Lecture – 22

Pole Placement Observer Design

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Outline

- Philosophy of observer design
- Full-order observer
- Reduced (Minimum) order observer

Philosophy of Observer Design

- In practice all the state variables are not available for feedback. Possible reasons include:
  - Non-Availability of sensors
  - Expensive sensors
  - Available sensors are not acceptable (due to high noise, high power consumption etc.)
- A state observer estimates the state variables based on the measurements of the output over a period of time.
- The system should be "observable".

Full-order Observer Design

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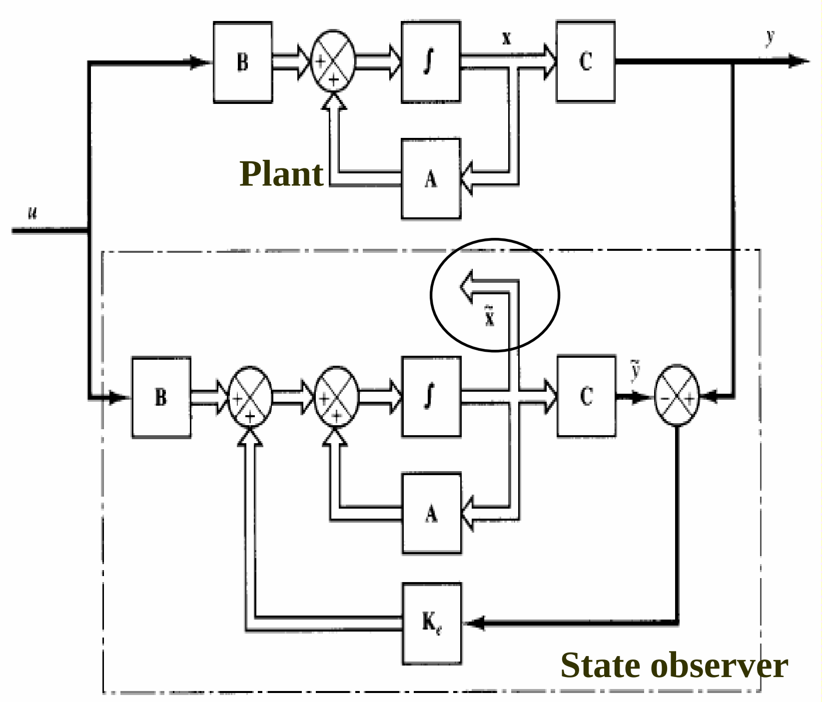
State Observer Block Diagram

Plant : $\dot{X} = AX + BU$

$$y = CX \quad \text{(single output)}$$



Let the observed state be $\tilde{X}$. Let the observer dynamics be

$$\dot{\tilde{X}} = \tilde{A}\tilde{X} + \tilde{B}U + K_e y$$

Error : $E \triangleq \left(X - \tilde{X} \right)$

Ref: K. Ogata: *Modern Control Engineering*, 3rd Ed., Prentice Hall, 1999

Observer Design: Concepts

Error Dynamics:

$$\dot{E} = \dot{X} - \dot{\tilde{X}}$$

$$= (AX + BU) - (\tilde{A}\tilde{X} + \tilde{B}U + K_e y)$$

Add and Substract $\tilde{A}X$ and substitute $y = CX$

$$= AX - \tilde{A}X + \tilde{A}X \; - \; \tilde{A}\tilde{X} + BU - \tilde{B}U - K_e CX$$

$$= (A - \tilde{A})X + \tilde{A}(X - \tilde{X}) + (B - \tilde{B})U - K_e CX$$

$$\therefore \dot{E} = \tilde{A}E + (A - \tilde{A} - K_e C)X + (B - \tilde{B})U$$

Strategy: 1. Make the error dynamics independent of X

2. Eliminate the effect of U from eror dynamics

Observer Design: Concepts

- This leads to:

$$\tilde{A} = A - K_e C$$

$$\tilde{B} = B$$

- Error dynamics: $\dot{E} = \tilde{A}E = \left(A - K_e C\right)E$

- Observer dynamics

$$\boxed{\dot{\tilde{X}} = A\tilde{X} + BU + K_e \underbrace{\left(y - C\tilde{X}\right)}_{\text{Residue}}}$$

Observer Design: Full Order

- Goal: Obtain gain K_e such that the error dynamics are asymptotically stable with sufficient speed of response.

- $\tilde{A}^T = A^T - C^T K_e^T$. Hence the problem here becomes the same as the pole placement problem!

Necessary and sufficient condition for the existence of K_e:

The system should be completely observable!

Comparison with Pole Placement Design

Controller Design	**Observer Design**
- Dynamics $\dot{X} = (A - BK)X$	- Dynamics $\dot{E} = \tilde{A}E = (A - K_e C)E$
- Objective $X(t) \to 0, \quad \text{as} \quad t \to \infty$	- Objective $E(t) \to 0, \quad \text{as} \quad t \to \infty$
	- Notice that $\lambda(A - K_e C) = \lambda\left[(A - K_e C)^T\right]$ $= \lambda\left(A^T - C^T K_e^T\right)$

Observer Design as a Dual Problem

Consider the dual problem with *input* v and *output* y^*

$$\dot{Z} = A^T Z + C^T v$$

$$y^* = B^T Z$$

Pole placement design for this problem

with desired observer roots at $\mu_1 \cdots\cdots \mu_n$ yields

$$\left| sI - \left(A^T - C^T K_o \right) \right| = \left(s - \mu_1 \right) \cdots \left(s - \mu_n \right)$$

Now equating observer characteristic equation

to the RHS of the above equation

we get $K_e = K_o^T$

Observer Design: Method – 1

- For systems of low order *($n \leq 3$)*
- Check Observability
- Define $K_e = [k_1 \;\; k_2 \;\; k_3]^T$
- Substitute this gain in the desired characteristic polynomial equation

$$\left| sI - (A - K_e C) \right| = (s - \mu_1) \cdots (s - \mu_n)$$

- Solve for the ***gain elements*** by equating the like powers on both sides

Observer Design: Method – 2

Step:1 $|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$

find a_i's

Step:2 $(s-\mu_1)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$

find α_i's

Step:3 Follow a similar approach as in pole placement control design (i.e. Bass-Gura approach) to compute the observer gain.

Observer Design: Method – 2

$$K_e = \left(WN^T\right)^{-1} \begin{bmatrix} (\alpha_n - a_n) \\ (\alpha_{n-1} - a_{n-1}) \\ \vdots \\ (\alpha_1 - a_1) \end{bmatrix}$$

Where $N = \begin{bmatrix} C^T & A^T C^T & \cdots\cdots\cdots (A^T)^{n\text{-}1} C^T \end{bmatrix}$

$$W = \begin{bmatrix} a_{n-1} & \cdots & a_1 & 1 \\ \vdots & & \iddots & 0 \\ a_1 & \iddots & \cdots & \vdots \\ 1 & \cdots & \cdots & 0 \end{bmatrix}$$

Observer Design: Method – 3
Ackerman's Formula

$$K_e = \phi(A) \begin{bmatrix} C \\ CA \\ \vdots \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

$$\phi(A) = A^n + \alpha_1 A^{n-1} + \cdots\cdots\cdots + \alpha_{n-1} A + \alpha_n I$$

Example: Observer Design

$$A=\begin{bmatrix}0 & 20.6\\ 1 & 0\end{bmatrix}; B=\begin{bmatrix}0\\ 1\end{bmatrix}; C=\begin{bmatrix}0 & 1\end{bmatrix}$$

Assume the desired eigen values of the observer

$$\mu_1=-1.8+2.4j;\ \mu_2=-1.8-2.4j$$

Step : 1 observability $n=2$

$$\begin{bmatrix}C^T & A^T C^T\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};\quad rank=2$$

Step : 2 Characteristic equation

$$|sI-A|=\begin{vmatrix}s & -20.6\\ -1 & s\end{vmatrix}=s^2-20.6=s^2+a_1 s+a_2=0$$

Example: Observer Design

$$a_1 = 0; \quad a_2 = -20.6$$

Step : 3 Desired Characteristic Equation

$$(s+1.8-2.4j)(s+1.8+2.4j) = s^2 + 3.6s + 9 = s^2 + \alpha_1 s + \alpha_2 = 0$$

$$\alpha_1 = 3.6; \quad \alpha_2 = 9$$

Step : 4 Observer gain

$$K_e = (WN^T)^{-1}\begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9+20.6 \\ 3.6-0 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 29.6 \\ 3.6 \end{bmatrix}$$

Separation Principle

System dynamics $\dot{X} = AX + BU$

$$y = CX$$

State feedback control based on observed state is $U = -K\,\tilde{X}$

State equation $\dot{X} = AX - BK\tilde{X} = \left(A - BK\right)X + BK\left(X - \tilde{X}\right)$

error $E(t) = X - \tilde{X}$

hence $\dot{X} = \left(A - BK\right)X + BKE$

observer error equation

$$\dot{E} = \left(A - K_e C\right)E$$

Separation Principle

Combined equation: $\begin{bmatrix} \dot{X} \\ \dot{E} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-K_eC \end{bmatrix} \begin{bmatrix} X \\ E \end{bmatrix}$

Characteristic equation for the Observer-State-Feedback system

$$\begin{vmatrix} sI-A+BK & -BK \\ 0 & sI-A+K_eC \end{vmatrix} = 0$$

$$\left|sI-A+BK\right|\left|sI-A+K_eC\right| = 0$$

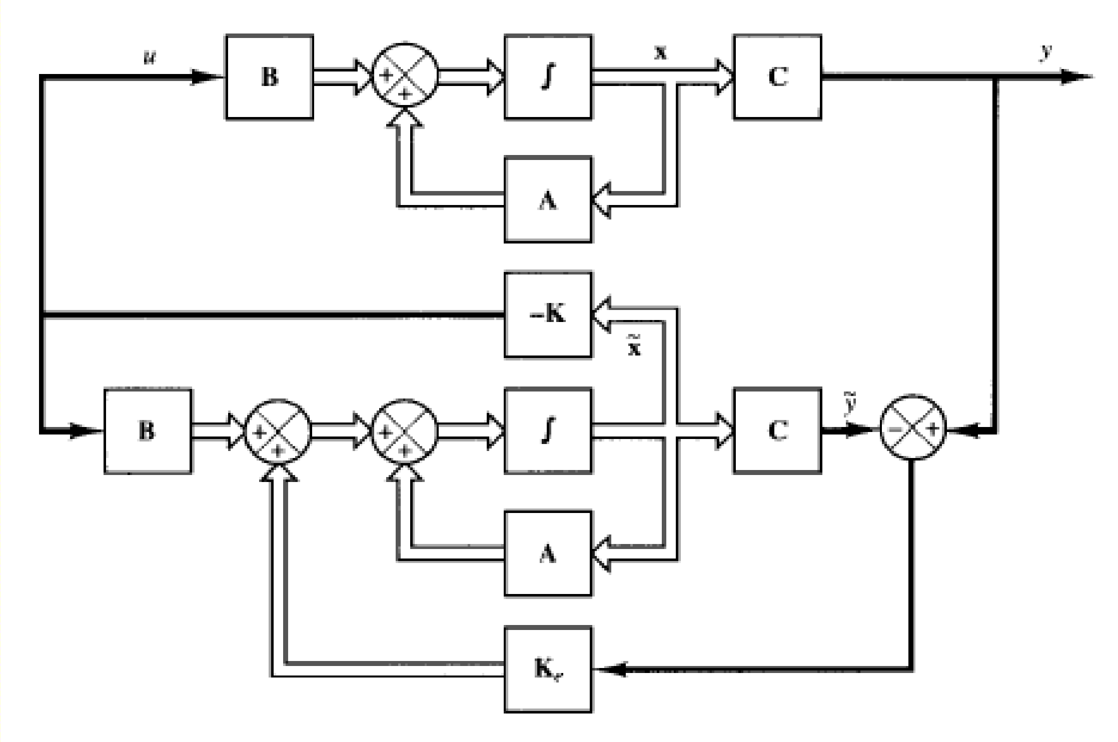
Poles due to controller

Poles due to Observer

Hence Observer design and Pole placement are independent of each other!

This is known as “Separation Theorem”.

Closed Loop System



Ref: K. Ogata: *Modern Control Engineering*, 3rd Ed., Prentice Hall, 1999

Fig: Observed State feedback Control System

Reduced-order Observer Design

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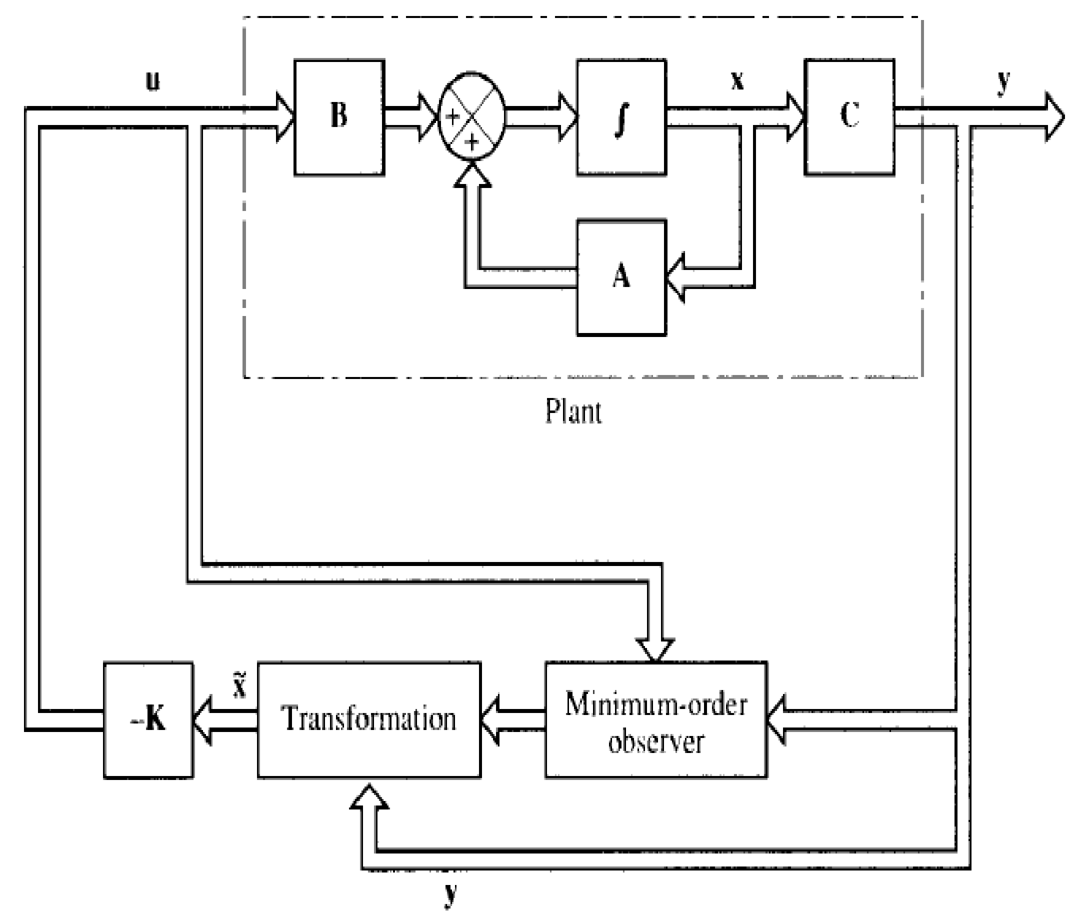
Reduced Order Observer

- Some of the state variables may be accurately measured .
- Suppose X is an *n - vector* and the output *y* is an *m - vector* that can be measured .
- We need to estimate only (*n-m*) state variables.
- The reduced-order observer becomes (*n-m*)th order observer.

Block diagram:

State feedback control with minimum order observer



Ref : *K. Ogata: Modern Control Engineering, 3rd Ed., Prentice Hall, 1999*

State Equation for the Reduced order observer

Let $m = 1,$ $\qquad \dot{X} = AX + Bu$

$$y = CX$$

$$\begin{bmatrix} \dot{x}_a \\ \dot{X}_b \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ X_b \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ X_b \end{bmatrix}$$

$$x_a = scalar \;\;,\;\; X_b = (n-1)\; vector$$

State Equation for the Reduced order observer

- The equation for the measured portion of the state,

$$\dot{x}_a = A_{aa} x_a + A_{ab} X_b + B_a u$$

$$\dot{x}_a - A_{aa} x_a - B_a u = A_{ab} X_b$$

- The equation for the unmeasured portion of the state,

$$\dot{X}_b = A_{ba} x_a + A_{bb} X_b + B_b u$$

- Terms $A_{ba} x_a$ and $B_b u$ are "$\underline{\text{known quantities}}$"

Full order and Reduced order observer comparison

- State/output equation for the full order observer :

$$\dot{X} = AX + Bu$$

$$y = CX$$

- State/output equation for the reduced order observer:

$$\dot{X}_b = A_{bb} X_b + A_{ba} x_a + B_b u$$

$$\dot{x}_a - A_{aa} x_a - B_a u = A_{ab} X_b$$

Full order and Reduced order observer comparison

Full – Order State Observer	Reduced Order State observer
$\tilde{X}$	$\tilde{X}_b$
A	A_{bb}
Bu	$A_{ba}x_a + B_b u$
y	$\dot{x}_a - A_{aa}x_a - B_a u$
C	A_{ab}
$K_e(n \times 1\ matrix)$	$K_e[(n-1) \times 1\ matrix]$

Fig : List of Necessary Substitutions for Writing the Observer Equation for the Reduced Order State Observer.

Observer Equation

- Full order Observer equation :

$$\dot{\tilde{X}} = (A - K_e C)\tilde{X} + Bu + K_e y$$

- Making substitutions from the table,

$$\dot{\tilde{X}}_b = (A_{bb} - K_e A_{ab})\tilde{X}_b + A_{ba} x_a + B_b u + K_e(\dot{x}_a - A_{aa} x_a - B_a u)$$

$i.e.$

$$\begin{aligned}\dot{\tilde{X}}_b - K_e \dot{x}_a &= (A_{bb} - K_e A_{ab})\tilde{X}_b + (A_{ba} - K_e A_{aa})y + (B_b - K_e B_a)u \\ &= (A_{bb} - K_e A_{ab})(\tilde{X}_b - K_e y) \\ &\quad + \left[(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}\right]y + (B_b - K_e B_a)u\end{aligned}$$

Observer Equation

- Define

$$X_b - K_e y = \left(X_b - K_e x_a \right) \triangleq \eta$$

$$\tilde{X}_b - K_e y = \left(\tilde{X}_b - K_e x_a \right) \triangleq \tilde{\eta}$$

- Then $\dot{\tilde{\eta}} = (A_{bb} - K_e A_{ab})\tilde{\eta} +$

$$\left[(A_{bb} - K_e A_{ab}) K_e + A_{ba} - K_e A_{aa} \right] y + (B_b - K_e B_a) u$$

This is reduced order observer.

Observer Error Equation

We have:

$$\dot{X}_b = A_{bb}X_b + \left(A_{ba}x_a + B_b u\right)$$

$$\dot{\tilde{X}}_b = \left(A_{bb} - K_e A_{ab}\right)\tilde{X}_b + \left(A_{ba}x_a + B_b u\right) + K_e A_{ab} X_b$$

Subtracting:

$$\dot{X}_b - \dot{\tilde{X}}_b = \left(A_{bb}X_b - K_e A_{ab} X_b\right) - \left(A_{bb} - K_e A_{ab}\right)\tilde{X}_b$$

$$= \left(A_{bb} - K_e A_{ab}\right)\underbrace{\left(X_b - \tilde{X}_b\right)}_{E}$$

$$i.e. \quad \dot{E} = \left(A_{bb} - K_e A_{ab}\right)E$$

where $E \triangleq \left(X_b - \tilde{X}_b\right) = \left(\eta - \tilde{\eta}\right)$

Gain Matrix Computation

Necessary Condition

The error dynamics can be chosen provided the rank of matrix

$$\begin{bmatrix} A_{ab} \\ A_{ab} A_{bb} \\ . \\ . \\ A_{ab} A_{bb}^{n\text{-}2} \end{bmatrix}$$

is $(n-1)$. This is complete observability condition

Characteristic Equation:

$$\left| sI - A_{bb} + K_e A_{ab} \right| = (s-\mu_1)(s-\mu_2)........(s-\mu_{n\text{-}1})$$

$$= s^{n-1} + \hat{\alpha}_1 s^{n-2} + ... + \hat{\alpha}_{n\text{-}2} s + \hat{\alpha}_{n\text{-}1} = 0$$

where $\mu_1, \mu_2, ...\mu_{n\text{-}1}$ are desired eigenvalues of error dynamics

The Characteristic Equation

$$K_e = \hat{Q}\begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ . \\ . \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix} = (\hat{W}\hat{N}^T)^{-1}\begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ . \\ . \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix}$$

where

$$\hat{N} = \left[\, A_{ab}^T \mid A_{bb}^T A_{ab}^T \mid ..... \mid (A_{bb}^T)^{n-2} A_{ab}^T \,\right]: \quad (n-1)\times(n-1) \text{ matrix.}$$

$$\hat{W} = \begin{bmatrix} \hat{a}_{n-2} & \hat{a}_{n-3} & .... \; \hat{a}_1 & 1 \\ \hat{a}_{n-3} & \hat{a}_{n-4} & .... \; 1 & 0 \\ . & . & . & . \\ . & . & . & . \\ \hat{a}_1 & 1 & 0 & 0 \\ 1 & 0 \; . \; . \; . & 0 & 0 \end{bmatrix}: \quad (n-1)\times(n-1) \text{ matrix .}$$

The Characteristic Equation

- $\hat{a}_1, \hat{a}_2, \ldots\ldots \hat{a}_{n\text{-}2}$ are coefficients in the characteristic equation

  $$\left|sI - A_{bb}\right| = s^{n\text{-}1} + \hat{a}_1 s^{n\text{-}2} + \ldots. + \hat{a}_{n\text{-}2} s + \hat{a}_{n\text{-}1} = 0.$$

- **Ackermann's formula :** $$K_e = \phi(A_{bb}) \begin{bmatrix} A_{ab} \\ A_{ab} A_{bb} \\ . \\ . \\ . \\ A_{ab} A_{bb}^{n-3} \\ A_{ab} A_{bb}^{n-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix}$$

where $\phi(A_{bb}) = A_{bb}^{n\text{-}1} + \hat{\alpha}_1 A_{bb}^{n\text{-}2} + \ldots.. + \hat{\alpha}_{n-2} A_{bb} + \hat{\alpha}_{n-1} I$

Separation Principle

- The system characteristic equation can be derived as

$$\left|sI - A + BK\right|\left|sI - A_{bb} + K_e A_{ab}\right| = 0$$

Poles due to pole placement

Poles due to reduced order Observer

- Therefore the pole-placement design and the design of the reduced order observer are independent of each other.

Example

Problem : Consider the system

$$\dot{X} = AX + Bu$$

$$y = CX$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Assume that the output y can be accurately measured. Design minimum order observer assuming that the desired eigen values are:

$$\mu_1 = -2 + j2\sqrt{3} \; , \; \mu_2 = -2 - j2\sqrt{3}$$

Example

Characteristic equation:

$$\left| sI - A_{bb} + K_e A_{ab} \right| = (s-\mu_1)(s-\mu_2)$$

$$= (s+2-j2\sqrt{3})(s+2+j2\sqrt{3}) = s^2 + 4s + 16 = 0$$

Ackermann's formula:

$$K_e = \phi(A_{bb}) \begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $\phi(A_{bb}) = A_{bb}^2 + \hat{\alpha}_1 A_{bb} + \hat{\alpha}_2 I = A_{bb}^2 + 4\,A_{bb} + 16I$

$$X = \begin{bmatrix} x_a \\ X_b \end{bmatrix} = \begin{bmatrix} x_a \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example

Here $A_{aa} = 0 \;,\; A_{ab} = [1 \quad 0], \quad A_{ba} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$

$$A_{bb} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}, \quad B_a = 0, \quad B_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence

$$K_e = \left\{ \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}^2 + 4\begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} + 16\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 \\ 22 & 17 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 17 \end{bmatrix}$$

Example

Observer equation:

$$\dot{\tilde{\eta}} = \left(A_{bb} - K_e A_{ab}\right)\tilde{\eta} + \left[\left(A_{bb} - K_e A_{ab}\right)K_e + A_{ba} - K_e A_{aa}\right]y$$

$$+\left(B_b - K_e B_a\right)u \quad \left(\text{Note: } \tilde{\eta} \triangleq \tilde{X}_b - K_e y = \tilde{X}_b - K_e x_1\right)$$

$$A_{bb} - K_e A_{ab} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} - \begin{bmatrix} -2 \\ 17 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -28 & -6 \end{bmatrix}$$

Substituting various values,

$$\begin{bmatrix} \dot{\tilde{\eta}}_2 \\ \dot{\tilde{\eta}}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -28 & -6 \end{bmatrix}\begin{bmatrix} \dot{\tilde{\eta}}_2 \\ \dot{\tilde{\eta}}_3 \end{bmatrix} + \begin{bmatrix} 13 \\ -52 \end{bmatrix}y + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Example :

$$\begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} - K_e y$$

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + K_e x_1$$

If the observed state feedback is used, then

$$u = -K\tilde{X} = -K \begin{bmatrix} x_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

where K is the state feedback matrix.

Comment

- Reduced order observers are computationally efficient.
- Reduced order observers may converge faster.
- Sometimes its advisable to use a full-order observer even if its possible to design a reduced-order observer.

References

- K. Ogata: *Modern Control Engineering*, 3rd Ed., Prentice Hall, 1999.
- B. Friedland: *Control System Design*, McGraw Hill, 1986.

Thanks for the Attention...!