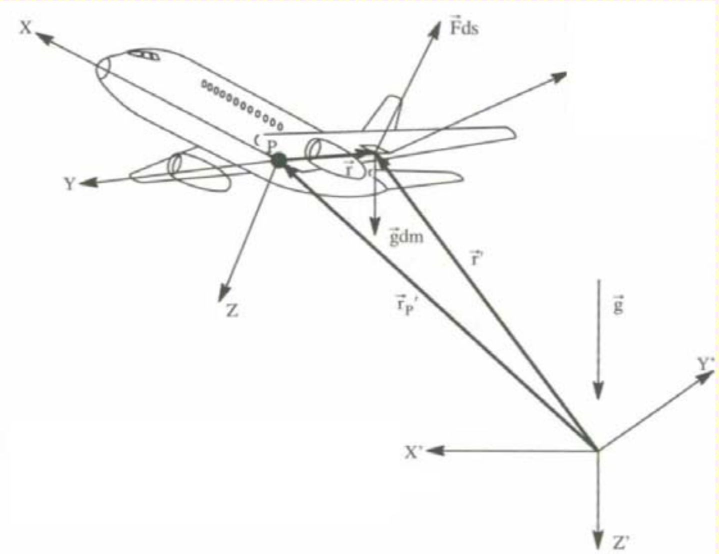
$$\dot{Q} = c_5 PR - c_6\left(P^2 - R^2\right) + c_7\left(M_A + M_T\right)$$

$$\dot{R} = c_8 PQ - c_2 QR + c_4\left(L_A + L_T\right) + c_9\left(N_A + N_T\right)$$

$$\dot{\Phi} = P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta$$

$$\dot{\Theta} = Q\cos\Phi - R\sin\Phi$$

$$\dot{\Psi} = \left(Q\sin\Phi + R\cos\Phi\right)\sec\Theta$$



$$\begin{bmatrix}\dot{X}'\\ \dot{Y}'\\ \dot{Z}'\end{bmatrix} = \begin{bmatrix}\cos\Psi & -\sin\Psi & 0\\ \sin\Psi & \cos\Psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\Theta & 0 & \sin\Theta\\ 0 & 1 & 0\\ -\sin\Theta & 0 & \cos\Theta\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\Phi & -\sin\Phi\\ 0 & \sin\Phi & \cos\Phi\end{bmatrix}\begin{bmatrix}U\\ V\\ W\end{bmatrix}$$

$\left[\text{Note: } \dot{h} = -\dot{Z}'\right]$

Example – 8: Airplane Dynamics, Six Degree-of-Freedom Nonlinear Model

Ref: Roskam J., Airplane Flight Dynamics and Automatic Controls, *1995*

$$F_{T_x} = \sum_{i=1}^{N} T_i \cos\Phi_{T_i} \cos\Psi_{T_i} \qquad L_T = -\sum_{i=1}^{N} \left(T_i \cos\Phi_{T_i} \sin\Psi_{T_i}\right) z_{T_i} - \sum_{i=1}^{N} \left(T_i \sin\Phi_{T_i}\right) y_{T_i}$$

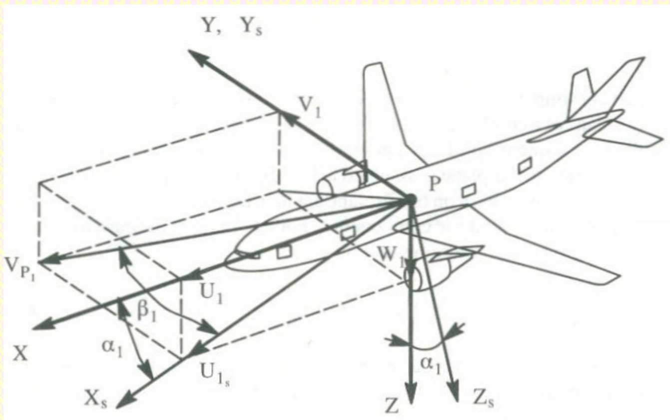
$$F_{T_y} = \sum_{i=1}^{N} T_i \cos\Phi_{T_i} \sin\Psi_{T_i} \qquad M_T = \sum_{i=1}^{N} \left(T_i \cos\Phi_{T_i} \cos\Psi_{T_i}\right) z_{T_i} + \sum_{i=1}^{N} \left(T_i \sin\Phi_{T_i}\right) x_{T_i}$$

$$T(\alpha) \triangleq \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$F_{T_z} = -\sum_{i=1}^{N} T_i \sin\Phi_{T_i} \qquad N_T = -\sum_{i=1}^{N} \left(T_i \cos\Phi_{T_i} \cos\Psi_{T_i}\right) y_{T_i} + \sum_{i=1}^{N} \left(T_i \cos\Phi_{T_i} \sin\Psi_{T_i}\right) x_{T_i}$$

$$\begin{bmatrix} F_{A_X} \\ F_{A_Z} \end{bmatrix} = T(\alpha) \begin{bmatrix} F_{A_{X_S}} \\ F_{A_{Z_S}} \end{bmatrix} = T(\alpha)(-\bar{q}S) \left(\begin{bmatrix} C_{D_O} & C_{D_\alpha} & C_{D_{i_h}} \\ C_{L_O} & C_{L_\alpha} & C_{L_{i_h}} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ i_h \end{bmatrix} + \begin{bmatrix} C_{D_{\delta_E}} \\ C_{L_{\delta_E}} \end{bmatrix} \delta_E \right)$$

$$\begin{bmatrix} L_A \\ N_A \end{bmatrix} = T(\alpha) \begin{bmatrix} L_{A_S} \\ N_{A_S} \end{bmatrix} = T(\alpha)\, \bar{q}Sb \left(\begin{bmatrix} C_{l_\beta} \\ C_{n_\beta} \end{bmatrix} \beta + \begin{bmatrix} C_{l_{\delta_A}} & C_{l_{\delta_R}} \\ C_{n_{\delta_A}} & C_{n_{\delta_R}} \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} \right)$$



$$F_{A_Y} = \bar{q}S\, C_Y = \bar{q}S \left(C_{Y_\beta} \beta + \begin{bmatrix} C_{Y_{\delta_A}} & C_{Y_{\delta_R}} \end{bmatrix} \begin{bmatrix} \delta_A \\ \delta_R \end{bmatrix} \right)$$

$$M_A = \bar{q}S\bar{c}\, C_m = \bar{q}S\bar{c} \begin{bmatrix} C_{m_o} & C_{m_\alpha} & C_{m_{i_h}} \end{bmatrix} \begin{bmatrix} 1 & \alpha & i_h \end{bmatrix}^T + C_{m_{\delta_E}} \delta_E$$

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Thanks for the Attention...!