Lecture – 19

Stability of Linear Time Invariant Systems

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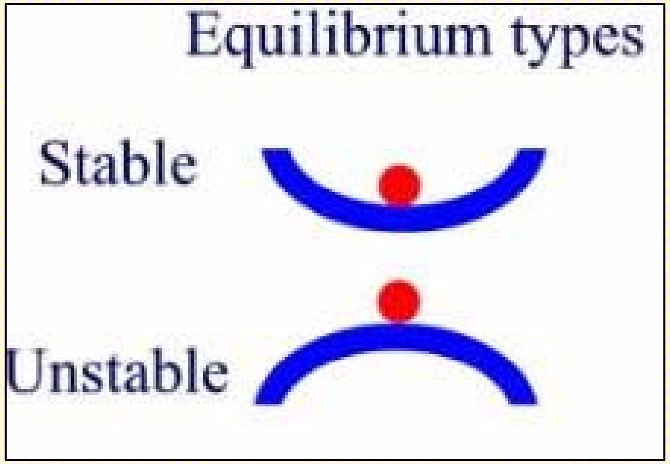
Stability of Linear Systems

Definition: If a system in equilibrium is disturbed and the system returns back to the equilibrium point with time, then the equilibrium point is said to be stable.



Stability of Linear Time Invariant (LTI) Systems

System:

$$\dot{X} = AX, \quad X(0) = X_0$$

Question:

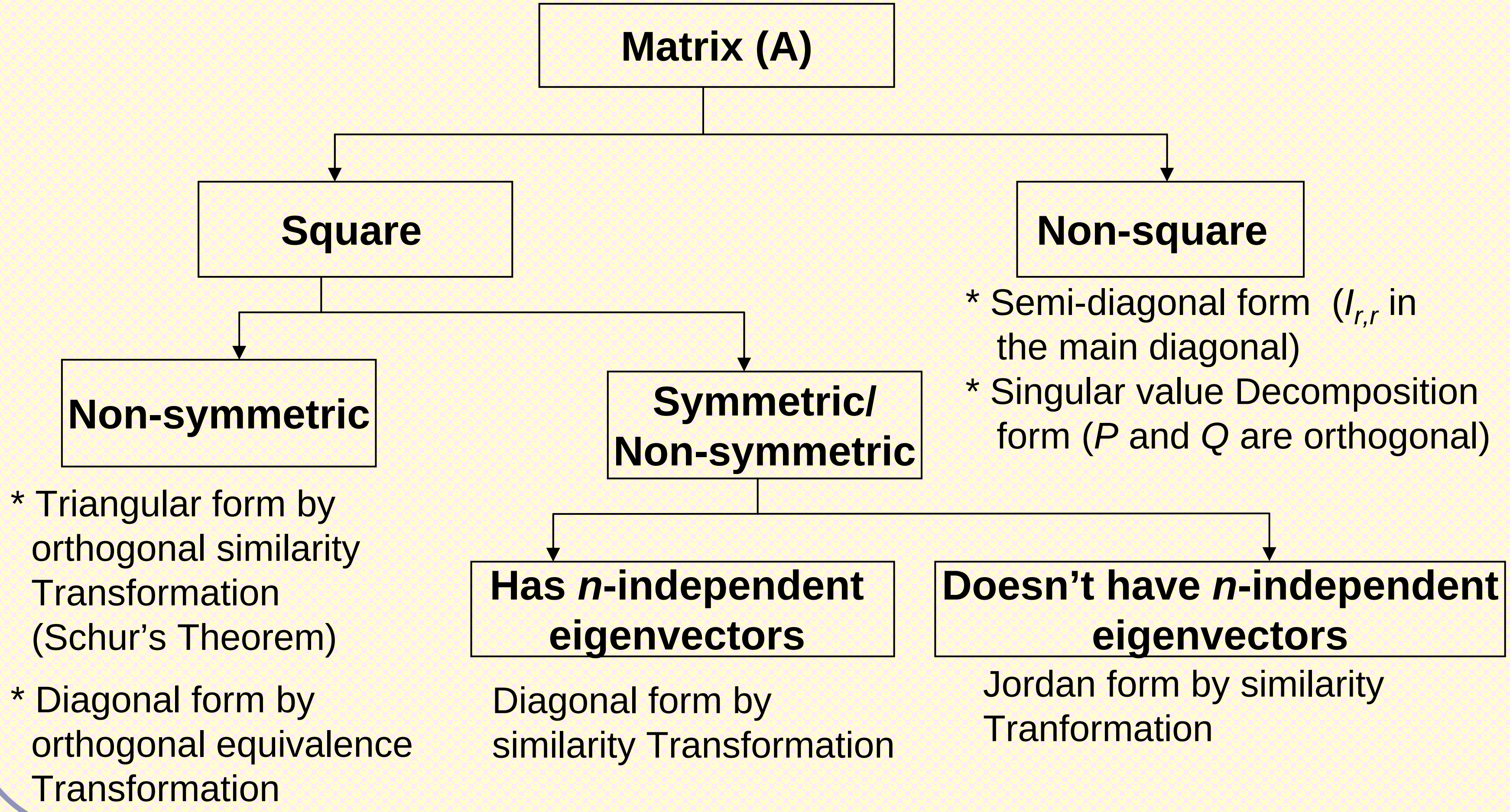
Can we conclude about nature of the solution, without solving the system model?

Answer: YES!

Definition: Eigenvalues of *A : "*Poles" of the system!

The nature of the solution is governed only by the locations of its poles

Summary of Matrix Transformations



Similarity Transformation

Definition: If $A_{n\times n}$ and $B_{n\times n}$ are nonsingular matrices and $P_{n\times n}$ is a non-singular matrix such that $B = P^{-1}AP$, then A and B are "similar".

Simplest forms possible:

- **Diagonal form**

  (if there are *n* linearly independent eigenvectors)
- **Jordan form**

  (if the number of linearly independent eigenvectors are less then *n*)

Stability Analysis - Special Case: A_{nxn} has n linearly independent eigenvectors

A is similar to a diagonal matrix D.

$$A = PDP^{-1}$$

$$A^2 = PD^2P^{-1}, \quad A^3 = PD^3P^{-1}, \quad \cdots$$

$$D = diag\left(\lambda_1, \quad \cdots \quad, \lambda_n\right)$$

$$P = \begin{bmatrix} p_1 & \cdots & p_n \\ \downarrow & & \downarrow \end{bmatrix}$$

Special Case:
A_{nxn} has n linearly independent eigenvectors

Solution:

$$X(t) = e^{At} X_0$$

$$= \left(I + At + A^2 t^2 / 2! + A^3 t^3 / 3! + \cdots \right) X_0$$

$$= \left(PP^{-1} + PDP^{-1} t + PD^2 P^{-1} t^2 / 2! + \cdots \right) X_0$$

$$= P \left(I + Dt + D^2 t^2 / 2! + \cdots \right) P^{-1} X_0$$

$$= P \left(e^{Dt} \right) P^{-1} X_0$$

$$= P \left[diag \left(1 + \lambda_i t + \lambda_i^2 t^2 / 2! + \cdots \right) \right] C$$

Special Case:
A_{nxn} has n linearly independent eigenvectors

Solution:

$$X(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} p_i \quad \text{(Modal form)}$$

Conclusion

The nature of solution depends only on the location of poles!

All poles in the LH plane: Stable System

One pole in the RH plane: Unstable System

General Case: A_{nxn} does not have n linearly independent eigenvectors

A is similar to a block-diagonal Jordan matrix J.

$$J = diag\left(J_1, \quad \cdots \quad , J_p\right)$$

$$A = PJP^{-1}$$

$$A^2 = PJ^2P^{-1}, \quad A^3 = PJ^3P^{-1}, \quad \cdots$$

$$J^2 = diag\left(J_1^2, \quad \cdots \quad , J_p^2\right)$$

$$J^3 = diag\left(J_1^3, \quad \cdots \quad , J_p^3\right)$$

$$\vdots$$

General Case: A_{nxn} does not have n linearly independent eigenvectors

Solution: $$X(t) = e^{At} X_0$$
$$= \left(I + At + A^2 t^2 / 2! + A^3 t^3 / 3! + \cdots \right) X_0$$
$$= \left(PP^{-1} + PJP^{-1} t + PJ^2 P^{-1} t^2 / 2! + \cdots \right) X_0$$
$$= P \left(I + Jt + J^2 t^2 / 2! + \cdots \right) P^{-1} X_0$$
$$= P \left(e^{Jt} \right) P^{-1} X_0$$

$$e^{Jt} = diag\left(e^{J_1 t}, \quad \cdots \quad , e^{J_p t} \right)$$

General Case: A_{nxn} does not have n linearly independent eigenvectors

Let $\hat{J}$ be a particular $r \times r$ Jordan block with eigenvalue λ

$$\hat{J}t = \lambda t\, I + Et$$

$$E = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad E^2 = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \cdots \quad , E^{r-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$E^r = E^{r+1} = \quad \cdots \quad = 0$$

General Case: A_{nxn} does not have n linearly independent eigenvectors

$$e^{\hat{J}t} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{1}{2!}t^2 & \cdots & \frac{1}{(r-1)!}t^{r-1} \\ 0 & 1 & t & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \frac{1}{2!}t^2 \\ \vdots & \vdots & \ddots & 1 & t \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$X(t) = e^{At}X_0 = Pe^{Jt}C = \begin{bmatrix} p_1 & \cdots & p_p \\ \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} e^{J_1 t}C_1 \\ \vdots \\ e^{J_p t}C_p \end{bmatrix}$$

General Case: A_{nxn} does not have n linearly independent eigenvectors

Let λ_1 be repeated r_1 times.

Then $C_1 = \begin{bmatrix} c_{1_1} & \cdots & c_{1_{r_1}} \end{bmatrix}^T$

$$P_1 = \begin{bmatrix} p_1 & p_2 & \cdots & p_{r_1} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

p_1 : Eigenvector

$p_2, \cdots, p_{r_1}$: Generalized Eigenvectors

General Case: A_{nxn} does not have n linearly independent eigenvectors

$$e^{J_1 t} C_1 = \underbrace{e^{\lambda_1 t}}_{\textbf{Exponential}} \underbrace{\begin{bmatrix} c_{1_1} + c_{1_2} t + c_{1_3}\left(t^2/2!\right) + \quad \cdots \quad + c_{1_{r_1}}\left(t^{(r_1-1)}/(r_1-1)!\right) \\ c_{1_2} + c_{1_3} t + \quad \cdots \quad + c_{1_{r_1}}\left(t^{(r_1-2)}/(r_1-2)!\right) \\ \vdots \\ c_{1_{r_1}} \end{bmatrix}}_{\textbf{Polynomial}}$$

Similar expressions can be obtained for $P_i e^{J_i t} c_i, \quad i = 2, 3, \cdots$

Exponential term will eventually dominate the polynomial term!

Stability of Linear Systems

Conclusion

The nature of solution depends only on the location of poles!

All poles in the LH plane: Stable System

One pole in the RH plane: Unstable System

Stabilizing Control Design

Closed loop system: $\dot{X} = AX + BU$

$$U = -KX$$

$$\dot{X} = A_{CL} X, \text{ where } A_{CL} = (A - BK)$$

- Closed loop system is stable if Eigenvalues of A_{CL} satisfy the stability condition

- For stabilizing controller K needs to be selected in such a way that the eigenvalues of A_{CL} should be in the left half plane

Thanks for the Attention...!

Controllability of Linear Time Invariant Systems

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Controllability

- A system is said to be *controllable* at time t_0 if it is possible by means of an ***unconstrained control vector*** to transfer the system from any initial state X_0 to any other state *in a* ***finite interval of time***
- Controllability depends upon the system matrix *A* and the control influence matrix *B*

Condition for Controllability: (single input case)

System: $$\dot{X} = AX + Bu$$

Solution: $$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Assuming $X(t_1) = 0,$

$$0 = e^{At_1}X(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau$$

$$X(0) = -\int_0^{t_1} e^{-A\tau}Bu(\tau)\,d\tau$$

Condition for Controllability: (single input case)

$$e^{-A\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau)\, A^k \quad \left(\text{Sylvester's formula}\right)$$

$$X(0) = -\int_0^{t_1} e^{-A\tau} Bu(\tau)\, d\tau = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} \alpha_k(\tau)\, u(\tau)\, d\tau$$

$$= -\sum_{k=0}^{n-1} A^k B \beta_k$$ where $\beta_k \triangleq \int_0^{t_1} \alpha_k(\tau)\, u(\tau)\, d\tau$

$$= -\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}^T$$

This system should have a non-trivial solution for $\begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}^T$

Controllability

Result:

If the rank of $C_B \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is n, then the system is controllable.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$C_B = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}$$

$rank\left(C_B\right) = 2 \quad \therefore$ The system is controllable.

Output Controllability

Result:

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

$$X \in \mathbb{R}^n, \quad U \in \mathbb{R}^m, \quad Y \in \mathbb{R}^p$$

If the rank of $C_B \triangleq \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B & D \end{bmatrix}$ is p, then the system is output controllable.

Note: The presence of DU term in the output equation always helps to establish output controllability.

Observability of Linear Time Invariant Systems

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Observability

- A system is said to be *observable* at time t_0 if, with the system in state $X(t_0)$,it is possible to determine this state from the observation of the output over a finite interval of time
- Observability depends upon the system matrix A and the output matrix C

Observability

Result:

If the rank of $O_B \triangleq \begin{bmatrix} C^T & A^T C^T & \cdots & \left(A^T\right)^{n-1} C^T \end{bmatrix}$ is n,

then the system is observable.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$O_B = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$rank\left(O_B\right) = 1 \neq 2 \quad \therefore$ The system is NOT observable.

Controllability and Observability in Transfer Function Domain

- The system is both controllable and observable if there is no Pole-Zero cancellation.

- **Note:** The cancelled pole-zero pair suppresses part of the information about the system

Principle of Duality

System $\boldsymbol{S_1}$: $\dot{X} = AX + BU$ $\qquad C_B = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

$Y_1 = CX$ $\qquad O_B = \begin{bmatrix} C^T & A^TC^T & {A^T}^2C^T & \cdots & {A^T}^{n-1}C^T \end{bmatrix}$

System $\boldsymbol{S_2}$: $\dot{Z} = A^TZ + C^TV$ $\qquad C_B = \begin{bmatrix} C^T & A^TC^T & {A^T}^2C^T & \cdots & {A^T}^{n-1}C^T \end{bmatrix}$

$Y_2 = B^TZ$ $\qquad O_B = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$

The principle of duality states that the system $\boldsymbol{S_1}$ is controllable if and only if system $\boldsymbol{S_2}$ is observable; and vice-versa!

Hence, the problem of observer design for a system is actually a problem of control design for its dual system.

Stabilizability and Detectability

- Stabilizable system: Uncontrollable system in which uncontrollable part is stable

- Detectable system: Unobservable system in which the unobservable subsystem is stable

Where do uncontrollable or unobservable systems arise?

- Redundant state variables

- Physically uncontrollable system

- Too much symmetry

References

- K. Ogata: *Modern Control Engineering*, 3rd Ed., Prentice Hall, 1999.
- B. Friedland: *Control System Design*, McGraw Hill, 1986.

Thanks for the Attention...!