



# BIOMATHEMATICS

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## Lecture 20

# **Applications of calculus and vector algebra in biology**

# Diffusion

$t = 0$

tube



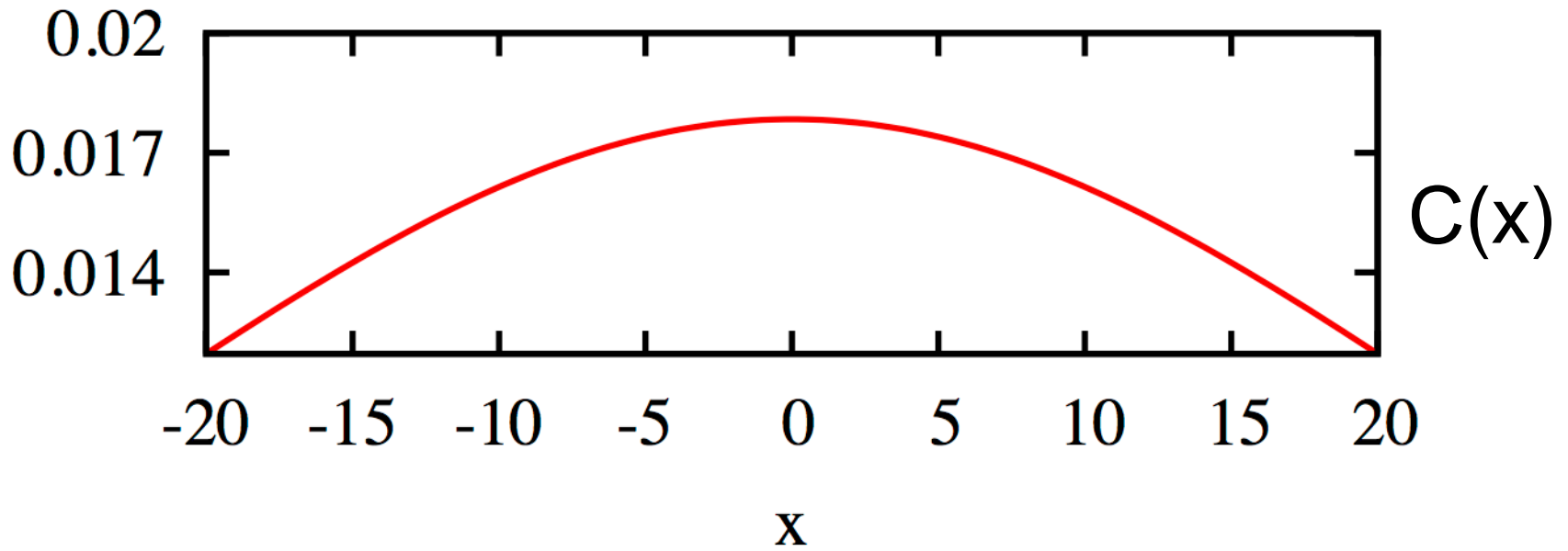
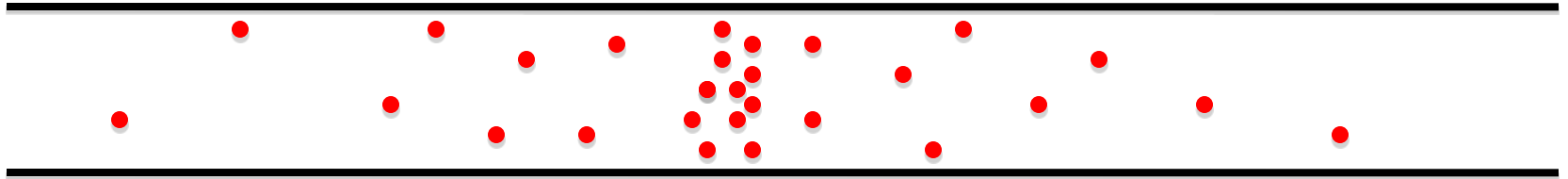
$C(x)$



$x=0$

$t = 10 \text{ min}$

tube



# Diffusion equation

$$\frac{\partial C(x, t)}{\partial t} = D \frac{\partial^2 C(x, t)}{\partial x^2}$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} x \tilde{C}(x) dx$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \tilde{C}(x) dx$$

$$\langle x^2 \rangle = 2Dt$$

$$\langle x \rangle = ?$$



Again,

$$\frac{\partial \tilde{C}}{\partial t} = D \frac{\partial^2 \tilde{C}}{\partial x^2}$$

Now,

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} x \tilde{C}(x) dx = D \int_{-\infty}^{+\infty} x \frac{\partial^2 \tilde{C}(x)}{\partial x^2} dx$$

$$\frac{\partial}{\partial t} \langle x \rangle = Dx \left. \frac{\partial \tilde{C}}{\partial x} \right|_{-\infty}^{+\infty} - D \int_{-\infty}^{+\infty} \frac{\partial \tilde{C}}{\partial x} dx$$

$$\frac{\partial}{\partial t} \langle x \rangle = -D \int_{-\infty}^{+\infty} \frac{\partial \tilde{C}}{\partial x} dx$$

$$\frac{\partial}{\partial t} \langle x \rangle = 0$$

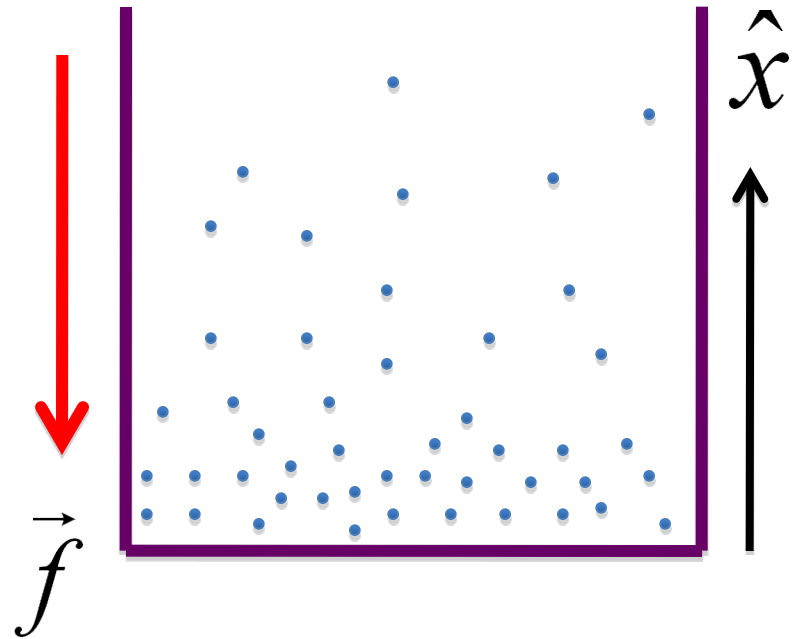
# Einstein's Relation

Einstein 1905

# Particles under an external field

$$\vec{f} = -g\hat{x}$$

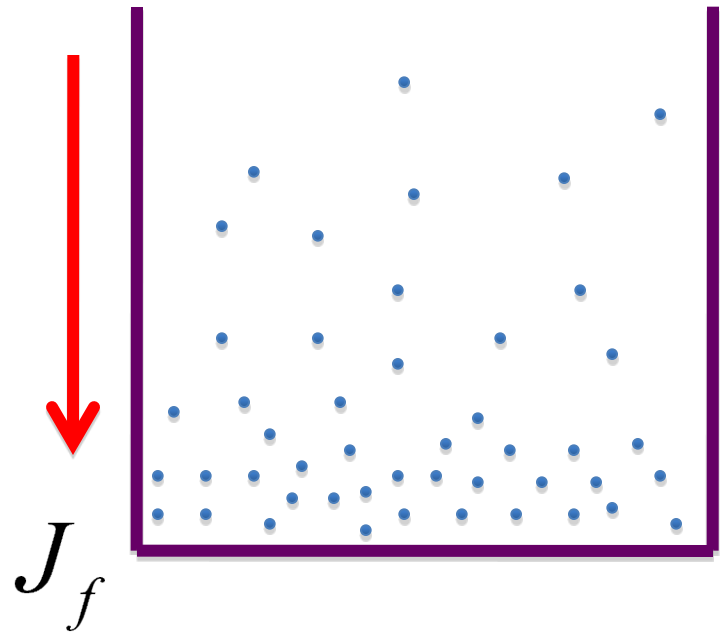
$$C(x) \propto \exp\left(-\frac{gx}{k_B T}\right)$$



## Current due to the force

$$\vec{J}_f = C\vec{v}$$

$$\vec{v} = \frac{\vec{f}}{6\pi\eta a}$$

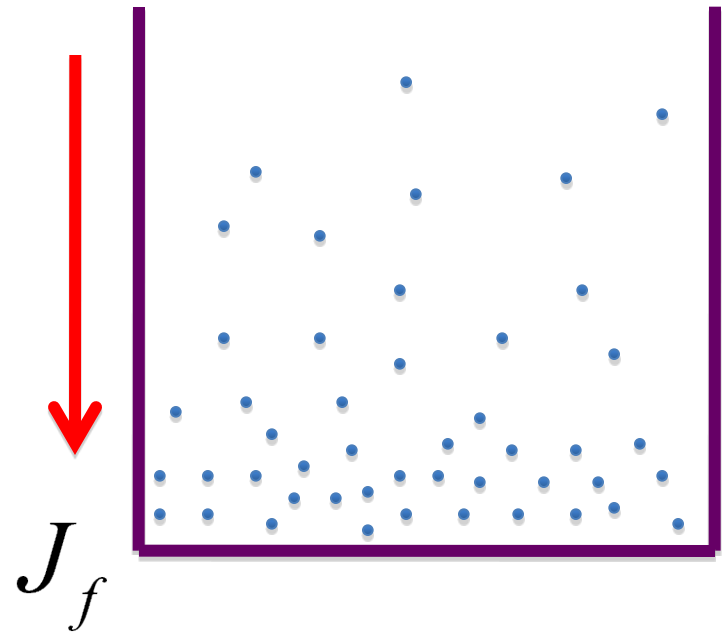


# Current due to the force

$$\vec{J}_f = C\vec{v}$$

$$\vec{v} = \frac{-g\hat{x}}{6\pi\eta a}$$

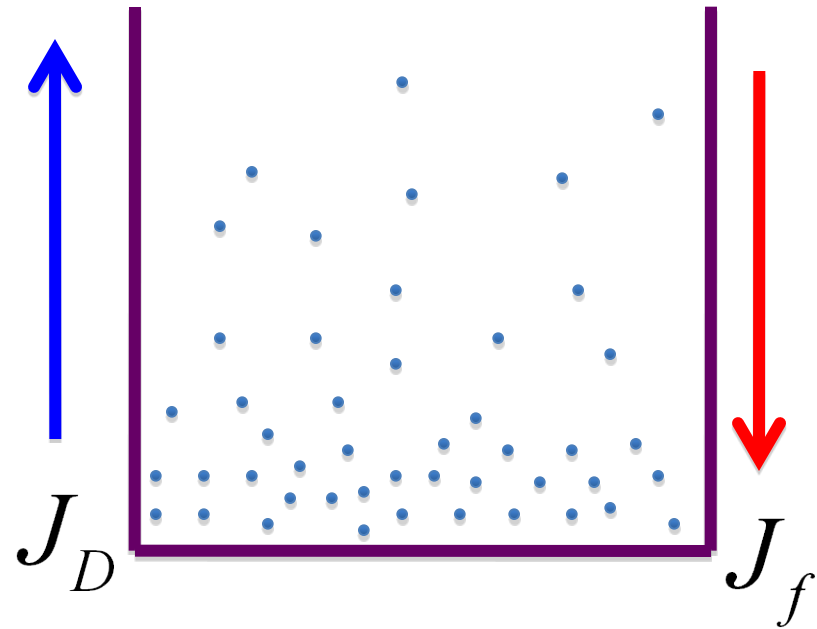
$$\vec{J}_f = -C \frac{g\hat{x}}{6\pi\eta a}$$



## Diffusion

$$\vec{J}_D = -D\vec{\nabla}C$$

$$\vec{J}_D = -D\frac{\partial C}{\partial x}\hat{x}$$

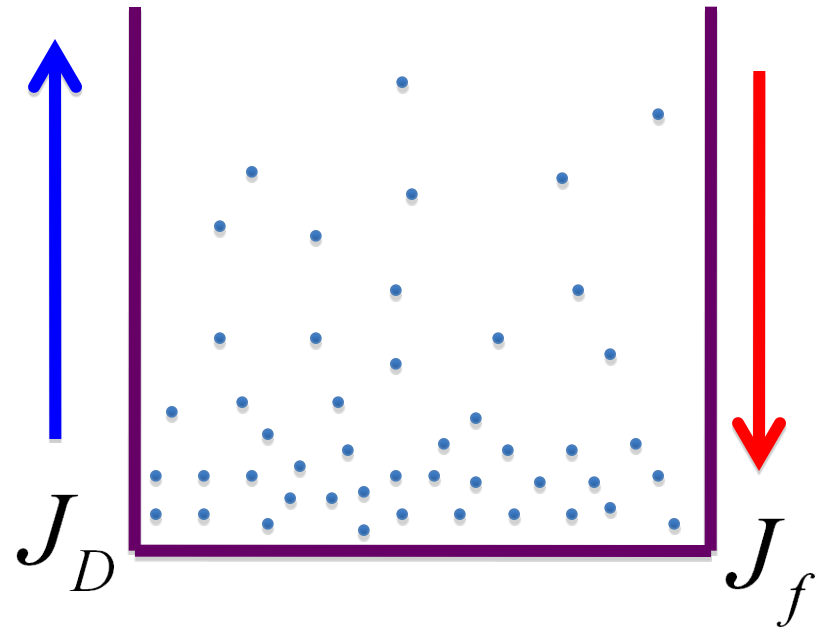


## Diffusion

$$\vec{J}_D = -D \frac{\partial C}{\partial x} \hat{x}$$

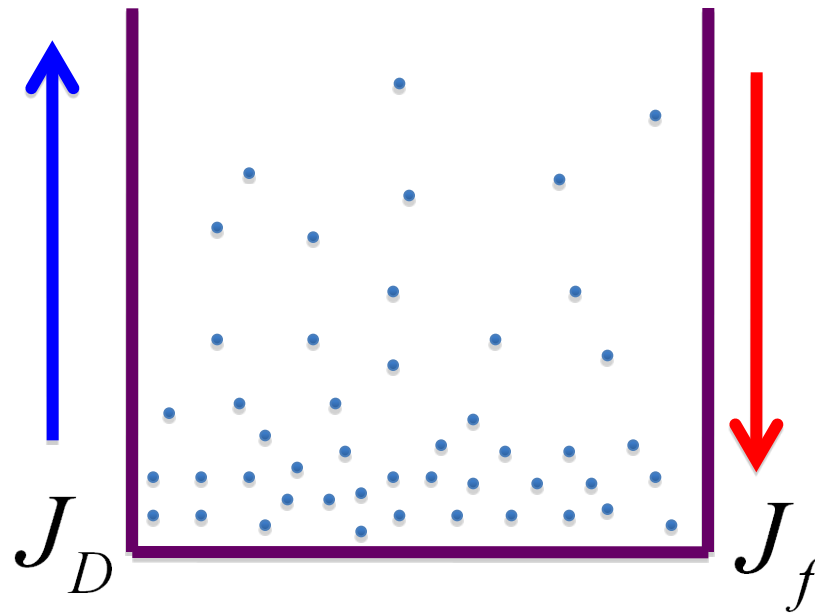
$$C \propto \exp\left(-\frac{gx}{k_B T}\right)$$

$$\vec{J}_D = DC \frac{g}{k_B T} \hat{x}$$

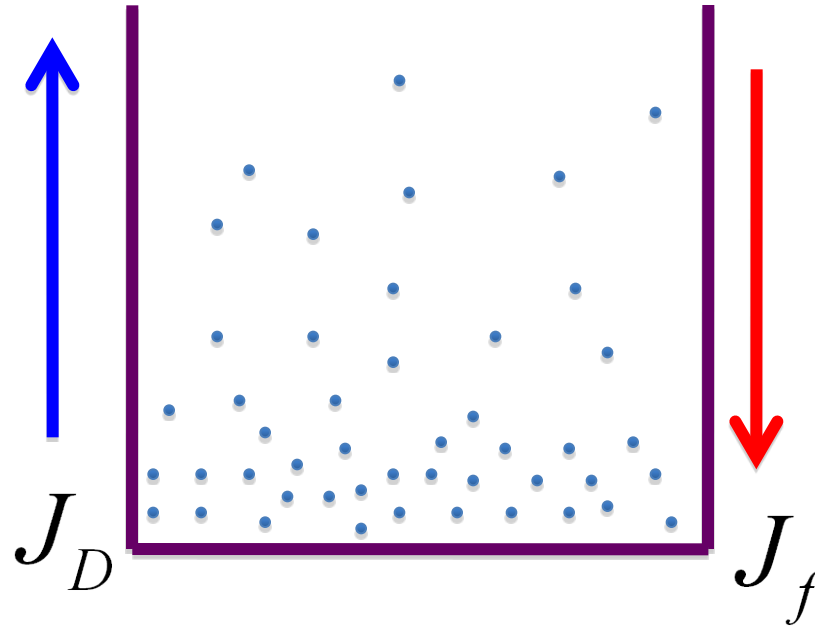




When both the currents balance,  
we reach equilibrium.



## Equilibrium=net current zero



$$\vec{J}_D + \vec{J}_E = 0$$

$$\vec{J}_D = -\vec{J}_E$$

## Einstein's relation

$$\vec{J}_D = DC \frac{g}{k_B T} \hat{x}$$

$$\vec{J}_f = -C \frac{g \hat{x}}{6 \pi \eta a}$$

$$\vec{J}_D = -\vec{J}_f$$

$$DC \frac{g}{k_B T} = C \frac{g}{6 \pi \eta a}$$

$$\Rightarrow D = \frac{k_B T}{6 \pi \eta a}$$

# Summary

Two important relations

$$\langle x^2 \rangle = 2Dt$$

$$D = \frac{k_B T}{6\pi\eta a}$$