Lecture – 20

Controllability and Observability of Linear Time Invariant Systems

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Evaluation of Matrix Exponential e^{At}

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Method – 1: Power-series

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

- This method is useful and accurate only if the series truncates naturally. Otherwise, series truncation introduces approximation error.
- Direct computation of e^{At} as power series is computationally inefficient as well.

Method – 2: Using Laplace Transform

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right]$$

- This method results in closed form expressions for e^{At}, can be quite useful for small matrices.
- Numerical algorithms exist to evaluate

$(sI - A)^{-1}$. However, its inverse still need to be found.

- Can be quite cumbersome for large matrices.

Method – 3: Using Similarity Transform

(Provided the matrix can be diagonalizable)

$$
\begin{aligned}
e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\
&= PP^{-1} + PDP^{-1}t + \frac{\left(PDP^{-1}\right)\left(PDP^{-1}\right)t^2}{2!} + \cdots \\
&= P\left(I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots \right)P^{-1} \\
&= P\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}P^{-1}
\end{aligned}
$$

Similarity Transformation:

$$A = PDP^{-1}$$

Method – 4: Sylvester's Formula

Case – 1: Distinct Eigenvalues

e^At satisfies the following determinant equation:

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} & e^{\lambda_2 t} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} & e^{\lambda_n t} \\ I & A & A^2 & \cdots & A^{n-1} & \underbrace{e^{At}}_{\text{Ultimate aim}} \end{vmatrix} = \mathbf{0}$$

i.e.

$$e^{At} = \alpha_0(t) I + \alpha_1(t) A + \alpha_2(t) A^2 + \cdots + \alpha_{n-1}(t) A^{n-1}$$

Method – 4: Sylvester's Formula

Case – 1: Distinct Eigenvalues

The coefficients $\alpha_0(t), \alpha_1(t), \cdots, \alpha_{n-1}(t)$ can be determined from the following set of equations:

$$\alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{n-1}(t)\lambda_1^{n-1} = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_2 + \alpha_2(t)\lambda_2^2 + \cdots + \alpha_{n-1}(t)\lambda_2^{n-1} = e^{\lambda_2 t}$$

$$\vdots$$

$$\alpha_0(t) + \alpha_1(t)\lambda_n + \alpha_2(t)\lambda_n^2 + \cdots + \alpha_{n-1}(t)\lambda_n^{n-1} = e^{\lambda_n t}$$

Method – 4: Sylvester's Formula

Case – 2: Repeated Eigenvalues

e^{At} satisfies the following determinant equation:

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & \frac{(n-1)(n-2)}{2}\lambda_1^{n-3} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (n-1)\lambda_1^{n-2} & t\,e^{\lambda_1 t} \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{n-1} & e^{\lambda_1 t} \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{n-1} & e^{\lambda_4 t} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \lambda_n^3 & \cdots & \lambda_n^{n-1} & e^{\lambda_n t} \\ I & A & A^2 & A^3 & \cdots & A^{n-1} & e^{At} \end{vmatrix} = \mathbf{0}$$

Eigenvalues:

$$\underbrace{\lambda_1, \lambda_1, \lambda_1}_{\text{3 times}}, \lambda_4, \cdots, \lambda_n$$

i.e.

$$e^{At} = \alpha_0(t) I + \alpha_1(t) A + \alpha_2(t) A^2 + \cdots + \alpha_{n-1}(t) A^{n-1}$$

Method – 4: Sylvester's Formula

Case – 2: Repeated Eigenvalues

The coefficients $\alpha_0(t), \alpha_1(t), \cdots, \alpha_{n-1}(t)$

can be determined from:

$$\alpha_2(t) + 3\alpha_3(t)\lambda_1 + \cdots + \frac{(n-1)(n-2)}{2}\alpha_{n-1}(t)\lambda_1^{n-3} = \frac{t^2}{2}e^{\lambda_1 t}$$

$$\alpha_1(t) + 2\alpha_2(t)\lambda_1 + 3\alpha_3(t)\lambda_1^2 + \cdots + (n-1)\alpha_{n-1}(t)\lambda_1^{n-2} = te^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{n-1}(t)\lambda_1^{n-1} = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_4 + \alpha_2(t)\lambda_4^2 + \cdots + \alpha_{n-1}(t)\lambda_4^{n-1} = e^{\lambda_4 t}$$

$$\vdots$$

$$\alpha_0(t) + \alpha_1(t)\lambda_n + \alpha_2(t)\lambda_n^2 + \cdots + \alpha_{n-1}(t)\lambda_n^{n-1} = e^{\lambda_1 t}$$

Method – 4: Sylvester's Formula Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \qquad \lambda_{1,2} = 0, -2$$

To compute e^{At} using Sylvester's formula, we have

$$\begin{vmatrix} 1 & \lambda_1 & e^{\lambda_1 t} \\ 1 & \lambda_2 & e^{\lambda_2 t} \\ I & A & e^{At} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & e^{-2t} \\ I & A & e^{At} \end{vmatrix} = \mathbf{0}$$

Expanding the determinant

$$-2e^{At} + A + 2I - Ae^{-2t} = 0$$

$$e^{At} = \frac{1}{2}\left(A + 2I - Ae^{-2t}\right) = \begin{bmatrix} 1 & \frac{1}{2}\left(1 - e^{-2t}\right) \\ 0 & e^{-2t} \end{bmatrix}$$

Controllability of Linear Time Invariant Systems

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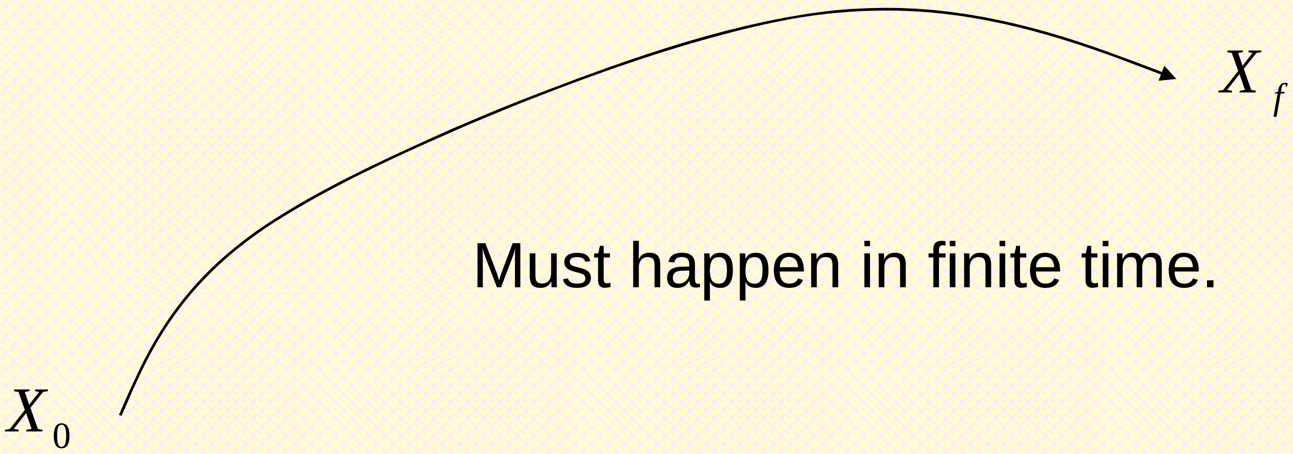
Controllability

- A system is said to be *controllable* at time t_0 if it is possible by means of an ***unconstrained control vector*** to transfer the system from any initial state X_0 to any other state *in a **finite interval of time***
- Controllability depends upon the system matrix *A* and the control influence matrix *B*

Graphical Meaning



Must happen in finite time.

Condition for Controllability: (single input case)

System: $$\dot{X} = AX + Bu$$

Solution: $$X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

Assuming $X(t_1) = 0$,

$$0 = e^{At_1} X(0) + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau$$

$$X(0) = -\int_0^{t_1} e^{-A\tau} Bu(\tau)\, d\tau$$

Condition for Controllability: (single input case)

$$e^{-A\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau)\, A^k \quad \text{(Sylvester's formula)}$$

$$X(0) = -\int_0^{t_1} e^{-A\tau} Bu(\tau)\, d\tau = -\sum_{k=0}^{n-1} A^k B \int_0^{t_1} \alpha_k(\tau)\, u(\tau)\, d\tau$$

$$= -\sum_{k=0}^{n-1} A^k B \beta_k$$

where $\beta_k \triangleq \int_0^{t_1} \alpha_k(\tau)\, u(\tau)\, d\tau$

$$= -\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}^T$$

This system should have a non-trivial solution for $\begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}^T$

Controllability

Result: If the rank of $C_B \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is n, then the system is controllable.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$C_B = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}$$

$$rank\left(C_B\right) = 2 \quad \therefore \text{ The system is controllable.}$$

Output Controllability

Result:

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

$$X \in \mathbb{R}^n, \quad U \in \mathbb{R}^m, \quad Y \in \mathbb{R}^p$$

If the rank of $C_B \triangleq \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B & D \end{bmatrix}$ is p, then the system is output controllable.

Note: The presence of DU term in the output equation always helps to establish output controllability.

Observability of Linear Time Invariant Systems

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Observability

- A system is said to be *observable* at time t_0 if, with the system in state $X(t_0)$,it is possible to determine this state from the observation of the output over a finite interval of time
- Observability depends upon the system matrix A and the output matrix C

Observability

Result: If the rank of $O_B \triangleq \begin{bmatrix} C^T & A^T C^T & \cdots & \left(A^T\right)^{n-1} C^T \end{bmatrix}$ is n, then the system is observable.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$O_B = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$rank\left(O_B\right) = 1 \neq 2 \quad \therefore$ The system is NOT observable.

Controllability and Observability in Transfer Function Domain

- The system is both controllable and observable if there is no Pole-Zero cancellation.

- **Note:** The cancelled pole-zero pair suppresses part of the information about the system

Principle of Duality

System $\boldsymbol{S_1}$: $\dot{X} = AX + BU$

$Y_1 = CX$

$$C_B = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$O_B = \begin{bmatrix} C^T & A^TC^T & A^{T^2}C^T & \cdots & A^{T^{n-1}}C^T \end{bmatrix}$$

System $\boldsymbol{S_2}$: $\dot{Z} = A^TZ + C^TV$

$Y_2 = B^TZ$

$$C_B = \begin{bmatrix} C^T & A^TC^T & A^{T^2}C^T & \cdots & A^{T^{n-1}}C^T \end{bmatrix}$$

$$O_B = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

The principle of duality states that the system $\boldsymbol{S_1}$ is controllable if and only if system $\boldsymbol{S_2}$ is observable; and vice-versa!

Hence, the problem of observer design for a system is actually a problem of control design for its dual system.

Stabilizability and Detectability

- Stabilizable system: Uncontrollable system in which uncontrollable part is stable
- Detectable system: Unobservable system in which the unobservable subsystem is stable

Example

Ref: B. Friedland, Control System Design, McGraw Hill, 1986

System Dynamics

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}}_{B} u$$

Output Equation

$$y = \underbrace{\begin{bmatrix} 7 & 6 & 4 & 2 \end{bmatrix}}_{C} X$$

Example

Ref: B. Friedland, Control System Design, McGraw Hill, 1986

Transfer Function:

$$\frac{y(s)}{u(s)} = C(sI - A)^{-1} B = \underbrace{\frac{(s+2)(s+3)(s+4)}{(s+1)(s+2)(s+3)(s+4)}}_{\text{pole-zero cancellation}} = \frac{1}{(s+1)}$$

Implication: What appears to be a fourth-order system, is actually a first-order system! Hence, there is either loss of controllability or observability (or both).

Question: Is this system stabilizable?

Example

Ref: B. Friedland, Control System Design, McGraw Hill, 1986

Define $\bar{X} = TX$. Then

$$\dot{\bar{X}} = T\dot{X} = T\left(AX + Bu\right)$$

$$\dot{\bar{X}} = \left(TAT^{-1}\right)\bar{X} + \left(TB\right)u$$

Let

$$T = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \quad TAT^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, TB = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Example

Ref: B. Friedland, Control System Design, McGraw Hill, 1986

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} -\bar{x}_1 + u \\ -2\bar{x}_2 \\ -3\bar{x}_3 + u \\ -4\bar{x}_4 \end{bmatrix}, \quad y = CX = CT^{-1}\bar{X} = \bar{x}_1 + \bar{x}_2$$

Implications:

$\bar{x}_1$: Affected by the input; visible in the output

$\bar{x}_2$: Unaffected by the input; visible in the output

$\bar{x}_3$: Affected by the input; Invisible in the output

$\bar{x}_4$: Unaffected by the input; Invisible in the output

Block Diagram:

Where do uncontrollable or unobservable systems arise?

- Redundant state variables
- Physically uncontrollable system
- Too much symmetry

References

- K. Ogata: *Modern Control Engineering*, 3rd Ed., Prentice Hall, 1999.
- B. Friedland: *Control System Design*, McGraw Hill, 1986.

Thanks for the Attention...!