Fundamentals of Transport Processes:

Why? What? How

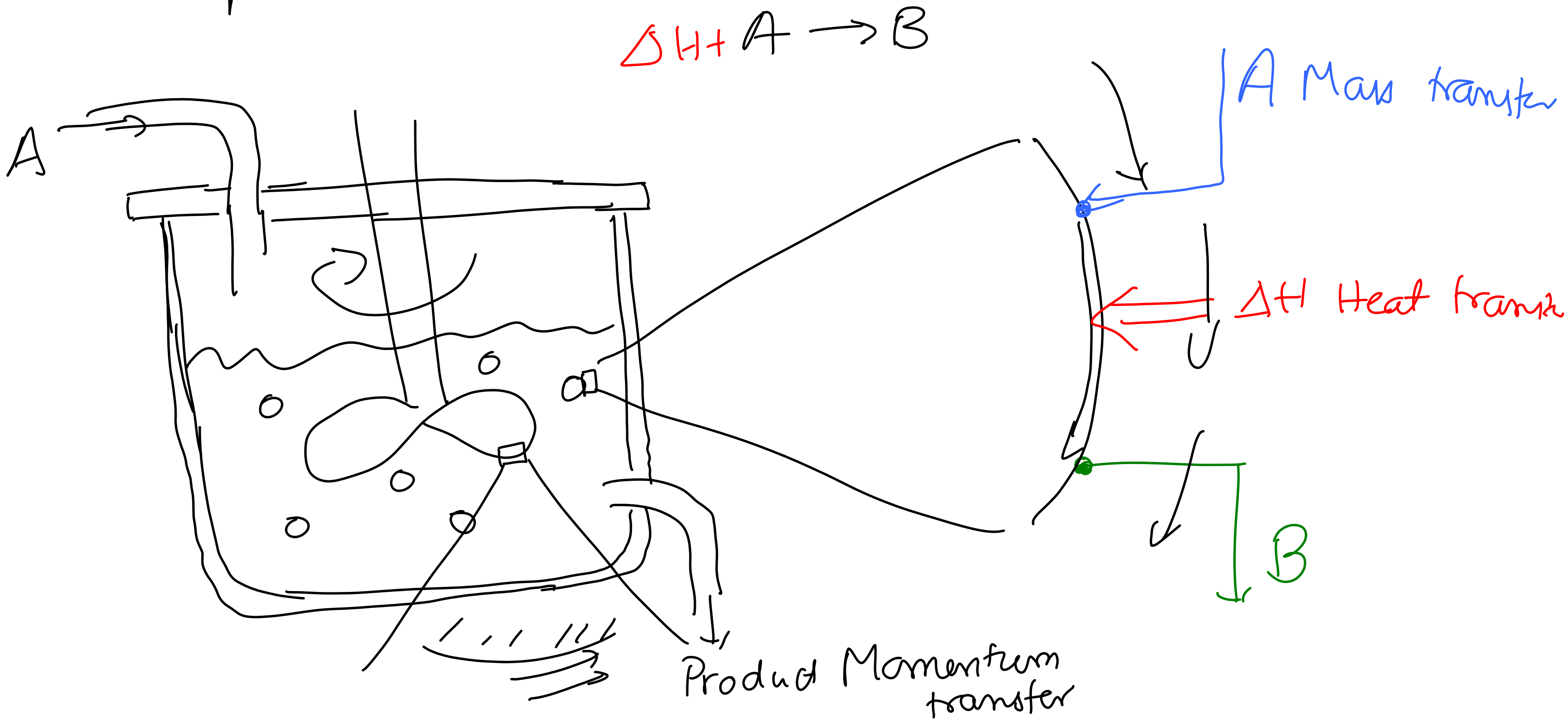
Raw Materials $\xrightarrow[\text{Economical}]{\text{Reliable}}$ Useful products

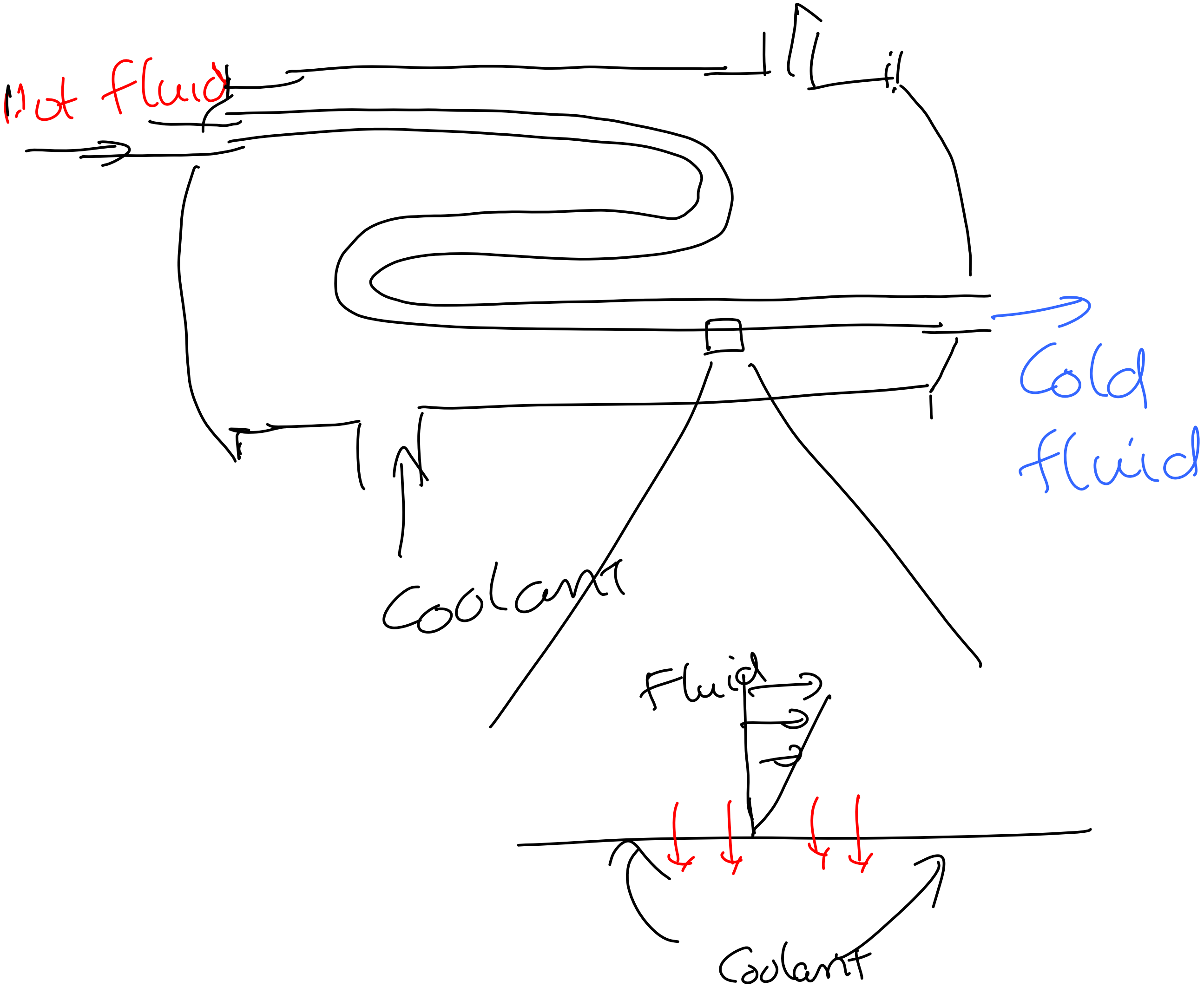
Chemical Reactions

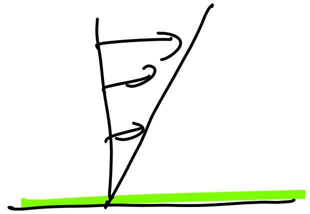
Physical Transformations: Heating, cooling, melting, evaporation, mixing, separations, etc.

Fluid systems:

Two-phase continuous stirred tank reactor:





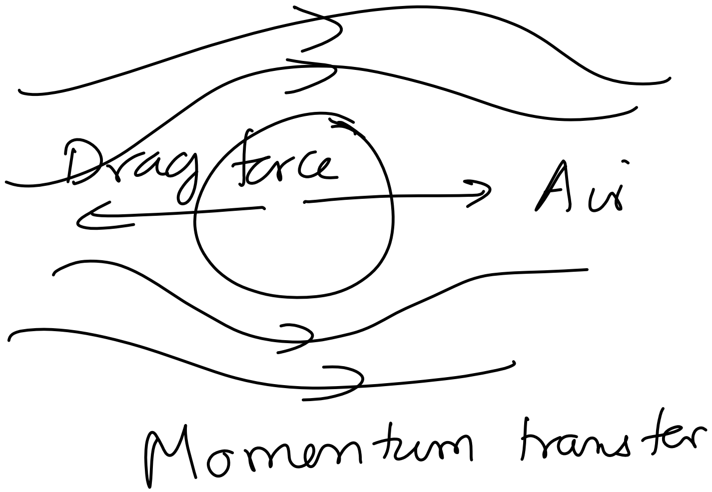



Momentum transfer rate/Area = Stress

# Spray drier:


Hot dry air


Mass transfer
Q Heat transfer


Drag force
Air
Momentum transfer

# What ??

Flux = Amount transferred per unit area per unit time

Mass flux $j$ = Mass transferred per unit area per unit time.

Heat flux $q$ = Heat transferred per unit area per unit time

Momentum flux $\tau$ = Momentum transferred per unit area per unit time.

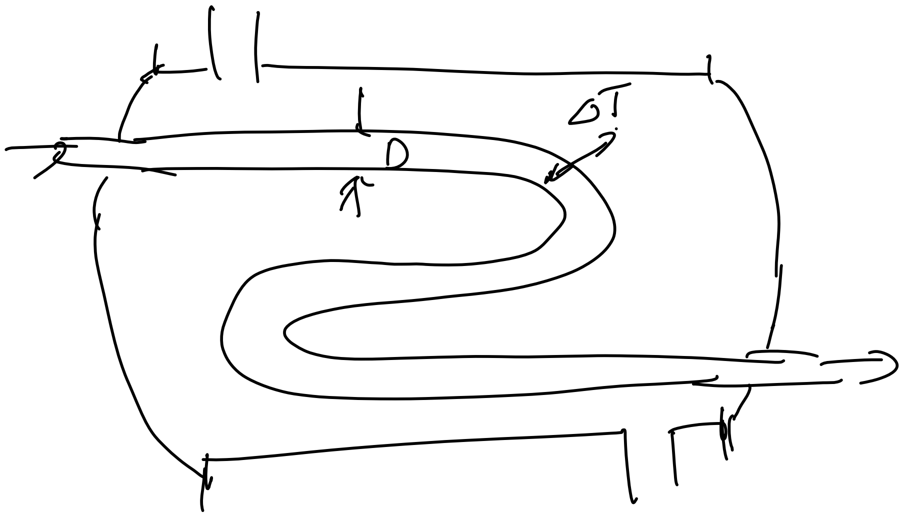
= Stress

Driving forces
Mass — Concentration difference
Heat — Temperature difference
Momentum - Velocity difference

Unit operations: $\Rightarrow$ Entire equipment

Correlations involving dimensionless variables

Dimensionless heat flux $Nu = \dfrac{qD}{k\Delta T}$

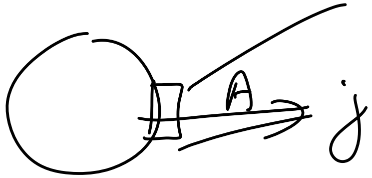



$$Nu = 1.86\, Re^{1/3} Pr^{1/3} (L/D)^{1/3} (\mu_w/\mu)^{0.14}$$

for $Re < 20{,}000$ laminar flow

$$Re = \left(\frac{\rho U D}{\mu}\right) \quad Pr = \left(\frac{C_p \mu}{k}\right)$$

$$Nu = 0.023\, Re^{0.8} Pr^{1/3} (\mu/\mu_w)^{0.8}$$




$$Sh = \frac{j\,D}{D_{AB}\,\Delta C}$$

$j$ = Flux

$D$ = diameter

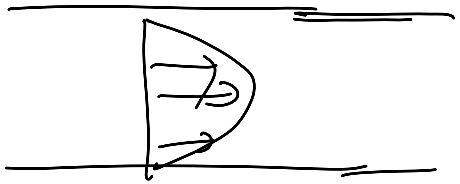
$D_{AB}$ = Diffusion coefficient

$\Delta C$ = Average concentration diff

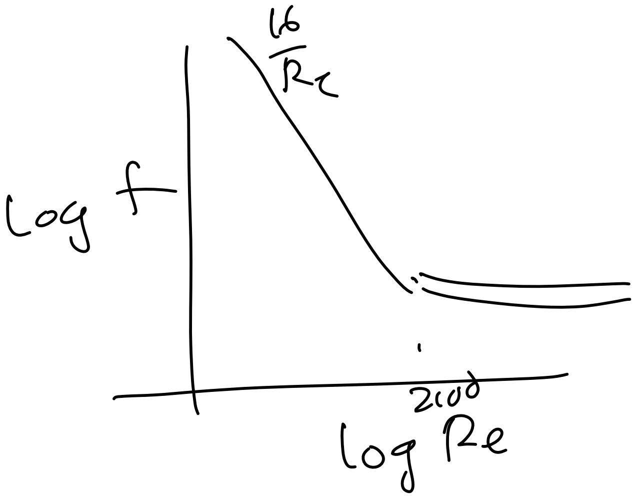
$Sh$ = function ( Re, Sc, dimensionless groups)

$$Sc = \left(\frac{\mu}{\rho D_{AB}}\right)$$

# Momentum transfer:



friction factor $f = \frac{\tau}{(\rho U^2/2)}$




$f$ = Function(Re)

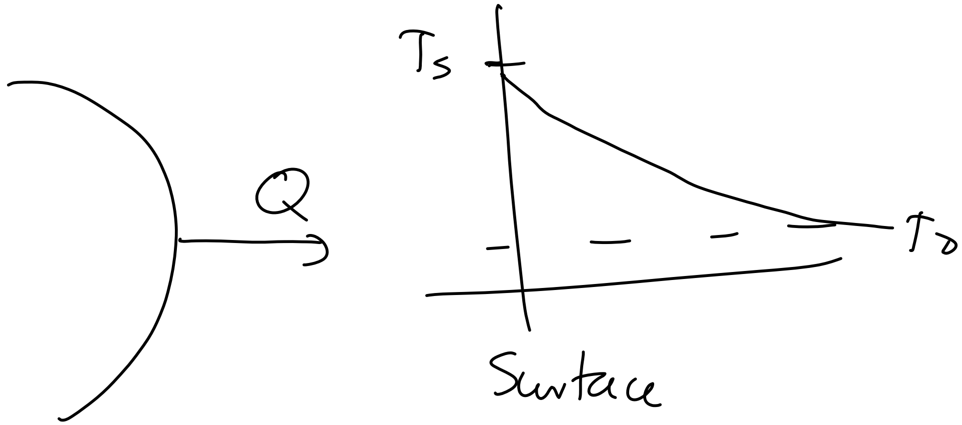
Low Reynolds number $Re < 2100$

$f = 16/Re$

High Reynolds number $Re > 2100$

$f$ = Function(Re)

Relations at the local (microscopic) level




Governing equations – Partial differential equation

Use physical insight to solve these equations in specific situations — Approximate, analytical

Convection

Transport due to mean fluid motion

How??

Diffusion

Transport due to the fluctuating motion of the molecules

# Dimensional Analysis

Height = 1.80 m

Time of a lecture = 1 hour

Fundamental units:

Mass (M) (kg, gm, ..)

Length (L) (m, cm, ..)

Time (T) (hr, sec, min, ...)

Temperature ($\Theta$) (°C, F, K, ...)

Amperes (A)

Candela

Velocity $LT^{-1}$

Acceleration $LT^{-2}$

Force $MLT^{-2}$

Work $ML^{2}T^{-2}$

Energy $ML^{2}T^{-2}$

Power $ML^{2}T^{-3}$

Pressure $ML^{-1}T^{-2}$

Stress $ML^{-1}T^{-2}$

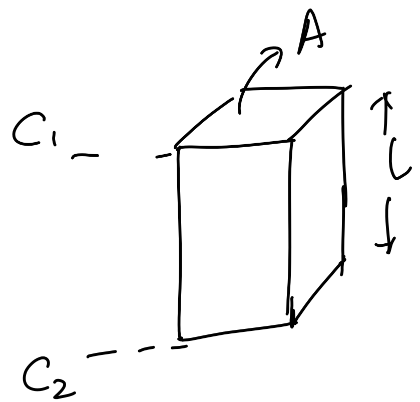
Viscosity $ML^{-1}T^{-1}$

Mass flux $ML^{-2}T^{-1}$

Diffusion coefficient $L^{2}T^{-1}$

Mass diffusion coefficient

Fick's law




$$j = \mathbb{D}\,\frac{\Delta c}{l}$$

$$ML^{-2}T^{-1} = \frac{[\mathbb{D}](ML^{-3})}{L}$$

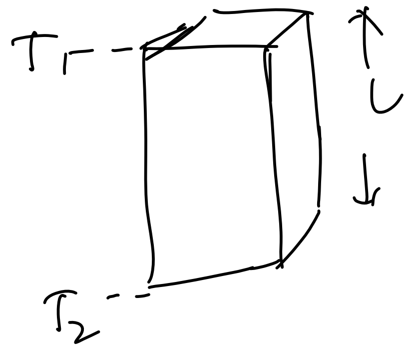
$$[\mathbb{D}] = L^{2}T^{-1}$$

Heat flux $q$ $MT^{-3}$

Specific Heat $C_p$ $L^2T^{-2}\Theta^{-1}$

Thermal conductivity $k$ $MLT^{-3}\Theta^{-1}$

Fourier's law:




$$q = -k\frac{\Delta T}{L}$$

$$MT^{-3} = \frac{[k]\Theta}{L}$$

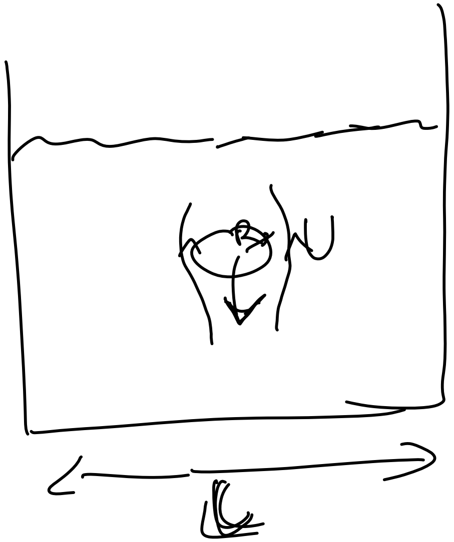
$$[k] = MLT^{-3}\Theta^{-1}$$

# Bukingham Pi Theorem:

$n$ dimensional quantities, $m$ dimensions, $(n-m)$ dimensionless groups.

Sphere settling in a fluid:

Drag force $F_D(U, R, \mu, \rho, L)$




| | |
|---|---|
| $F_D$ | $MLT^{-2}$ |
| $U$ | $LT^{-1}$ |
| $R$ | $L$ |
| $L$ | $L$ |
| $\mu$ | $ML^{-1}T^{-1}$ |
| $\rho$ | $ML^{-3}$ |

Quantities = 6

Dimensions = 3

Dimensionless groups = 3

$\Pi_1, \Pi_2, \Pi_3$

$\Pi_3 = (R/L)$

$$\left(\frac{F}{\mu R U}\right) = \text{Function}\left(\frac{\rho U R}{\mu}, \cancel{\frac{R}{L}}\right)$$

$$\Pi_1 = \frac{F}{\mu U R}$$

$$\Pi_2 = \left(\frac{\rho U R}{\mu}\right)$$

$$\Pi_3 = \left(R/L\right)$$

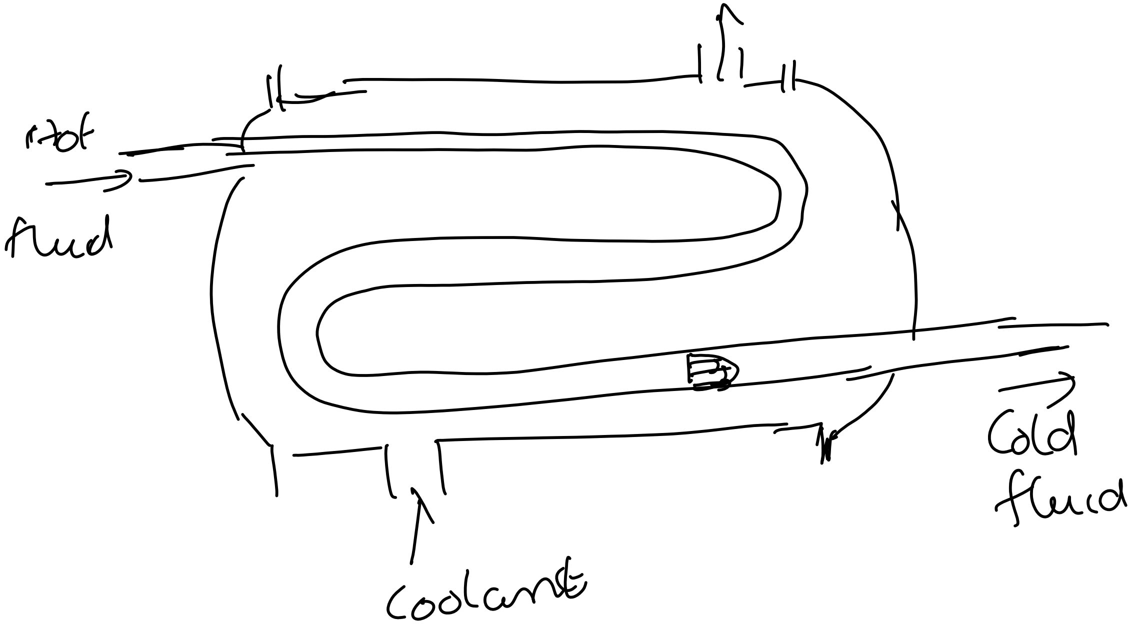
$$F = \mu R U \,\text{Function}\left(\frac{\rho U R}{\mu}\right)$$

Limit $Re \ll 1$,

$$F_D = \underbrace{C}_{6\pi} \mu R U$$ } stokes law

$Re \gg 1$,

$$\left(\frac{F}{\rho U^2 R^2}\right) = \text{constant}$$




$$q = \frac{\text{Heat transferred}}{\text{Area} \times \text{Time}}$$

$q$ = Heat flux $\quad H L^{-2} T^{-1}$

$C_p$ = Specific heat $\quad H M^{-1} \Theta^{-1}$

$k$ = Thermal conductivity $\quad H L^{-1} T^{-1} \Theta^{-1}$

$\Delta T$ = Temperature difference $\quad \Theta$

$D$ = Tube diameter $\quad L$

$\rho$ = Density of fluid $\quad M L^{-3}$

$\mu$ = Viscosity of fluid $\quad M L^{-1} T^{-1}$

$U$ = Fluid velocity $\quad L T^{-1}$

$L$ = Length of pipe $\quad L$

$$\left(\frac{qD}{k\Delta T}\right) = F\left(\frac{\rho U D}{\mu}, \frac{C_p \mu}{k}, \frac{D}{L}\right)$$

Nusselt number $= \left(\frac{qD}{k\Delta T}\right)$

Reynolds number $= \left(\frac{\rho U D}{\mu}\right)$

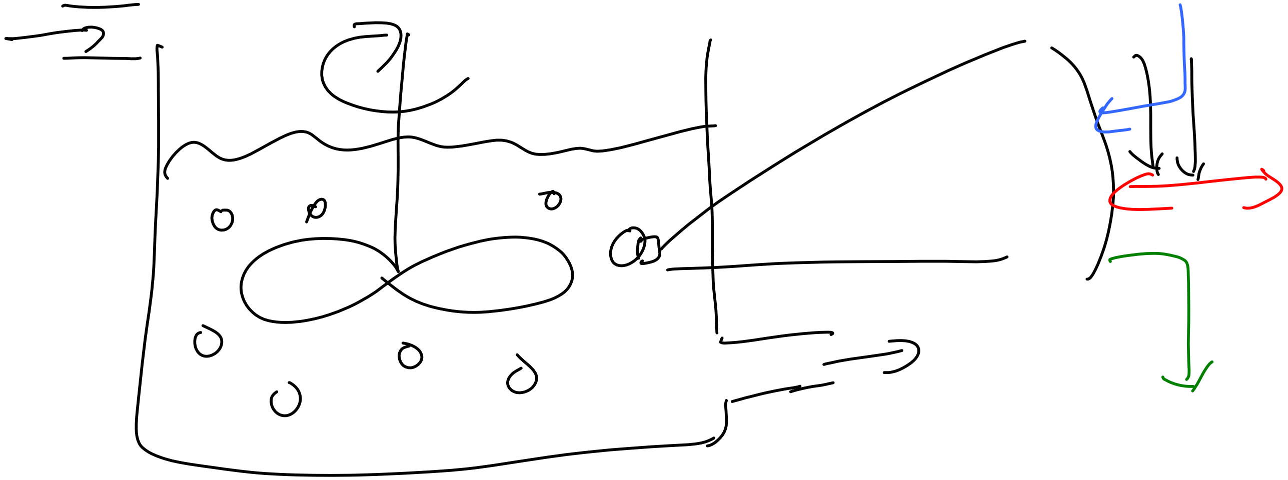
Prandtl number $= \left(\frac{C_p \mu}{k}\right)$

Laminar flow $Re < 2100$

$$Nu = 1.86\, Re^{1/3}\, Pr^{1/3} \left(D/L\right)^{1/3} \left(\frac{\mu}{\mu_w}\right)^{0.14}$$

Turbulent flow $Re > 20{,}000$

$$Nu = 0.023\, Re^{0.8}\, Pr^{1/3} \left(\mu/\mu_w\right)^{0.14}$$



$$j = ML^{-2}T^{-1}$$

$$\Delta c = ML^{-3}$$

$$\mathbb{D} = L^{2}T^{-1}$$

$$D = L$$

$$U = LT^{-1}$$

$$\rho = ML^{-3}$$

$$\mu = ML^{-1}T^{-1}$$

① $Re = \left(\dfrac{\rho U D}{\mu}\right)$

② $\dfrac{jD}{\mathbb{D}\Delta c}$ = Sherwood number

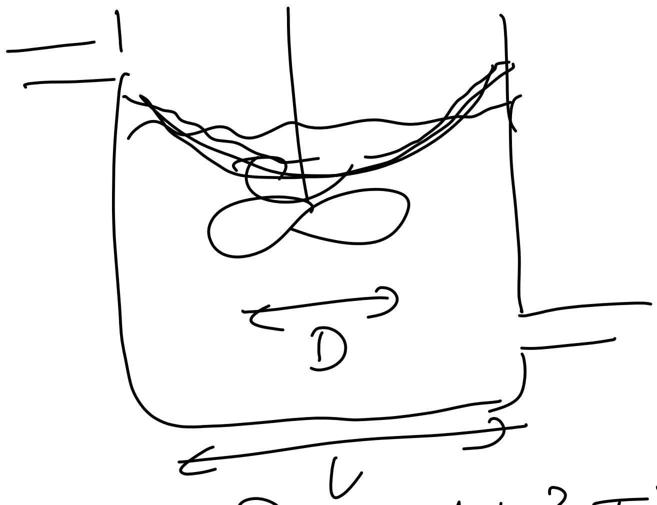
③ $\dfrac{\mu}{\rho \mathbb{D}} = \dfrac{\nu}{\mathbb{D}}$ = Schmidt number

$Sh = 2 + 0.6\, Re^{1/2} Sc^{1/3}$ Low Schmidt number & Reynolds number

$Sh = 1.24\, Re^{1/3} Sc^{1/3}$ High Sc, laminar flow

Nu

Pr




Power $P = ML^2T^{-3}$

$\Pi_5 = (L/D)$

$\Pi_1 = \dfrac{P}{\rho D^5 f^3}$ Power number

$\Pi_2 = \dfrac{\rho D^2 f}{\mu} =$ Reynolds number

$\Pi_3 = \dfrac{UD}{g} =$ Froude number

$\Pi_4 = \dfrac{\rho D^3 f^2}{\gamma} =$ Weber number

$P = ML^2T^{-3}$

$f = T^{-1}$

$D = L$

$\rho = ML^{-3}$

$\mu = ML^{-1}T^{-1}$

$L = L$

$g = LT^{-2}$

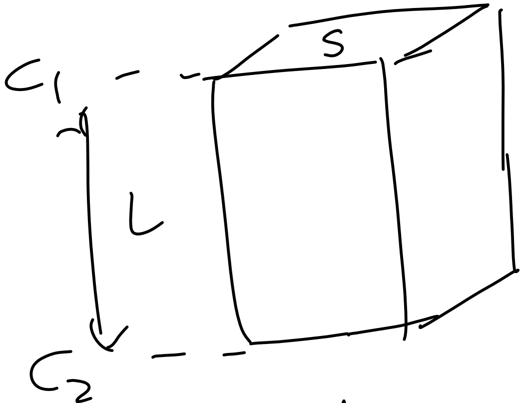
$\gamma = MT^{-2}$

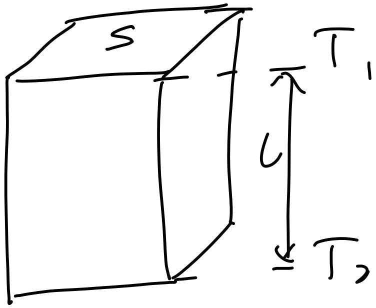
$Po = fn(Re, Fr, We, L/D)$

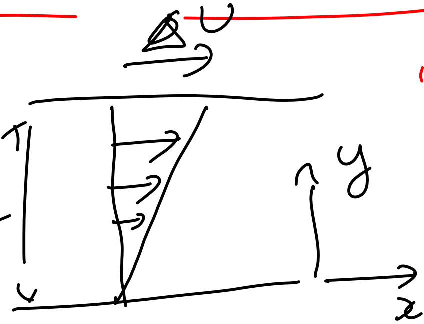
# Physical meaning of dimensionless numbers:

Flux of quantity per unit area per unit time = Diffusion coefficient × $\dfrac{\text{Change in density of quantity}}{\text{Unit length}}$

$\mathbb{D} = L^2 T^{-1}$




$$\tau_{xy} = \mu \frac{\Delta U}{L} = \frac{\mu}{\rho}\frac{\Delta(\rho U)}{L} = \nu \frac{\Delta(\rho U)}{L}$$

$$j = \mathbb{D}\frac{\Delta C}{L}$$

Fick's law of diffusion

$$q = k\frac{\Delta T}{L}$$ Fourier's law

$$= \left(\frac{k}{\rho C_p}\right)\frac{\Delta(\rho C_p T)}{L}$$

$$= \alpha \frac{\Delta(\rho C_p T)}{L}$$

$\alpha$ = Thermal diffusivity

$$\frac{\mu = ML^{-1}T^{-1}}{\rho = ML^{-3}} = L^2T^{-1}$$

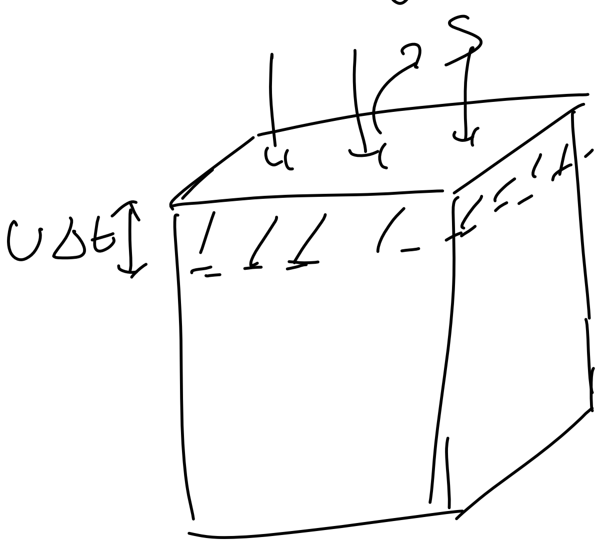
MASS DIFFUSIVITY $= \mathcal{D} = L^2T^{-1}$

THERMAL DIFFUSIVITY $= \alpha = \dfrac{k}{\rho C_p} = L^2T^{-1}$

MOMENTUM DIFFUSIVITY $= \nu = \dfrac{\mu}{\rho} = L^2T^{-1}$

Convection $U$

$t, \; t+\Delta t$




Volume in $= S \times U \times \Delta t$

Mass in $= C \times S \times U \times \Delta t$

Mass in / unit time $= C \times U \times S$

Flux = Mass in / unit area / time $= C \times U$

= Density of quantity × velocity

Mass transfer:

$$Sc = \left(\frac{\mu}{\rho \mathbb{D}}\right) = \left(\frac{\nu}{\mathbb{D}}\right) = \frac{\text{Momentum diffusion}}{\text{Mass diffusion}}$$

$$Pr = \left(\frac{C_p \mu}{k}\right) = \left(\frac{\mu}{\rho}\right)\left(\frac{\rho C_p}{k}\right) = \left(\frac{\nu}{\alpha}\right) = \frac{\text{Momentum diffusion}}{\text{Thermal diffusion}}$$

$$Re = \left(\frac{\rho U D}{\mu}\right) = \left(\frac{UD}{\nu}\right) = \frac{\text{Convection}}{\text{Momentum diffusion}}$$

$$Pe = \frac{UD}{\mathbb{D}} = Re \times Sc = \frac{\text{Convection}}{\text{Mass diffusion}}$$

$$Pe = \frac{UD}{\alpha} = Re \times Pr = \frac{\text{Convection}}{\text{Thermal diffusion}}$$

Dimensionless numbers involving surface tension

Capillary number $= \dfrac{\mu U}{\gamma} = \dfrac{\text{Ratio of viscosity}}{\text{Surface tension}}$

Weber number $= \dfrac{\rho U^2 D}{\gamma} = \dfrac{\text{Ratio of inertia}}{\text{Surface tension}}$

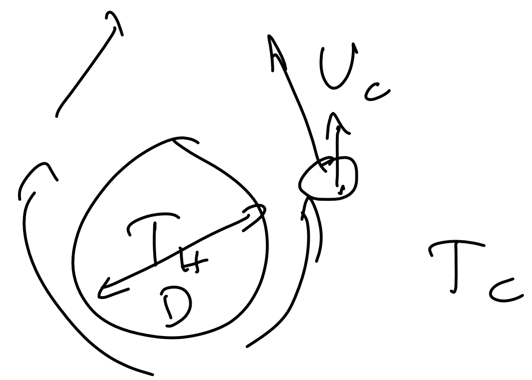
Dimensionless groups involving gravity:

$Fr = \dfrac{U^2}{gD}$

$= \dfrac{\text{Inertia}}{\text{gravity}}$

Bond number $= \dfrac{\rho g L^2}{\gamma} = \dfrac{\text{gravity}}{\text{surface tension}}$

# Natural convection:




$$Gr = \frac{\rho U_c D}{\mu}$$

$$= \left(\frac{\rho D}{\mu}\right)\left(\frac{f D^2}{\mu}\right)$$

$$= \left(\frac{\rho D}{\mu}\right)\left(\frac{D^2}{\mu}\right)\left(\rho \beta \Delta T g\right)$$

$$= \frac{\rho^2 D^3 \beta g \Delta T}{\mu^2}$$

$$f = \Delta \rho g$$

$$= \rho \beta \Delta T g$$

$$U_c = \left(\frac{f D^2}{\mu}\right)$$

$$Ra = \frac{U_c D}{\alpha} = \frac{U_c D}{k/\rho C_p}$$

$$= \frac{\rho C_p D}{k}\left(\frac{D^2}{\mu}\right)\left(\rho \beta \Delta T g\right)$$

$$= \frac{\rho^2 C_p D^3 \beta \Delta T g}{\mu k}$$

Non-dimensional fluxes:

$$Nu = \frac{qD}{k\Delta T} \qquad Sh = \frac{jD}{\mathbb{D}\Delta T}$$

Flow through pipes & channels:

Laminar $Nu = 1.86\, Re^{1/3} Pr^{1/3} (D/L)^{1/3} (\mu/\mu_w)^{0.14}$

$= 1.86\, Pe^{1/3} (D/L)^{1/3} (\mu/\mu_w)^{0.14}$

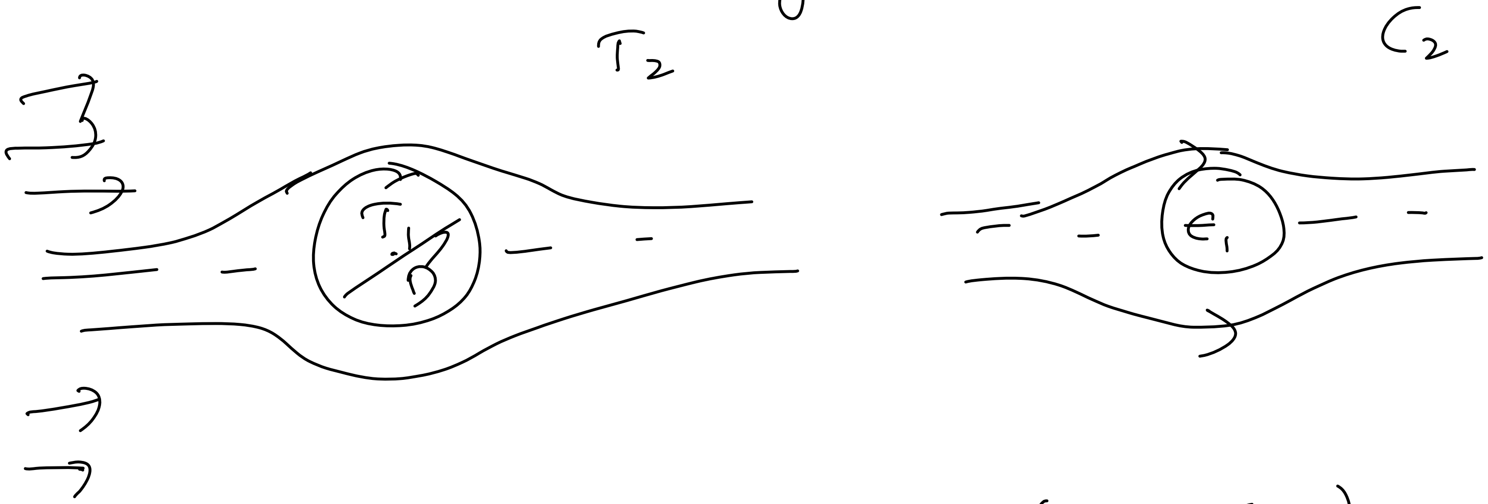
$Sh = 1.86\, Re^{1/3} Sc^{1/3} (D/L)^{1/3}$

Turbulent $Nu = 0.023\, Re^{0.8} Pr^{1/3} (\mu/\mu_w)^{0.14}$

Sieder-Tate relation

$Sh = 0.023\, Re^{0.8} Sc^{1/3} (\mu/\mu_w)^{0.14}$

Flow around objects:



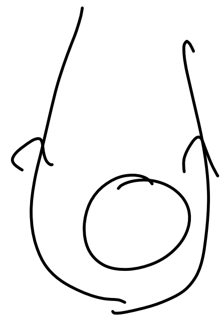

High Peclet number (laminar)

$$Nu = 1.24\, Re^{1/3}\, Pr^{1/3}$$

$$= 1.24\, Pe^{1/3}$$

$$Sh = 1.24\, Re^{1/3}\, Sc^{1/3}$$

Natural convection:



$$Nu = 2 + 0.59\,(Gr\,Pr)^{1/4}$$

very low $Gr\,Pr < 10^4$

For $Gr\,Pr$ between $10^4$ & $10^9$

$$Nu = 0.518\,(Gr\,Pr)^{1/4}$$

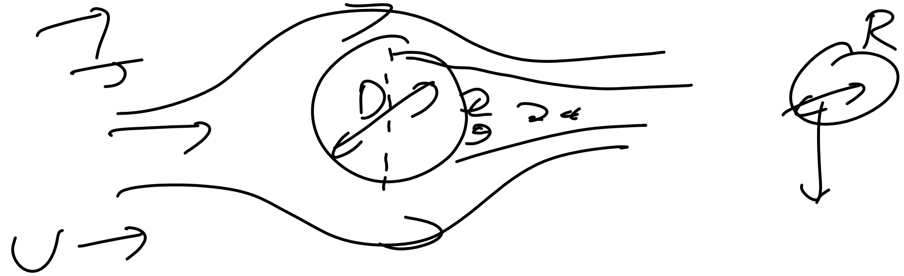
Limit $Pr \ll 1$

Limit $Pr \gg 1$

$$Nu \propto Pr^{1/2}\,Gr^{1/4}$$

# Flow around object

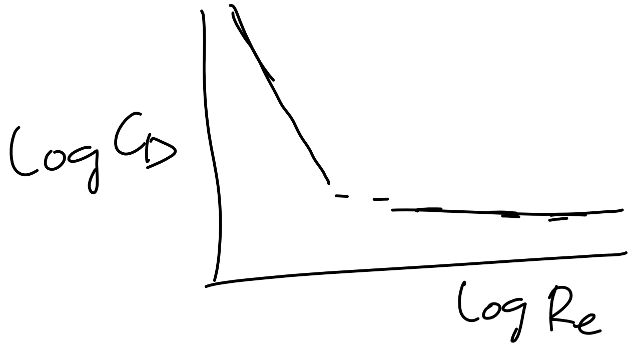
$$C_D = \frac{(F_D / A_p)}{\frac{1}{2}\rho U^2}$$




$$A_p = \pi R^2 = (\pi D^2/4)$$

$F_D = 6\pi\mu R U$ Stokes drag law

$$= 3\pi\mu D U$$

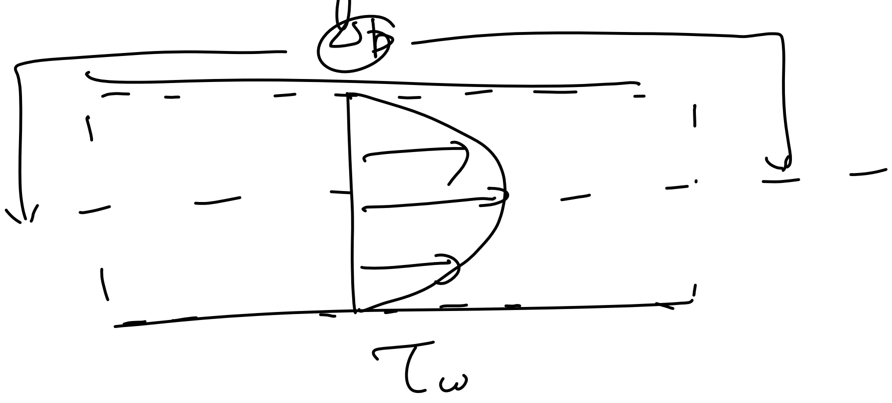



$$F_D/A_p = \frac{3\pi\mu D U}{\pi D^2/4} = \frac{12\mu U}{D}$$

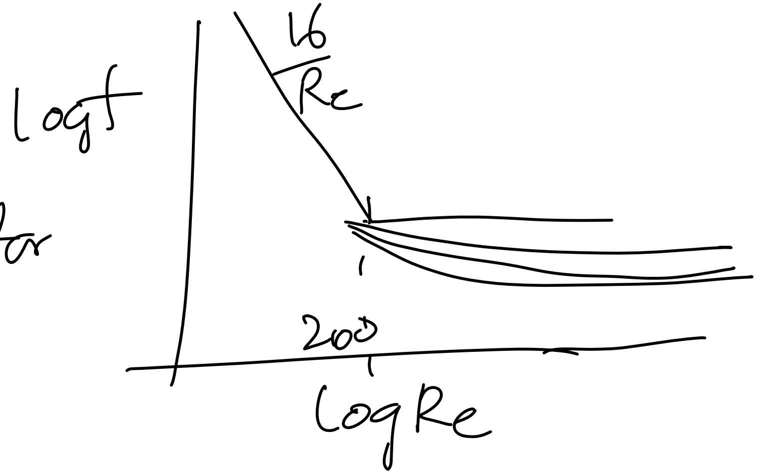
$$C_D = \frac{12\mu U}{D(\frac{1}{2}\rho U^2)} = \frac{24}{(\rho U D/\mu)} = \frac{24}{Re}$$

Flow through tubes:

$$Re = \frac{\rho U D}{\mu}$$




$Re < 2100 \Rightarrow$ laminar flow




$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \text{Friction factor}$$

$$\left(\tau_w 2\pi R L\right) = \Delta p \, \pi R^2$$

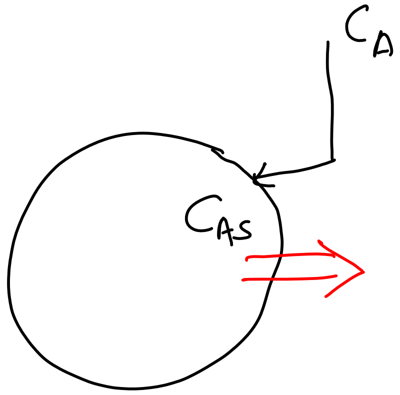
$$\frac{\Delta p}{L} = \frac{2\tau_w}{R} = \frac{4\tau_w}{D}$$

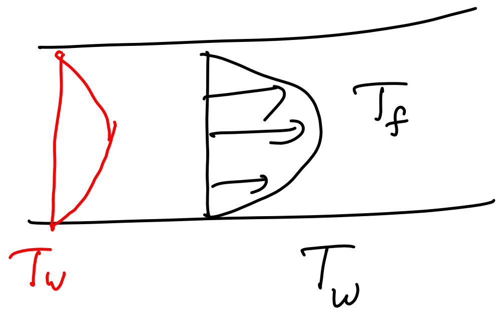
$$f = \left(\frac{\Delta p}{L}\right)\frac{D}{2\rho U^2}$$

$$\tau_w = \frac{c \mu U}{D} \Rightarrow f = c\left(\left(\frac{\mu U}{D}\right)\left(\frac{1}{\frac{1}{2}\rho U^2}\right)\right)$$

$$= 2c\left(\frac{\mu}{\rho U D}\right) = \frac{2c}{Re}$$

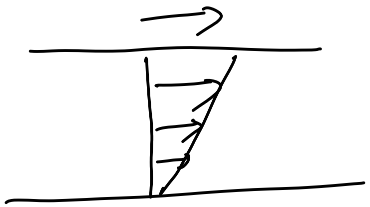
# CONTINUUM DESCRIPTION



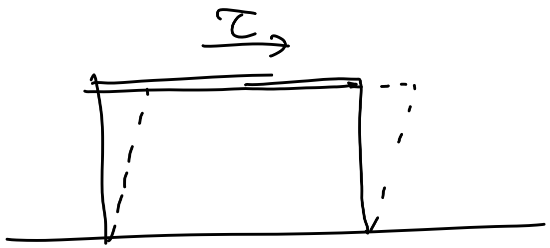



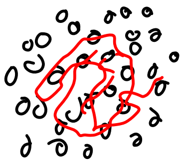

# FLUIDS

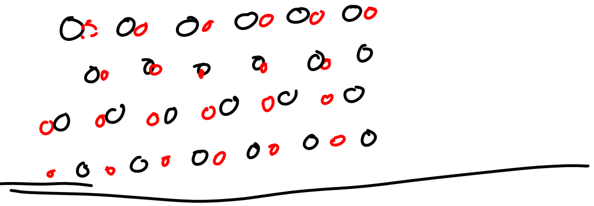
# FLUIDS



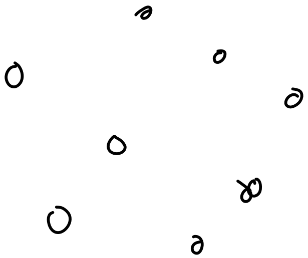
# SOLIDS



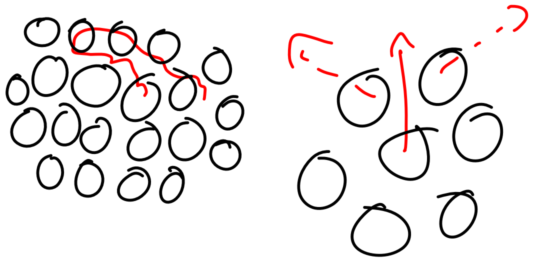





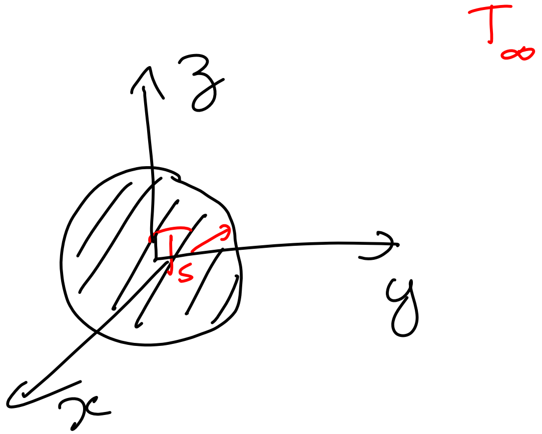
# GASES



# LIQUDS

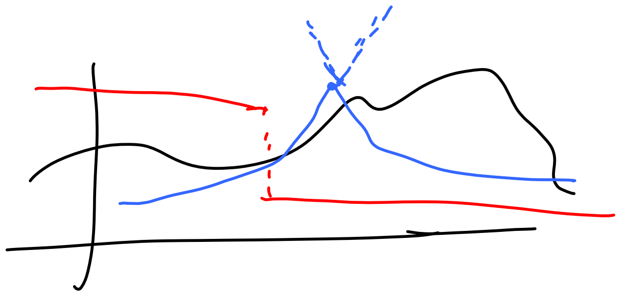


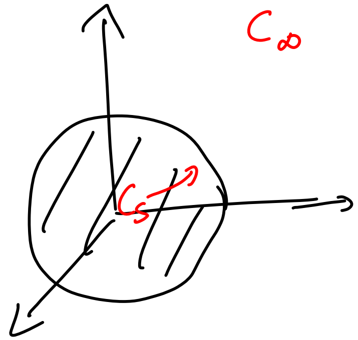
# Continuum description:




Temperature field

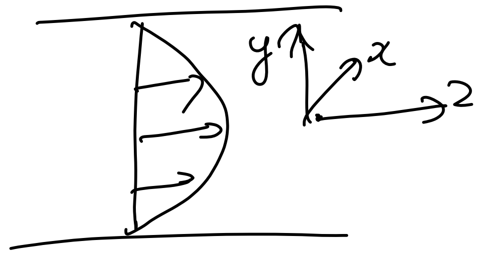
$T(x, y, z)$






Concentration field

$c(x, y, z)$

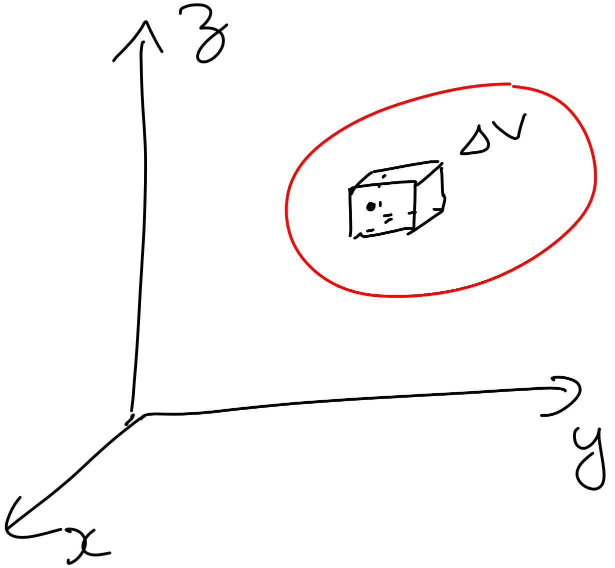



Density field

$\rho(x, y, z)$

Velocity field

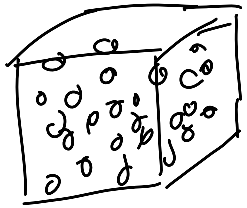
$u_z(x, y, z)$

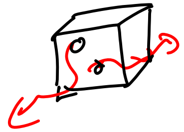


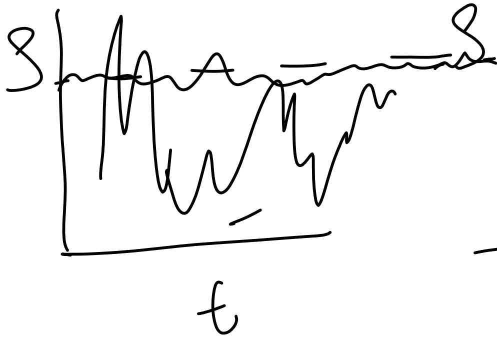

$\rho(x, y, z)$

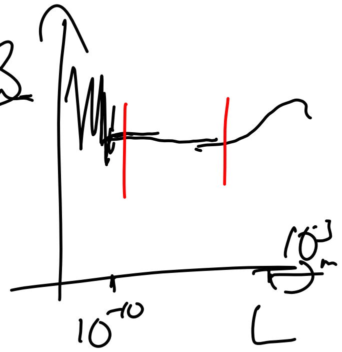
Mass within $\Delta V = nm$

Density $= \lim_{\Delta V \to 0} \frac{nm}{\Delta V}$








$d_{H_2} = 1.38\,\text{Å}$

$d_{O_2} = 3.8\,\text{Å}$

Microscopic scale

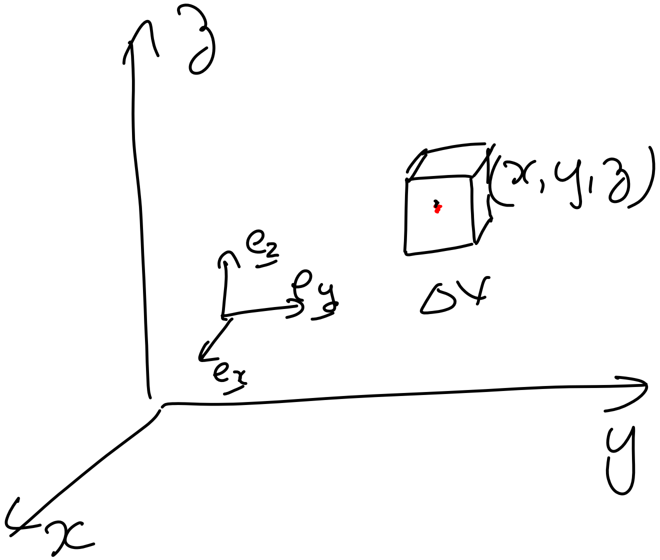
Liquids $\rightarrow 10^{-9} - 10^{-10}$ m

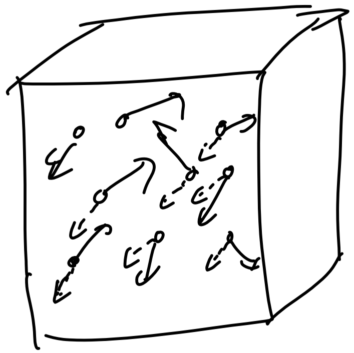
Gases $\rightarrow 10^{-6}$ m to $10^{-8}$ m

Macroscopic scale

1 mm $\quad 10^{-3}$ m

1 m $\quad$ 1 m



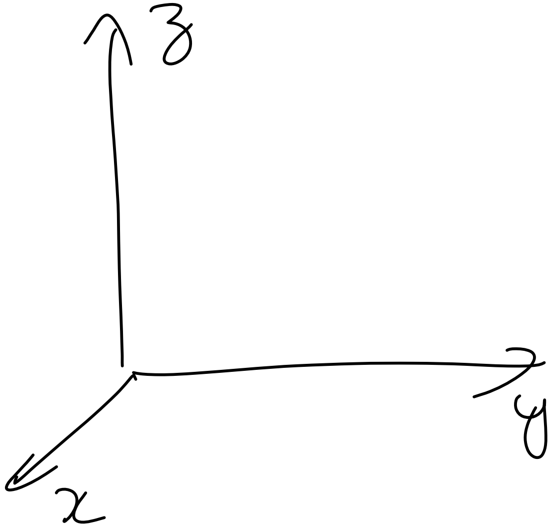
$$\boxed{\rho(x,y,z)} = \lim_{\Delta V \to 0} \left( \frac{n\, m}{\Delta V} \right)$$

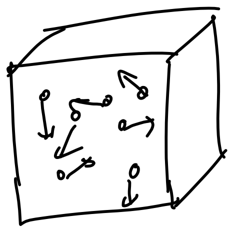
$$(\rho \underline{u}) = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n} m \underline{v}}{\Delta V}$$

$$\underline{u} = u_x \underline{e_x} + u_y \underline{e_y} + u_z \underline{e_z}$$

$$= \underline{e_x} \lim_{\Delta V \to 0} \frac{\sum m v_x}{\Delta V} + \underline{e_y} \lim_{\Delta V \to 0} \frac{\sum m v_y}{\Delta V}$$

$$+ \underline{e_z} \lim_{\Delta V \to 0} \frac{\sum m v_z}{\Delta V}$$



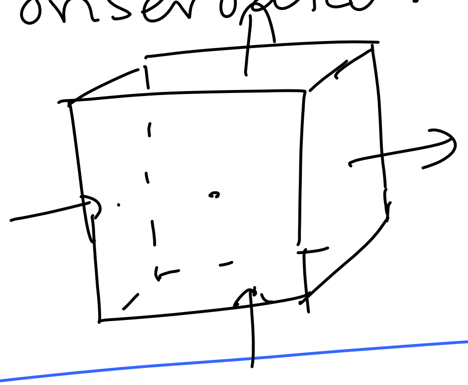
Fluctuating energy $= \frac{1}{2} m (\underline{v} - \underline{u})^2$

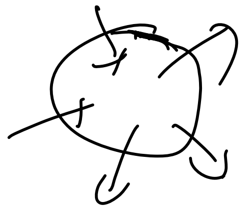
$$e = \frac{\lim_{\Delta V \to 0} \sum_{i=1}^{n} \frac{1}{2} m (\underline{v}_i - \underline{u})^2}{\Delta V} = \frac{3}{2} n k T$$

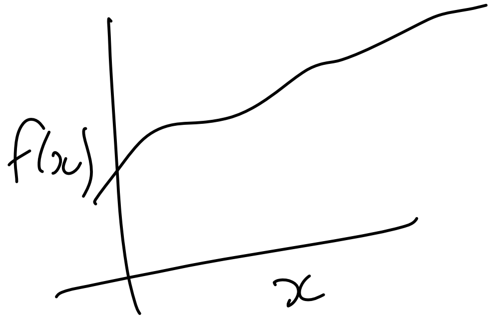
$$= m n C_v T$$

$$= \rho C_v T$$

# Conservation equations



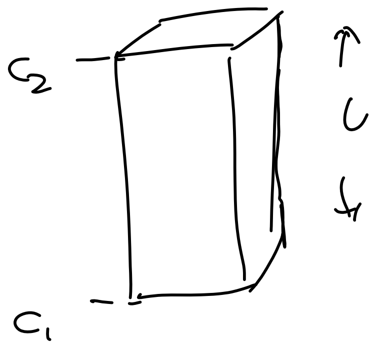


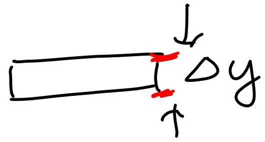



(Change in energy in a time $\Delta t$) = (Energy in) − (Energy out)

(Rate of change of momentum) = (Momentum in) − (Momentum out) + (Sum of all forces)

# Constitutive relations:




$$j = -D\left(\frac{\Delta c}{l}\right)$$

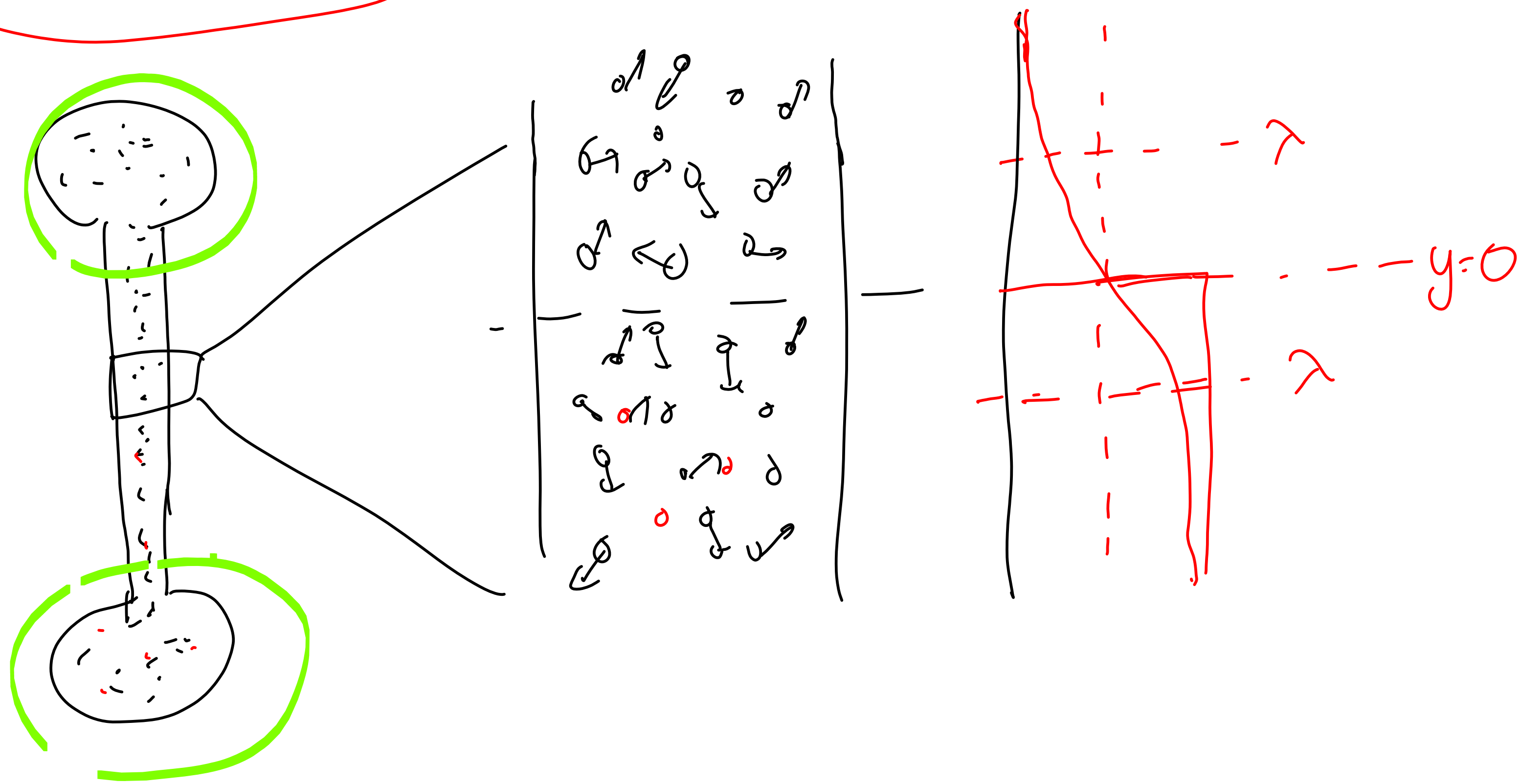
$$j_y = -D\,\frac{\Delta c}{\Delta y}$$

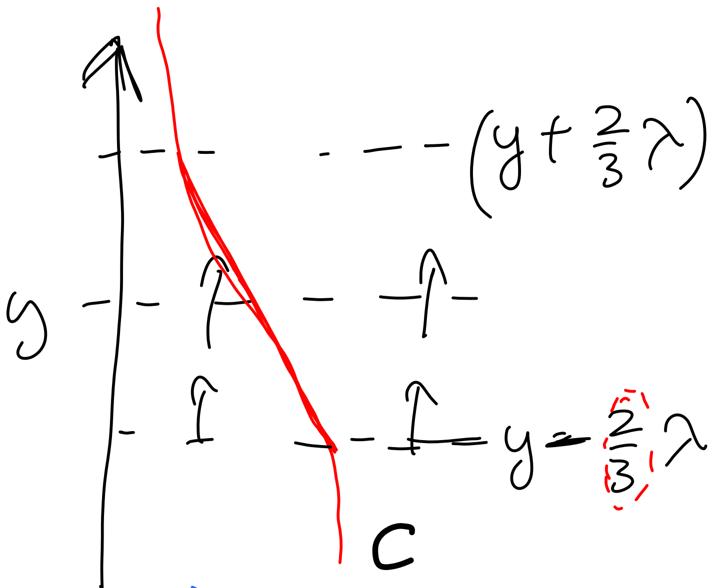
$$j_y = -D \lim_{\Delta y \to 0} \left.\frac{\partial c}{\partial y}\right|_{\substack{x=\text{const}\\ z=\text{const}\\ t=\text{const}}}$$

$$j_x = -D\,\frac{\partial c}{\partial x}\ ; \quad j_z = -D\,\frac{\partial c}{\partial z}$$

$$\vec{j} = e_x\, j_x + e_y\, j_y + e_z\, j_z$$
$$= -D\left[e_x \frac{\partial c}{\partial x} + e_y \frac{\partial c}{\partial y} + e_z \frac{\partial c}{\partial z}\right]$$

DIFFUSION:






$$c\left(y-\tfrac{2}{3}\lambda\right) = c(y) - \frac{2}{3}\lambda \frac{dc}{dy}\Big|_y + \frac{2}{3}\frac{\lambda^2}{2!}\frac{d^2c}{dy^2}$$

$$c\left(y+\tfrac{2}{3}\lambda\right) = c(y) + \frac{2}{3}\lambda \frac{dc}{dy}\Big|_y + \cdots$$

$$j_y = \frac{1}{4}v_{rms}\left[c(y) - \frac{2}{3}\lambda\frac{dc}{dy}\Big|_y - c(y) - \frac{2}{3}\lambda\frac{dc}{dy}\Big|_y\right]$$

$$= -\frac{1}{3}v_{rms}\,\lambda\frac{dc}{dy}\Big|_y$$

$$j_+ = \frac{1}{4}c\left(y-\tfrac{2}{3}\lambda\right)v_{rms}$$

$$j_- = \frac{1}{4}c\left(y+\tfrac{2}{3}\lambda\right)v_{rms}$$

$$j_y = j_+ - j_- = \frac{1}{4}v_{rms}\left[c\left(y-\tfrac{2}{3}\lambda\right) - c\left(y+\tfrac{2}{3}\lambda\right)\right]$$

$$D \simeq \frac{1}{3}v_{rms}\lambda$$

$$\frac{dc}{dy} \sim \frac{c}{L}$$

$$\lambda\frac{dc}{dy} \sim \frac{\lambda c}{L}$$

$$\propto \lambda^2\frac{d^2c}{dy^2}$$

$$\sim \frac{\lambda^2 c}{L^2}$$

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT$$

Oxygen Mass $= 32 \times 10^{-3}$ kg

$v_{rms} = 321$ m/s

$$v_{mean} = \sqrt{\frac{8kT}{\pi m}}$$

$k = 1.38 \times 10^{-23}$ J/k

$T = 300K$ (room temperature)

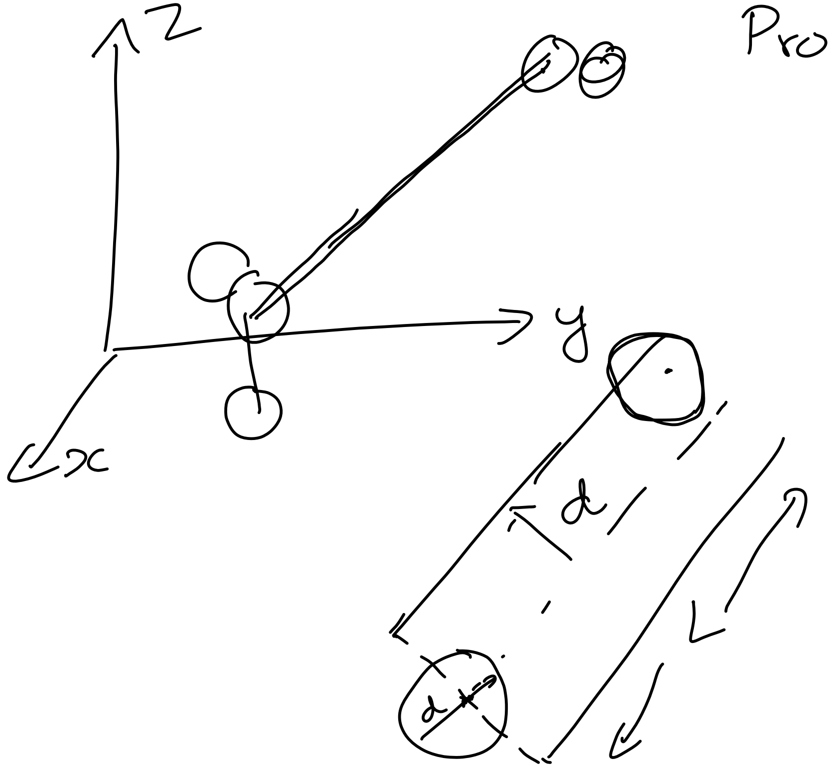
$kT \cong 4 \times 10^{-21}$ J

Hydrogen: Mass $= 2 \times 10^{-3}$ kg for $6.023 \times 10^{23}$ molecules

$$m = \frac{2 \times 10^{-3}}{6.023 \times 10^{-23}} \text{ kg} = 3.32 \times 10^{-27} \text{ kg}$$

$$v_{rms} = \sqrt{\frac{3kT}{m}} = 1.29 \times 10^{3} \text{ m/s}$$

Mean free path:




Volume of cylinder $= (\pi d^2 L)$

Probability of finding a second molecule

$$= (n \pi d^2 L)$$

$$n \pi d^2 \lambda \sim 1$$

$$\lambda \cong \frac{1}{\pi n d^2} = \frac{1}{\sqrt{2}\, \pi n d^2}$$

$$\lambda = \frac{1}{\sqrt{2}\pi n d^2}$$

$$D \simeq 10^{-5}\,m^2/s$$

$H_2, He$ $1.132 \times 10^{-4}\,m^2/s$

$O_2, H_2 \simeq 1.8 \times 10^{-5}\,m^2/s$

$$n = \left(\frac{p}{kT}\right) = \frac{1 \times 10^5\,N/m^2}{4 \times 10^{-21}\,J} = 2.5 \times 10^{25}\,\text{molecules}/m^3$$

$$\lambda = \frac{1}{\sqrt{2}\pi n d^2}$$

Hydrogen $d = 1.38\,\mathring{A} = 1.38 \times 10^{-10}\,m$

$\lambda = 0.5 \times 10^{-6}\,m \simeq 0.5\,\mu$

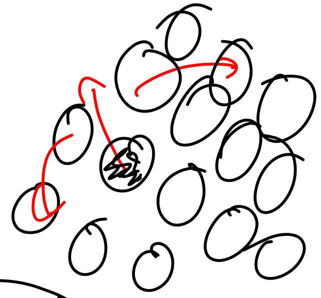
Oxygen & nitrogen, $d = 3.7 - 3.8\,\mathring{A}$

$\lambda = 6 \times 10^{-8}\,m$

$$D = \frac{1}{3} v_{rms} \lambda = 6 \times 10^{-4}\,m^2/s \text{ (Hydrogen)}$$

$$= 2 \times 10^{-5}\,m^2/s \text{ (Oxygen, nitrogen)}$$

$$D = \frac{3}{8nd^2}\left(\frac{kT}{\pi m}\right)^{1/2}$$



$$D_{12} = \frac{3}{8n_{12}d_{12}^2}\left(\frac{kT(m_1+m_2)}{\pi m_1 m_2}\right)^{1/2}$$

$$D = \frac{kT}{3\pi\mu d} \quad \text{Stokes-Einstein relation}$$

$$d_{12} = (d_1+d_2)/2 \; ; \; n_{12} = \sqrt{n_1 n_2}$$

Liquids

$$v_{rms} = \left(\frac{3kT}{m}\right)^{1/2}$$

Expect $\cancel{D_{liquid} \cong \frac{1}{100} D_{gases}}$

$$D_{liquid} = 10^{-9}\ m^2/s \text{ (small molecules)}$$

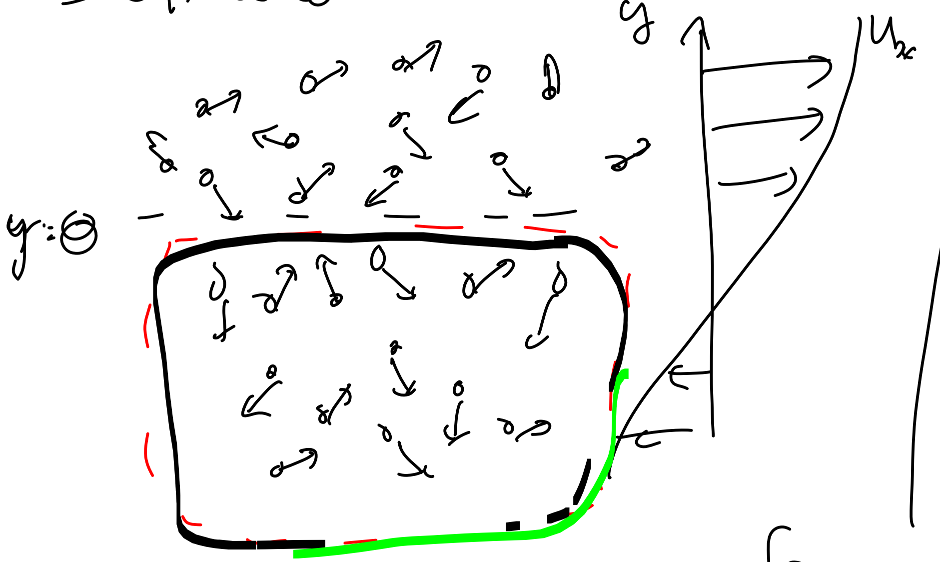
$$= 10^{-11} - 10^{-13}\ m^2/s \text{ (large molecules)}$$

Gases

$$D = a \lambda v_{rms} \propto T^{1/2}$$

$$\lambda = \frac{1}{\sqrt{2}\pi n d^2} \qquad v_{rms} = \sqrt{\frac{kT}{m}}$$

Diffusion of momentum:




$$\begin{pmatrix}\text{Rate of}\\ \text{change of}\\ \text{momentum}\end{pmatrix} = \dot{j}^{m}_{y-} - \dot{j}^{m}_{y+}$$

$$\begin{pmatrix}\text{Flux of}\\ \text{momentum}\\ \text{from below to}\\ \text{above}\end{pmatrix} = \dot{j}^{m}_{y+} = \left[\left[n m u_x\left(y-\tfrac{2}{3}\lambda\right)\right] v_{rms}\, a\right]$$

$$\dot{j}^{m}_{y-} = \left[\left[n m u_x\left(y+\tfrac{2}{3}\lambda\right)\right] v_{rms}\, a\right]$$

$$\begin{pmatrix}\text{net flux of}\\ \text{momentum}\end{pmatrix} = \dot{j}^{m}_{y-} - \dot{j}^{m}_{y+} = \left(n m u_x\left(y+\tfrac{2}{3}\lambda\right) - n m u_x\left(y-\tfrac{2}{3}\lambda\right)\right) v_{rms}\, a$$

$$= a\, v_{rms}\, n m \left[\left(u_x(y) + \tfrac{2}{3}\lambda \left.\frac{du_x}{dy}\right|_y\right) - \left(u_x(y) - \tfrac{2}{3}\lambda \left.\frac{du_x}{dy}\right|_y\right)\right]$$

$$\tau_{xy} = \frac{4}{3}\, a\, v_{rms}\, n m \lambda \frac{du_x}{dy}$$

$$\tau_{xy} = \mu \frac{du_x}{dy}$$

$$\mu \equiv v_{rms}\, n m \lambda$$

$$\mu = A\, n\, m\, v_{rms}\, \lambda$$

$$\text{Momentum density} = \rho u_x$$

$$\tau_{xy} = \mu \left(\frac{du_x}{dy}\right) = \frac{\mu}{\rho}\left(\frac{d}{dy}(\rho u_x)\right)$$

$$\left(\mu/\rho\right) = \nu = \text{Kinematic viscosity}$$

$$\frac{\mu}{\rho} = \nu = A\, v_{rms}\, \lambda$$

$$\mu = A\, n\, m\, v_{rms}\, \lambda = A\, n\, m \sqrt{\frac{3kT}{m}} \left(\frac{1}{\sqrt{2}\,\pi\, n\, d^2}\right)$$

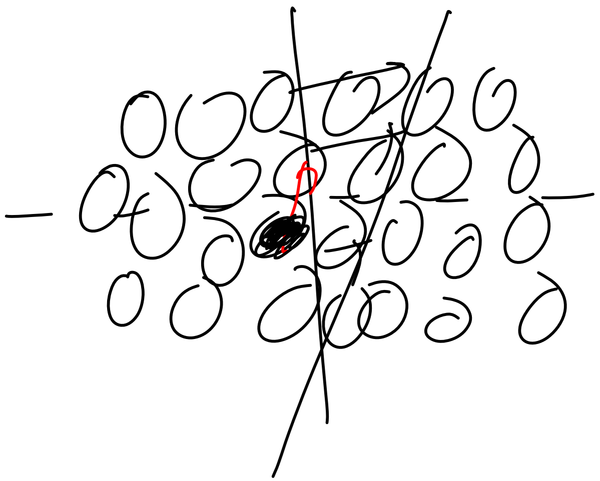
$$= \frac{5}{16 d^2}\left(\frac{mkT}{\pi}\right)^{1/2}$$

$$\nu = \frac{\mu}{nm} = \frac{5}{16nd^2}\left(\frac{kT}{\pi m}\right)^{1/2}$$

$$D = \frac{3}{8nd^2}\left(\frac{kT}{\pi m}\right)^{1/2}$$

$$Sc = \frac{\nu}{D} = \frac{5}{6}$$

Momentum transport in liquids:

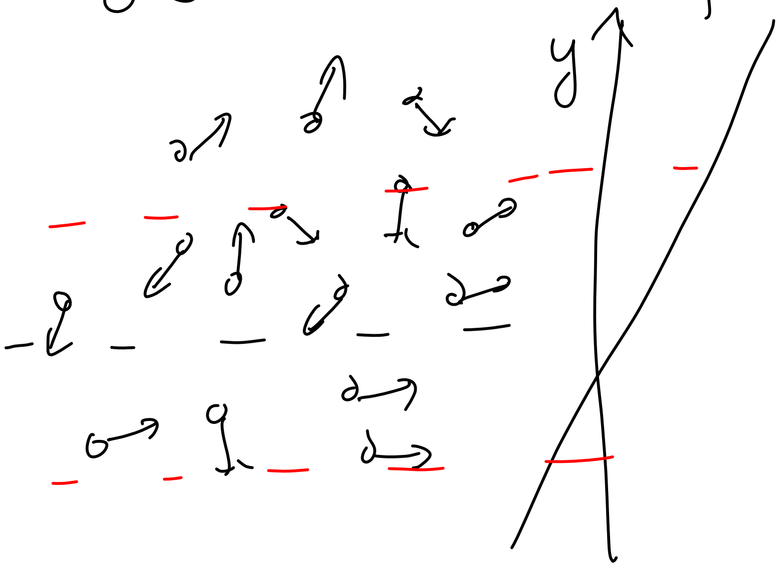


$$\nu_{water} = 10^{-6}\, m^2/s$$

$$\nu_{air} = 1.5 \times 10^{-5}\, m^2/s$$

$$Sc = \frac{\nu}{D} \cong 10^3$$

Energy diffusion: Gases:




$$j_+ \cong \frac{1}{4} e\left(y - \frac{2}{3}\lambda\right) v_{rms}$$

$$j_- \cong \frac{1}{4} e\left(y + \frac{2}{3}\lambda\right) v_{rms}$$

Net flux $\bar{j} = j_+ - j_-$

$$= \frac{1}{4} v_{rms} \left( e\left(y - \frac{2}{3}\lambda\right) - e\left(y + \frac{2}{3}\lambda\right)\right)$$

$$= \frac{1}{4} v_{rms} \left[ e(y) - \frac{2}{3}\lambda \frac{de}{dy}\bigg|_y - e(y) - \frac{2}{3}\frac{de}{dy}\bigg|_y \lambda \right]$$

$$= \frac{1}{3} v_{rms}\, \lambda \left(\frac{de}{dy}\right)$$

$$q = -k \frac{dT}{dy}$$

$$\alpha = \frac{k}{\rho C_p} = \frac{C_v}{C_p} \lambda v_{rms}$$

$$q = -\frac{1}{3} \lambda v_{rms} \frac{d}{dy}(\rho C_v T)$$

$$= -\frac{1}{3} \lambda v_{rms} \rho C_v \frac{dT}{dy}$$

$$k \cong \lambda v_{rms} n m C_v$$

$$\cong \frac{1}{n d^2} \sqrt{\frac{3kT}{m}}\, n m C_v$$

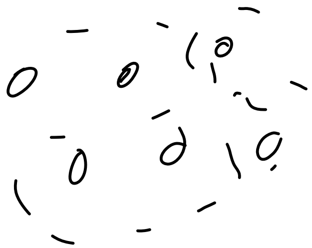
$$k = \frac{75}{64 d^2} \left(\frac{k^3 T}{\pi m}\right)^{3/2} = \frac{5}{2} C_v \mu$$

$$Pr = \frac{C_p \mu}{k} = \frac{2}{5} \frac{C_p}{C_v} = \frac{2}{3}$$

Larger molecules $Pr \rightarrow 1$

# Thermal Conduction in liquids:

Liquid metals



$$Pr = \frac{\text{Momentum diffusion}}{\text{Thermal diffusion}}$$

$\ll 1$

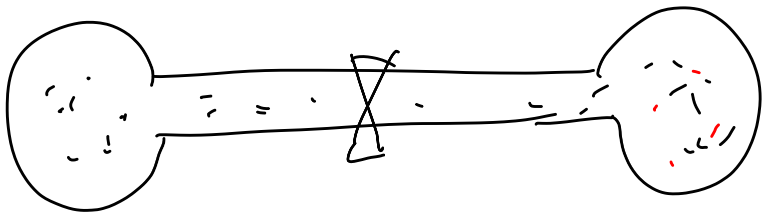
Liquid mercury $Pr = 0.015$

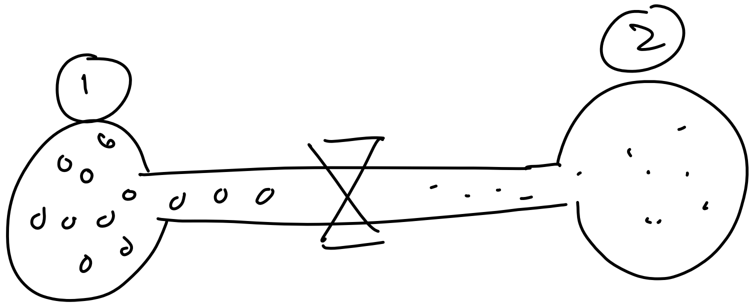
Large organic molecules

$10^2 < Pr < 10^4$

Water $Pr \cong 7$

# Multicomponent diffusion:



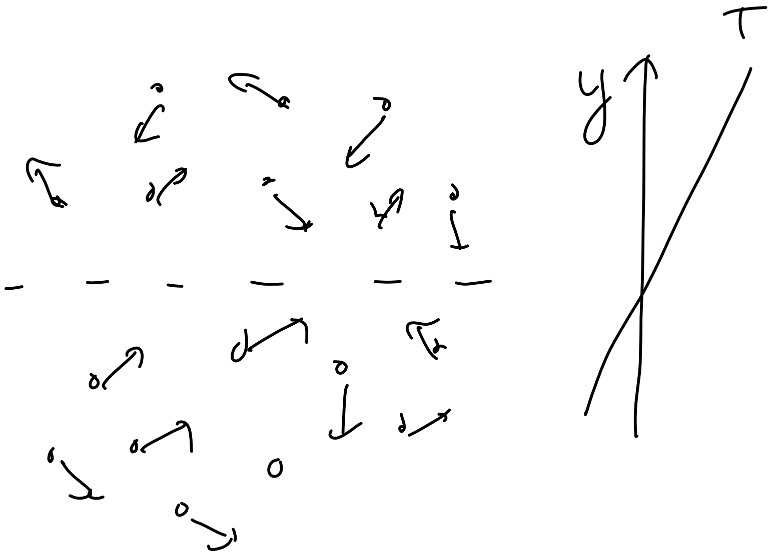



$$n_1 = j_1 + C_1 v_{cm}$$

$$n_2 = j_2 + C_2 v_{cm}$$

$$(n_1 + n_2) = (C_1 + C_2) v_{cm}$$

$$j_1 = -D \frac{dC_1}{dy}$$





$$j_+ = \frac{1}{4} v_{rms}\left(y - \frac{2}{3}\lambda\right) c\left(y - \frac{2}{3}\lambda\right)$$

$$= \frac{1}{4} v_{rms}\left(y - \frac{2}{3}\lambda\right) c(y)$$

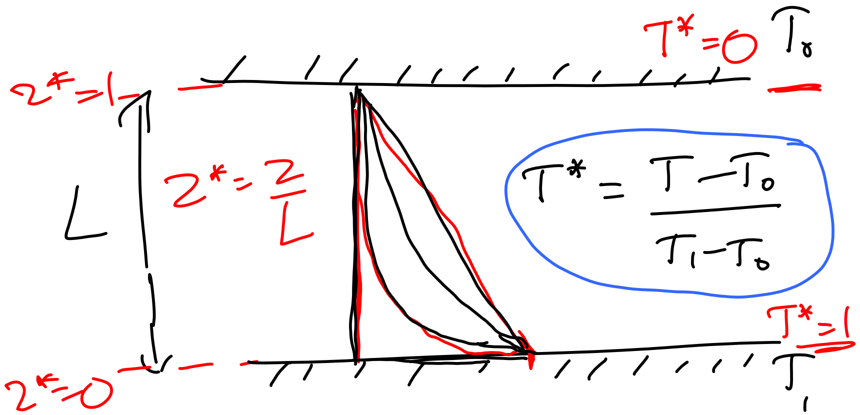
$$j_- = \frac{1}{4} v_{rms}\left(y + \frac{2}{3}\lambda\right) c(y)$$

$$j_+ - j_- = \frac{1}{4} c \left[ v_{rms}\left(y - \frac{2}{3}\lambda\right) - v_{rms}\left(y + \frac{2}{3}\lambda\right) \right]$$

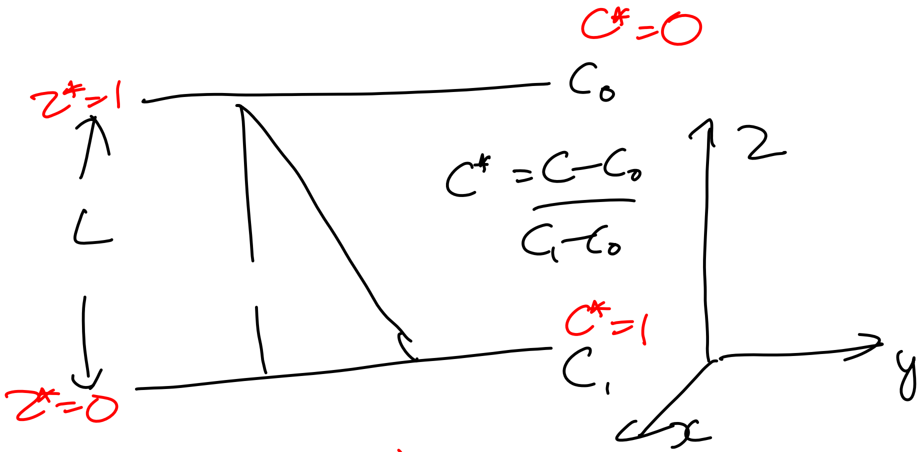
$$= \frac{1}{3} c \frac{d v_{rms}}{dy} = \frac{1}{3} c \frac{d}{dy}\left(\sqrt{\frac{3kT}{m}}\right)$$

$$= \frac{1}{3} c \sqrt{\frac{3kT}{m}} \frac{1}{2T} \frac{dT}{dy}$$

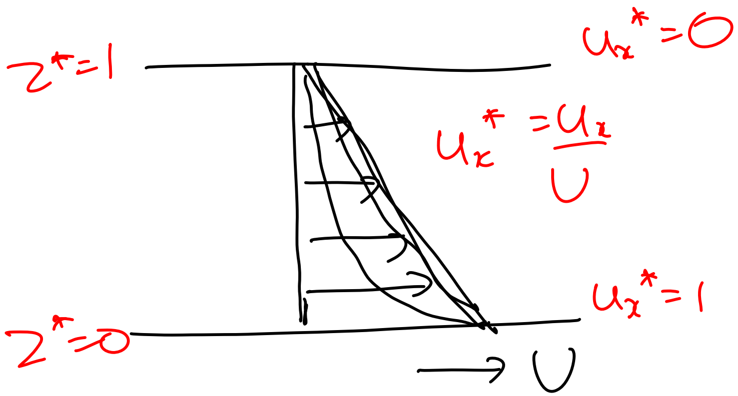
# Unidirectional Transport




$$T^* = \frac{T - T_0}{T_1 - T_0}$$



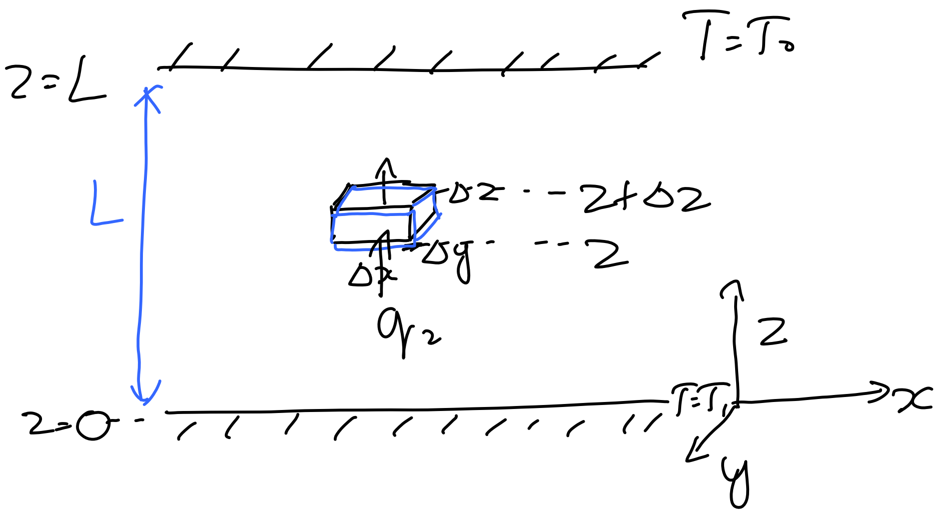

$$T^* = C^* = u_x^* = (1 - z^*)$$




$$\frac{\partial^2 T}{\partial z^2} = 0$$

$$\frac{\partial^2 T^*}{\partial z^{*2}} = 0$$

# Shell balance:




Energy at time t $= (e\Delta x\Delta y\Delta z)\big|_t$

Energy at $t+\Delta t = (e\Delta x\Delta y\Delta z)\big|_{t+\Delta t}$

Change in energy

$$= [e(x,y,z,t+\Delta t) - e(x,y,z,t)]\Delta x\Delta y\Delta z$$

$$= [\rho C_p T(x,y,z,t+\Delta t) - \rho C_p T(x,y,z,t)]\Delta x\Delta y\Delta z$$

$$\begin{pmatrix}\text{Change in}\\ \text{energy in}\\ \text{time } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Energy}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{Energy}\\ \text{out}\end{pmatrix} + \begin{pmatrix}\text{Source of}\\ \text{energy}\end{pmatrix}$$

Energy in $= q_z\big|_z \Delta x\Delta y\Delta t$

Energy out $= q_z\big|_{z+\Delta z}\Delta x\Delta y\Delta t$

Source of energy $= S_e\,\Delta x\Delta y\Delta z\Delta t$

$$= S_e(z,t)\,\Delta x\Delta y\Delta z\Delta t$$

$$\left[\rho C_p T(x,y,z,t+\Delta t) - \rho C_p T(x,y,z,t)\right]\Delta x \Delta y \Delta z =$$

$$q_z\big|_z \Delta x \Delta y \Delta t - q_z\big|_{z+\Delta z} \Delta x \Delta y \Delta t + S_e \Delta x \Delta y \Delta z \Delta t$$

Divide by $\Delta x \Delta y \Delta z \Delta t$

$$\rho C_p \left[\frac{T(x,y,z,t+\Delta t) - T(x,y,z,t)}{\Delta t}\right] = \frac{q_z\big|_z - q_z\big|_{z+\Delta z}}{\Delta z} + S_e$$

$$\rho C_p \left[\frac{T(x,y,z,t+\Delta t) - T(x,y,z,t)}{\Delta t}\right] = -\left(\frac{q_z\big|_{z+\Delta z} - q_z\big|_z}{\Delta z}\right) + S_e$$

Take limit $\Delta t \to 0$, $\Delta z \to 0$

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_z}{\partial z} + S_e$$

$T(z+\Delta z)$

$T(z)$ $\quad \Delta z$

$$q_z = -k\left[\frac{T(z+\Delta z) - T(z)}{\Delta z}\right]$$

$$= -k\,\frac{\partial T}{\partial z}$$

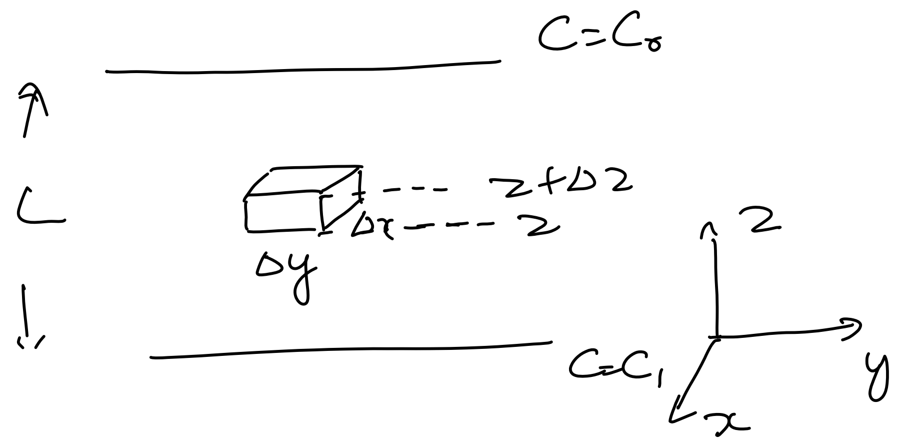
$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_z}{\partial z} = -\frac{\partial}{\partial z}\left(-k\frac{\partial T}{\partial z}\right)$$

$$= k\,\frac{\partial^2 T}{\partial z^2} + S_e$$

$$\frac{\partial T}{\partial t} = \alpha\,\frac{\partial^2 T}{\partial z^2} + \frac{S_e}{\rho C_p}$$

where $\alpha$ = thermal diffusivity

Concentration diffusion:




$$\begin{pmatrix}\text{Change in}\\ \text{mass in}\\ \text{time } \Delta t\end{pmatrix} = C(x,y,z,t+\Delta t)\Delta x\Delta y\Delta z - C(x,y,z,t)\Delta x\Delta y\Delta z$$

$$\begin{pmatrix}\text{Change in}\\ \text{mass in time}\\ \Delta t\end{pmatrix} = \begin{pmatrix}\text{Mass in}\end{pmatrix} - \begin{pmatrix}\text{Mass out}\end{pmatrix} + \begin{pmatrix}\text{Source}\\ \text{of mass}\end{pmatrix}$$

Mass in $= j_z\big|_z \Delta x \Delta y \Delta t$

Mass out $= j_z\big|_{z+\Delta z} \Delta x \Delta y \Delta t$

Source of mass $= S\, \Delta x \Delta y \Delta z \Delta t$

$$\left[C(x,y,z,t+\Delta t) - C(x,y,z,t)\right]\Delta x \Delta y \Delta z = j_z\Big|_{z}\Delta x \Delta y \Delta t - j_z\Big|_{z+\Delta z}\Delta x \Delta y \Delta t + S\,\Delta x \Delta y \Delta z \Delta t$$

Divide by $\Delta x\ \Delta y\ \Delta z\ \Delta t$

$$\frac{c(x,y,z,t+\Delta t) - c(x,y,z,t)}{\Delta t} = \frac{j_z\big|_z - j_z\big|_{z+\Delta z}}{\Delta z} + S$$

$$= -\left(\frac{j_z(z+\Delta z) - j_z(z)}{\Delta z}\right) + S$$

Limit $\Delta t \to 0,\ \Delta z \to 0$

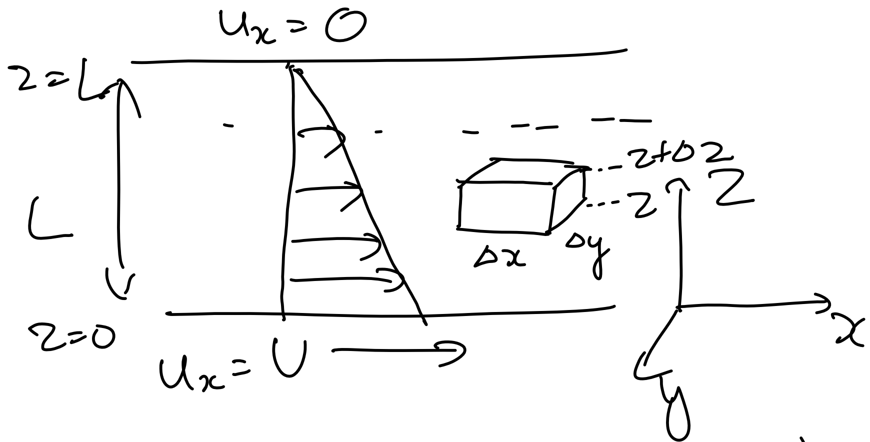
$$\frac{\partial c}{\partial t} = -\frac{\partial j_z}{\partial z} + S$$

$$j_z = -D\frac{\partial c}{\partial z}$$

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial z^2} + S$$

Momentum diffusion: Momentum in the volume $\Delta x \Delta y \Delta z$

$$= \rho u_x(x,y,z)\Delta x \Delta y \Delta z$$




Rate of change of momentum

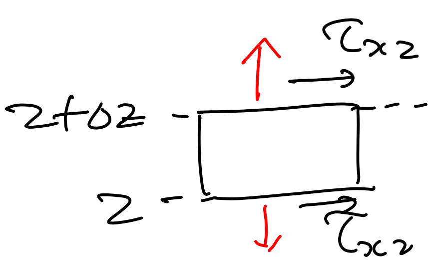
$$= \frac{[\rho u_x(x,y,z,t+\Delta t) - \rho u_x(x,y,z,t)]\Delta x \Delta y \Delta z}{\Delta t}$$

$$\left(\begin{array}{c}\text{Rate of}\\ \text{change of}\\ \text{momentum}\end{array}\right) = \left(\begin{array}{c}\text{Sum of}\\ \text{body}\\ \text{forces}\end{array}\right) + \left(\begin{array}{c}\text{Sum of}\\ \text{surface}\\ \text{forces}\end{array}\right)$$

Body force $= f_x \Delta x \Delta y \Delta z$

Gravitational $= \rho g_x \Delta x \Delta y \Delta z$

$$f_x = \rho g_x$$




Unit normal = Unit vector perpendicular to surface

$\tau_{xz}$ = Force/Area in the $x$ direction acting at a surface with outward unit normal in $z$ direction

Force on top surface $= \tau_{xz}|_{z+\Delta z} \Delta x \Delta y$

Force on bottom surface $= -\tau_{xz}|_{z} \Delta x \Delta y$

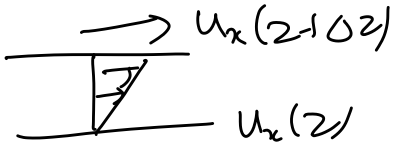
$$\left[\frac{\rho u_x(x,y,z,t+\Delta t)-\rho u_x(x,y,z,t)}{\Delta t}\right]\Delta x\Delta y\Delta z$$

$$= \tau_{xz}\big|_{z+\Delta z}\Delta x\Delta y - \tau_{xz}\big|_{z}\Delta x\Delta y + f_x\Delta x\Delta y\Delta z$$

Divide throughout by $\Delta x\Delta y\Delta z$

$$\frac{\rho u_x(x,y,z,t+\Delta t)-\rho u_x(x,y,z,t)}{\Delta t} = \frac{\left(\tau_{xz}\big|_{z+\Delta z}-\tau_{xz}\big|_{z}\right)}{\Delta z}+f_x$$

$$\frac{\partial(\rho u_x)}{\partial t} = \frac{\partial \tau_{xz}}{\partial z} \quad\Big|\quad \rho\frac{\partial u_x}{\partial t} = \frac{\partial \tau_{xz}}{\partial z}+f_x$$

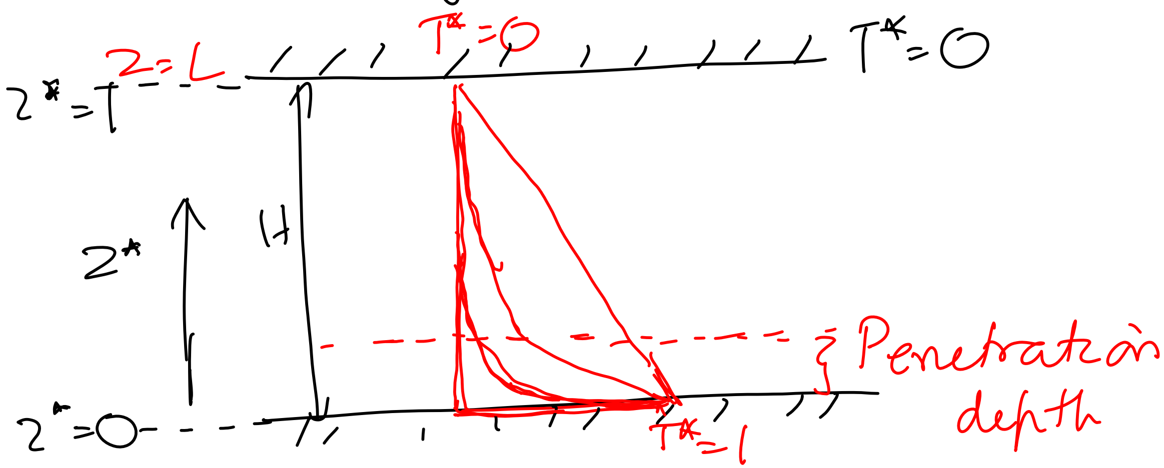



$$\tau_{xz} = \mu\frac{\Delta u_x}{\Delta z} = \mu\frac{\partial u_x}{\partial z}$$

$$\rho\frac{\partial u_x}{\partial t} = \frac{\partial}{\partial z}\left(\mu\frac{\partial u_x}{\partial z}\right) = \mu\frac{\partial^2 u_x}{\partial z^2}+f_x$$

$$\frac{\partial u_x}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u_x}{\partial z^2} = \nu\frac{\partial^2 u_x}{\partial z^2}+\frac{f_x}{\rho}$$

# Unsteady diffusion:




$(t, z, \alpha)$ $(L, T)$

$\xi = \dfrac{z}{\sqrt{\alpha t}}$ Similarity variable

For $t<0$, $T^*=0$ everywhere

At $t \geq 0$, $T^*=1$ at $z^*=0$

$T^*=0$ as $z \to \infty$

$$\frac{\partial T^*}{\partial t} = \alpha \frac{\partial^2 T^*}{\partial z^2}$$

$$\frac{\partial T^*}{\partial t} = \alpha \frac{\partial^2 T^*}{\partial z^2} \qquad \xi = \frac{z}{\sqrt{\alpha t}}$$

$$\frac{\partial T^*}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial T^*}{\partial \xi} = \frac{-z}{2\sqrt{\alpha}\, t^{3/2}}\frac{\partial T^*}{\partial \xi} = -\frac{\xi}{2t}\frac{\partial T^*}{\partial \xi}$$

$$\frac{\partial T^*}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial T}{\partial \xi} = \frac{1}{\sqrt{\alpha t}}\frac{\partial T}{\partial \xi}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial T^*}{\partial z}\right) = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}\left(\frac{\partial T}{\partial \xi}\right) = \frac{1}{\alpha t}\frac{\partial^2 T}{\partial \xi^2}$$

$$-\frac{\xi}{2\cancel{t}}\frac{\partial T}{\partial \xi} = \frac{\cancel{\alpha}}{\cancel{\alpha}\cancel{t}}\frac{\partial^2 T}{\partial \xi^2}$$

$$\boxed{-\frac{\xi}{2}\frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \xi^2}}$$

Boundary condition

$z = 0,\ T^* = 1 \Rightarrow \xi = 0$

$z \to \infty,\ T^* = 0 \Rightarrow \xi \to \infty$

At $t = 0$ for $z > 0$, $T^* = 0 \Rightarrow \xi \to \infty$

$$-\frac{\eta}{2}\frac{\partial T^*}{\partial \eta} = \frac{\partial^2 T^*}{\partial \eta^2}$$

$$u = \frac{\partial T^*}{\partial \eta}$$

$$-\frac{\eta}{2} u = \frac{\partial u}{\partial \eta}$$

$$u = C\, e^{-\eta^2/4} = \frac{\partial T^*}{\partial \eta}$$

$$T^* = C \int_0^{\eta} d\eta'\, e^{-\eta'^2/4} + D$$

$T^* = 0$ as $\eta \to \infty$ ; $T^* = 1$ at $\eta = 0$

$$T^* = \left[1 - \frac{\int_0^{\eta} d\eta'\, e^{-\eta'^2/4}}{\int_0^{\infty} d\eta'\, e^{-\eta'^2/4}}\right]$$

$$T^* = \left[1 - \frac{\int_0^{z/\sqrt{\alpha t}} d\eta'\, e^{-\eta'^2/4}}{\int_0^{\infty} d\eta'\, e^{-\eta'^2/4}}\right]$$

$\sqrt{\pi}$

$t \ll (H^2/\alpha)$

$$\frac{z}{\sqrt{\alpha t}} \sim 1$$

Penetration depth $\sim \sqrt{\alpha t}$

$\sqrt{\alpha t} \ll H$

$$\boxed{t \ll (H^2/\alpha)}$$

Heat flux

$$q_z = -k\frac{\partial T}{\partial z} = -k(T_1-T_0)\frac{\partial T^*}{\partial z}$$

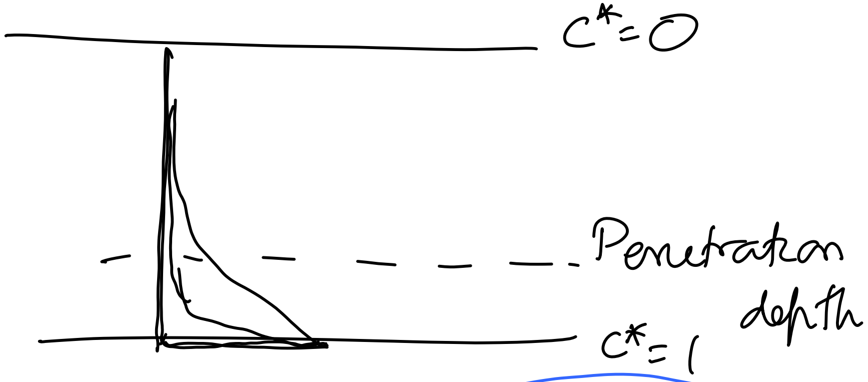
$$= -k(T_1-T_0)\frac{\partial \zeta_1}{\partial z}\frac{\partial T^*}{\partial \zeta_1} = -\frac{k(T_1-T_0)}{\sqrt{\alpha t}}\frac{\partial T^*}{\partial \zeta_1}$$

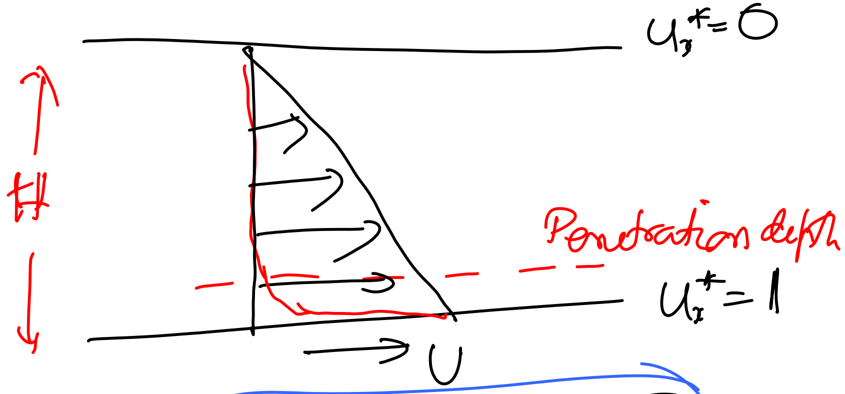
Heat flux at $z=0$ $(\zeta_1=0)$

$$q_z\Big|_{z=0} = \frac{-k(T_1-T_0)}{\sqrt{\alpha t}}\frac{\partial T}{\partial \zeta_1}\Big|_{\zeta_1=0}$$

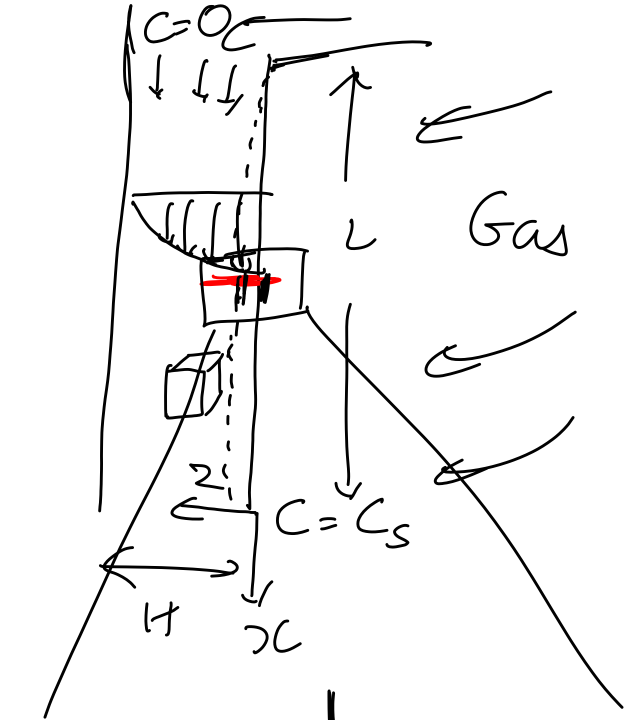
$$= -\frac{k(T_1-T_0)}{\sqrt{\alpha t}}\left(\frac{-1}{\int_0^\infty d\zeta_1' e^{-\zeta_1'^2/4}}\right)$$

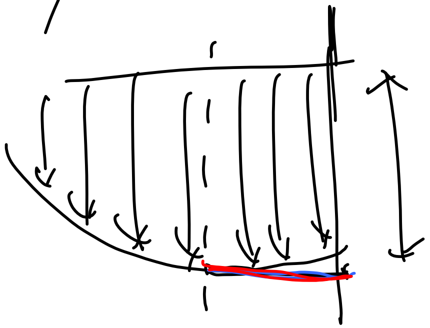
$$= \frac{k(T_0-T_\infty)}{\sqrt{\alpha t}\int_0^\infty d\zeta_1' e^{-\zeta_1'^2/4}}$$



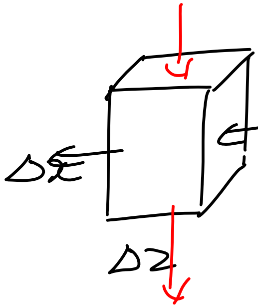

$$c^* = \left[ 1 - \frac{\int_0^{z/\sqrt{Dt}} d\eta' \, e^{-\eta'^2/4}}{\int_0^{\infty} d\eta' \, e^{-\eta'^2/4}} \right] \qquad u_x^* = \left[ 1 - \frac{\int_0^{z/\sqrt{\nu t}} d\eta' \, e^{-\eta'^2/4}}{\int_0^{\infty} d\eta' \, e^{-\eta'^2/4}} \right]$$



$U = \text{constant}$

1. Penetration depth $\ll H$
2. Velocity is a constant
3. Diffusion in $x$-direction is not important.



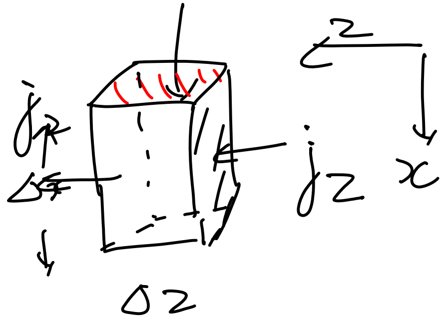

$$C^* = C/C_s$$

Boundary conditions

$C^* = 1$ at $z = 0$

$C^* = 0$ as $z \to \infty$

$C^* = 0$ at $x = 0$ for $z > 0$




$$\left(j_z\big|_z - j_z\big|_{z+\Delta z}\right)\Delta x\,\Delta y\,\Delta t + \left((Uc)\big|_x - (Uc)\big|_{x+\Delta x}\right)\Delta y\,\Delta z\,\Delta t = 0$$

$$(\text{Mass in}) - (\text{Mass out}) = 0$$

$$\begin{pmatrix}\text{Mass in due to}\\ \text{diffusion at } (x,y,z)\end{pmatrix} = j_z\big|_{(x,y,z)}\Delta x\,\Delta y\,\Delta t$$

$$\begin{pmatrix}\text{Mass out due to}\\ \text{diffusion at } x,y,z+\Delta z\end{pmatrix} = j_z\big|_{z+\Delta z}\Delta x\,\Delta y\,\Delta t$$

$$\begin{pmatrix}\text{Mass in due to}\\ \text{convection at } x,y,z\end{pmatrix} = (Uc)\big|_x\Delta y\,\Delta z\,\Delta t$$

$$\begin{pmatrix}\text{Mass out due to}\\ \text{convection at } x+\Delta x\end{pmatrix} = (Uc)\big|_{x+\Delta x}\Delta y\,\Delta z\,\Delta t$$

$$\frac{\left(j_z|_z - j_z|_{z+\Delta z}\right)}{\Delta z} + \frac{\left((Uc)|_x - (Uc)|_{x+\Delta x}\right)}{\Delta x} = 0$$

$$-\frac{\partial j_z}{\partial z} - \frac{\partial}{\partial x}(Uc) = 0$$

$$\frac{\partial}{\partial x}(Uc) = -\frac{\partial j_z}{\partial z}$$

$$U\frac{\partial c}{\partial x} = \frac{-\partial j_z}{\partial z}$$

$$j_z = -D\frac{\partial c}{\partial z}$$

$$U\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial z^2}$$

$$\boxed{U\frac{\partial c^*}{\partial x} = D\frac{\partial^2 c^*}{\partial z^2}}$$

B.C. $c^* = 1$ at $z = 0$

$c^* = 0$ as $z \to \infty$

$c^* = 0$ at $x = 0$

$$\xi = \frac{z}{\sqrt{Dx/U}}$$

$$C^* = \left[ 1 - \frac{\int_0^{2/\sqrt{xD/U}} d\alpha_1' \, e^{-\alpha_1'^2/4}}{\int_0^{\infty} d\alpha_1' \, e^{-\alpha_1'^2/4}} \right]$$

① Penetration depth $\ll H$

$\mathrm{Pe}_H \gg 1$

$$\sqrt{\frac{xD}{U}} \ll H$$

$$\frac{xD}{U} \ll H^2$$

$$\left(\frac{x}{H}\right) \ll \left(\frac{UH}{D}\right) \ll \mathrm{Pe}_H$$

$$\left(\frac{L}{H}\right) \ll \mathrm{Pe}_H$$

② Velocity is nearly constant

$$U(z) = U(z=0) + \cancel{z \frac{dU}{dz}\Big|_{z=0}} + \frac{z^2}{2}\frac{d^2U}{dz^2}\Big|_{z=0} + \cdots$$

$$U(z) - U(0) = \frac{z^2}{2}\frac{d^2U}{dz^2}\Big|_{z=0}$$

$$\frac{U(z) - U(0)}{U(0)} = \frac{z^2}{2U}\frac{d^2U}{dz^2}\Big|_{z=0} \ll 1$$

$$\frac{z^2}{2U}\frac{d^2U}{dz^2} \ll 1$$

$$\frac{\left(\sqrt{\frac{Dx}{U}}\right)^2}{2U}\left(\frac{U}{H^2}\right) \ll 1$$

$$\frac{Dx}{U} \ll H^2 \Longrightarrow \frac{x}{H} \ll \frac{UH}{D}$$

$$\ll Pe_H$$

Convective flux $\sim Uc$

Diffusive flux $\sim D\dfrac{\partial c}{\partial x} \simeq \dfrac{Dc}{x}$

$$\frac{Dc}{x} \ll Uc$$

$$\frac{Ux}{D} \gg 1 \Longrightarrow Pe_x \gg 1$$

---

Flux at interface:

$$j_z\Big|_{z=0} = -D\frac{\partial c}{\partial z}\Big|_{z=0} = -DC_s\frac{\partial c^*}{\partial z}\Big|_{z=0}$$

$$= -DC_s\frac{\partial \xi}{\partial z}\frac{\partial c^*}{\partial \xi}\Big|_{z=0}$$

$$= -DC_s\frac{1}{\sqrt{Dx/U}}\frac{\partial c^*}{\partial \xi}\Big|_{\xi=0}$$

$$c^* = \left[1 - \frac{\int_0^{\xi} d\xi'\, e^{-\xi'^2/4}}{\int_0^{\infty} d\xi'\, e^{-\xi'^2/4}}\right] \Longrightarrow \left[\frac{\partial c^*}{\partial \xi}\Big|_{\xi=0} = \frac{-1}{\int_0^{\infty} d\xi'\, e^{-\xi'^2/4}}\right]$$

$$j_2 = \frac{-D}{\sqrt{xD/U}} c_s \left( - \frac{1}{\int_0^\infty d\eta_1' \, e^{-\eta_1'^2/4}} \right)$$

$$= \frac{D c_s}{\sqrt{xD/U}} \frac{1}{\int_0^\infty d\eta_1' \, e^{-\eta_1'^2/4}}$$

$$= \left(\sqrt{\frac{UD}{x}}\right) c_s \frac{1}{\int_0^\infty d\eta_1' \, e^{-\eta_1'^2/4}}$$

$$\bar{j}_2 = \frac{1}{L} \int_0^L dx \; j_2(x)$$

$$= \frac{c_s \sqrt{UD}}{\int_0^\infty d\eta_1' \, e^{-\eta_1'^2/4}} \frac{1}{L} \int_0^L \frac{dx}{x^{1/2}} = \frac{c_s \sqrt{UD}}{\int_0^\infty d\eta_1' \, e^{-\eta_1'^2/4}} \left[\frac{2}{L^{1/2}}\right]$$

$$= \frac{2 c_s \sqrt{UD}}{L^{1/2} \int_0^\infty d\eta_1' \, e^{-\eta_1'^2/4}}$$

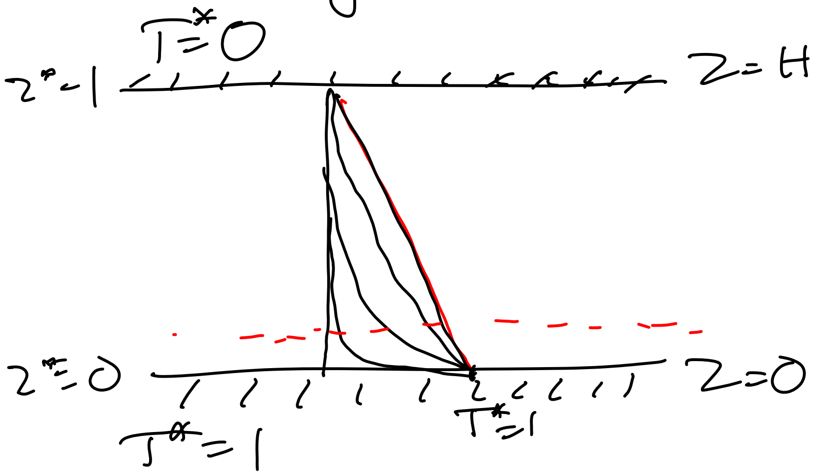
$$Nu = \frac{\bar{j}_z}{(DC_s/L)}$$

$$= \frac{2}{\int_0^\infty d\xi_1' \, e^{-\xi_1'^2/4}} \left(\frac{UL}{D}\right)^{1/2}$$

$$Sh = \frac{2}{\int_0^\infty d\xi_1' \, e^{-\xi_1'^2/4}} Pe_L^{1/2} = \overset{1.12}{\frac{2}{\int_0^\infty d\xi_1' \, e^{-\xi_1'^2/4}}} (Re\,Sc)^{1/2}$$

$$Nu = \frac{2}{\int_0^\infty d\xi_1' \, e^{-\xi_1'^2/4}} (Re\,Pr)^{1/2}$$

# Unsteady diffusion in a finite channel




$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$$T^* = \frac{T-T_0}{T_1-T_0} \qquad z^* = (z/H) \qquad t^* = \left(\frac{t\alpha}{H^2}\right)$$

$$\frac{\partial T^*}{\partial t} = \frac{\alpha}{H^2}\frac{\partial^2 T^*}{\partial z^{*2}}$$

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial z^{*2}}$$

At $z^*=0$, $T^*=1$

At $z^*=1$, $T^*=0$

At $t=0$, $T^*=0$ $z^*>0$

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial z^{*2}}$$

In the limit $t^* \to \infty$,

$$\frac{\partial^2 T^*}{\partial z^{*2}} = 0 \Rightarrow T_s^* = (1-z^*)$$

$$T^* = T_s^* + T_t^*$$

$$\frac{\partial^2 T_s^*}{\partial z^{*}} = 0 \qquad T_s^* = 1 \text{ at } z^* = 0$$

$$T_s^* = 0 \text{ at } z^* = 1$$

$$\frac{\partial(\cancel{T_s^*} + T_t^*)}{\partial t} = \frac{\partial^2(\cancel{T_s^*} + T_t)}{\partial z^{*2}} \qquad T_s^* + T_t^* = 1 \text{ at } z^* = 0$$

$$T_s^* + T_t^* = 0 \text{ at } z^* = 1$$

$$\frac{\partial T_t^*}{\partial t^*} = \frac{\partial^2 T_t^*}{\partial z^{*2}} \qquad T_t^* = 0 \text{ at } z^* = 0$$

$$T_t^* = 0 \text{ at } z^* = 1$$

At $t^* = 0$, $T^* = 0$ at all $z^* > 0$

$$T_t^* + T_s^* = 0 \text{ at all } z^* > 0$$

Initial condition $T_t^* = -T_s^*$ at $t^* = 0$ for all $z^*$

$$T_t^* = -(1 - z^*)$$

'Homogeneous boundary conditions'

Separation of variables:

$$T_t^*(z^*, t^*) = Z(z^*)\,\Theta(t^*)$$

$$\frac{\partial}{\partial t^*}(Z\Theta) = \frac{\partial^2}{\partial z^{*2}}(Z\Theta)$$

$$Z\frac{d\Theta}{dt^*} = \Theta\frac{d^2 Z}{dz^{*2}}$$

Divide by $Z\Theta$

$$\frac{1}{\Theta}\frac{d\Theta}{dt} = \frac{1}{Z}\frac{d^2 Z}{dz^{*2}}$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^{*2}} = \alpha$$

$$\frac{d^2 Z}{dz^{*2}} = \alpha Z$$

$$Z = A e^{\sqrt{\alpha} z^*} + B e^{-\sqrt{\alpha} z^*}$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^{*2}} = -\beta^2$$

$$\frac{d^2 Z}{dz^{*2}} = -\beta^2 Z$$

$$Z = A\sin(\beta z^*) + B\cos(\beta z^*)$$

BC $Z=0$ at $z^*=0$

$Z=0$ at $z^*=1$

$$A+B=0$$

$$Ae^{\sqrt{\alpha}}+Be^{-\sqrt{\alpha}}=0$$

$$A=0 \ \& \ B=0$$

BC $Z=0$ at $z^*=0$

$Z=0$ at $z^*=1$

$$B=0$$

$$A\sin(\beta z^*)=0$$

$$\beta_n = n\pi$$

where $n$ is integer

$$Z = A\sin(\beta_n z^*) = A\sin(n\pi z^*)$$

$$\frac{1}{\Theta}\frac{\partial\Theta}{\partial t^*} = -\beta^2 = -(n\pi)^2$$

$$\frac{\partial\Theta}{\partial t^*} = -(n\pi)^2\,\Theta$$

$$\Theta = e^{-(n\pi)^2 t^*}$$

$$T_t^* = \sum_{n=0}^{\infty} A_n \sin(n\pi z^*)\, e^{-(n^2\pi^2 t^*)}$$

'Orthogonality conditions'

$$\int_0^1 dz^* \sin(n\pi z^*) \sin(m\pi z^*) = \frac{1}{2} \text{ if } m=n$$
$$= 0 \text{ if } m \neq n$$
$$= \frac{\delta_{mn}}{2}$$

Initial condition:

$$T_t^* = -(1-z^*) \text{ at } t^*=0$$
$$= -T_s$$

$$\text{At } t^*=0 \, ; \, T_t^* = \sum_{n=0}^{\infty} A_n \sin(n\pi z^*) = -(1-z^*)$$

$$\sum_{n=0}^{\infty} A_n \int dz^* \sin(n\pi z^*) \sin(m\pi z^*) = -\int dz^* (1-z^*) \sin(m\pi z^*)$$

$$\sum_{n=0}^{\infty} A_n \frac{1}{2} \delta_{mn} = -\int dz^* (1-z^*) \sin(m\pi z^*)$$

$$\frac{1}{2} A_m = -\int dz^* (1-z^*) \sin(m\pi z^*)$$

$$A_m = -2 \int dz^* (1-z^*) \sin(m\pi z^*)$$
$$= -\frac{2}{m\pi} \text{ for odd } m$$
$$= 0 \text{ for even } m$$

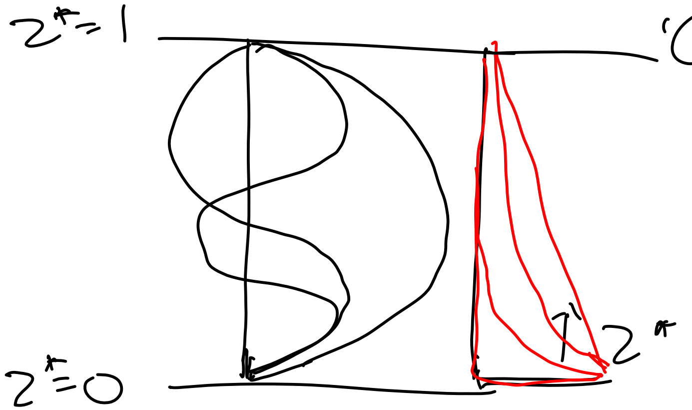
$$T_t^* = -\sum_{n=1,3,5..}^{\infty} \frac{2}{n\pi} \sin(n\pi z^*) e^{-n^2\pi^2 t^*}$$

$$T^* = (1 - z^*) - \sum_{n=1,3,5..}^{\infty} \frac{2}{n\pi} \sin(n\pi z^*) e^{-n^2\pi^2 t^*}$$

$$S_n = \sin(n\pi z^*) \qquad S_m = \sin(m\pi z^*)$$

$$\langle S_n, S_m \rangle = \int_0^1 dz^* \, S_n(z^*) S_m(z^*)$$

$$= \frac{\delta_{mn}}{2}$$




'Complete basis set'

$$T_t^* = \sum A_n S_n e^{-(n\pi)^2 t^*}$$

At time $t^* = 0$, $T_t^* = -(1-z^*)$

$$\sum_{n=0}^{\infty} A_n S_n = -(1-z^*)$$

$$\left\langle \sum_{n=0}^{\infty} A_n S_n , S_m \right\rangle = -\langle (1-z^*), S_m \rangle$$

$$\sum_{n=0}^{\infty} A_n \langle S_n , S_m \rangle = -\langle (1-z^*), S_m \rangle$$

$$\sum_{n=0}^{\infty} A_n \frac{\delta_{mn}}{2} = -\langle (1-z^*), S_m \rangle$$

$$\frac{A_m}{2} = -\langle (1-z^*), S_m \rangle$$

$\beta_n = n\pi \leftarrow$ Eigenvalues

$$T^*_{approx} \cong \sum_{n=1,3,\ldots}^{p} \frac{2}{n\pi} \sin(n\pi z^*)\, e^{-n^2\pi^2 t^*}$$

$$\text{Error} = T^*_t - T^*_{approx}$$

$$= \sum_{p+1}^{\infty} \frac{2}{n\pi} \sin(n\pi z^*)\, e^{-n^2\pi^2 t^*}$$

$$\leq \sum_{p+1}^{\infty} \frac{2}{n\pi}\, e^{-n^2\pi^2 t^*}$$

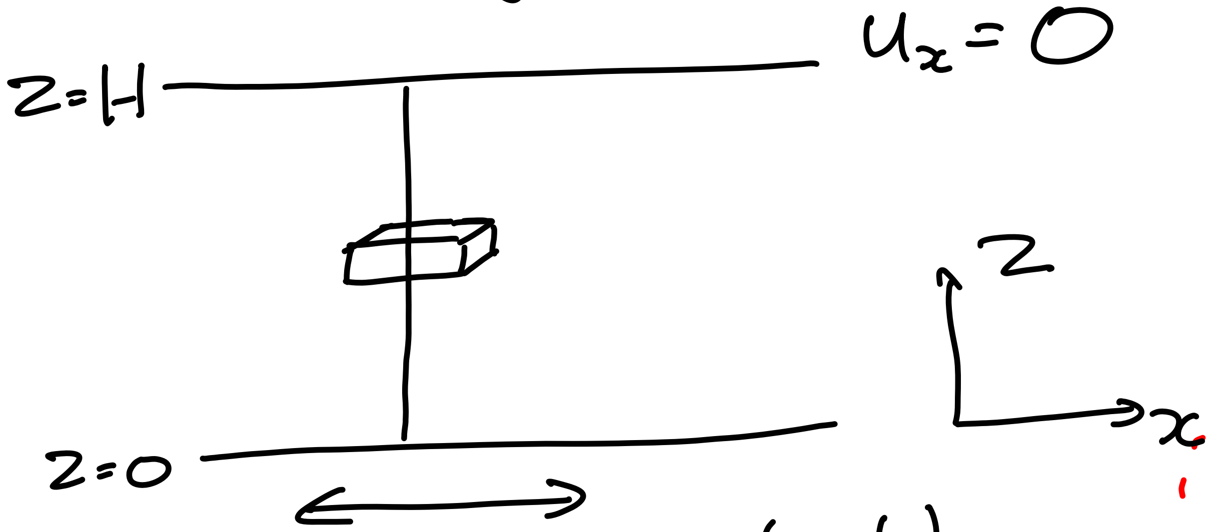
$$\leq = \int_{p+1}^{\infty} dn \left(\frac{2}{n\pi}\right) e^{-n^2\pi^2 t^*}$$

$$\text{Define } n^* = (n^2\pi^2 t^*)^{1/2} \Longrightarrow n = n^*/\pi t^{*1/2}$$

$$\text{Error} \leq \int_{p^*}^{\infty} dn^* \left(\frac{2}{n'}\right) e^{-n^{*2}}$$

$$\text{where } p^* = p\pi t^{*1/2}$$

# Oscillatory flow:




$$u_x = U\cos(\omega t)$$

$$z^* = (z/H)$$

$$u_x^* = (u_x/U)$$

$$t^* = \omega t$$

$$U\omega \frac{\partial u_x^*}{\partial t^*} = \frac{\nu U}{H^2} \frac{\partial^2 u_x^*}{\partial z^{*2}}$$

$$\left(\frac{\omega H^2}{\nu}\right) \frac{\partial u_x^*}{\partial t^*} = \frac{\partial^2 u_x^*}{\partial z^{*2}}$$

$Re_\omega$

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial z^2}$$

At $z = H$, $u_x = 0$

$z = 0$, $u_x = U\cos(\omega t)$

$$Re_\omega \frac{\partial u_x^*}{\partial t^*} = \frac{\partial^2 u_x^*}{\partial z^{*2}}$$

At $z^* = 1 \quad u_x^* = 0$

$z^* = 0 \quad u_x^* = \cos t^*$

$u_x^* = Re(u_x^+)$

$$Re_\omega \frac{\partial u_x^+}{\partial t^*} = \frac{\partial^2 u_x^+}{\partial z^{*2}}$$

At $z^* = 1, \quad u_x^+ = 0$

$z^* = 0, \quad u_x^+ = e^{it^*}$

$$Re_\omega \frac{\partial u_x^+}{\partial t^*} = \frac{\partial^2 u_x^+}{\partial z^{*2}}$$

At $z^* = 1, \quad u_x^+ = 0$

$z^* = 0, \quad u_x^+ = e^{it^*}$

$$u_x^+(z^*, t^*) = e^{it^*} \tilde{u}_x(z^*)$$

$$Re_\omega \tilde{u}_x(z)\, i \cancel{e^{it^*}} = \frac{\partial^2 \tilde{u}_x}{\partial z^{*2}} \cancel{e^{it^*}}$$

$$\frac{\partial^2 \tilde{u}_x}{\partial z^{*2}} = i Re_\omega \tilde{u}_x$$

At $z^* = 1$, $u_x^+ = 0 \Longrightarrow \tilde{u}_x = 0$

At $z^* = 0$, $u_x^+ = e^{it^*} \Longrightarrow \tilde{u}_x = 1$

$$u_x^+ = \tilde{u}_x(z)\, e^{it^*} ; \quad u_x^* = \mathrm{Real}(u_x^+)$$

$$\tilde{u}_x = A_1 e^{\sqrt{iRe_\omega}\, z^*} + A_2 e^{-\sqrt{iRe_\omega}\, z^*}$$

$$\tilde{u}_x = \left[ \frac{e^{\sqrt{iRe_\omega}\, z^*} - e^{\sqrt{iRe_\omega}\,(2 - z^*)}}{1 - e^{2\sqrt{iRe_\omega}}} \right]$$

$$u_x^+ = \left[ \frac{e^{\sqrt{iRe_\omega}\, z^*} - e^{\sqrt{iRe_\omega}\,(2 - z^*)}}{1 - e^{2\sqrt{iRe_\omega}}} \right] e^{it^*}$$

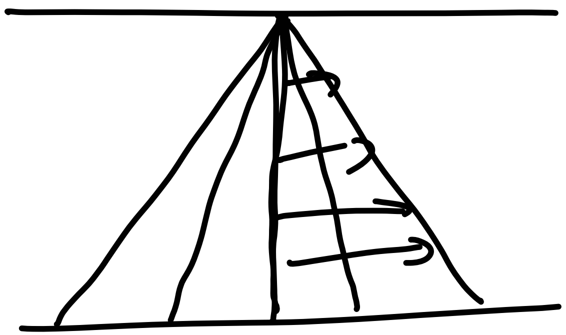
$$u_x^* = \mathrm{Real}(u_x^+)$$

Limit $Re_\omega \ll 1$

$$\tilde{u}_x = (1-z^*) \qquad u_x^{+} = (1-z^*)e^{it^*}$$

$$u_x^* = \boxed{(1-z^*)}\cos(t^*)$$

$$Re_\omega = \left(\frac{\omega H^2}{\nu}\right) = \left(\frac{H^2/\nu}{1/\omega}\right)$$



$$Re_\omega \gg 1$$

$$\tilde{u}_x(z^*) = e^{-\sqrt{i Re_\omega}\, z^*}$$

$$u_x^+(z^*) = e^{-\sqrt{i Re_\omega}\, z^*} e^{i t^*}$$

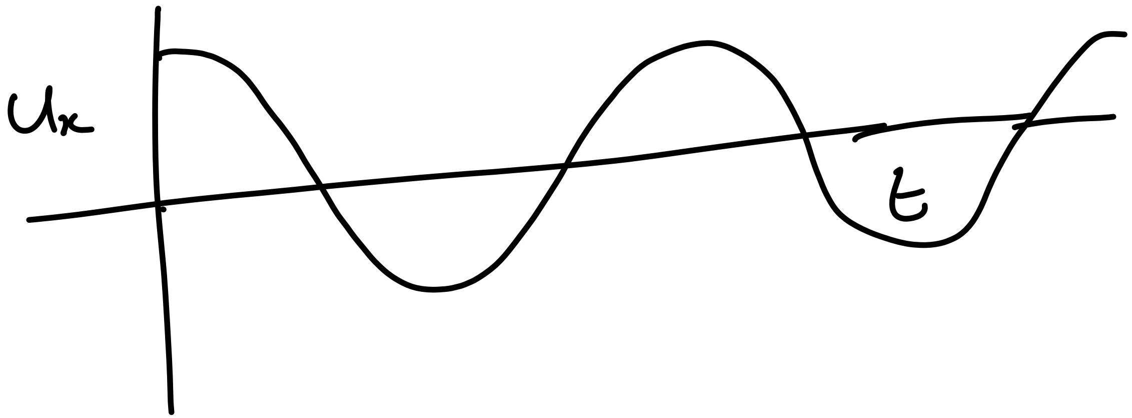
$$u_x^*(z^*) = e^{-\left(\sqrt{\frac{Re_\omega}{2}}\, z^*\right)} \left[ \cos\left(\sqrt{\frac{Re_\omega}{2}}\, z^*\right) \cos t^* - \sin\left(\sqrt{\frac{Re_\omega}{2}}\, z^*\right) \sin t^* \right]$$

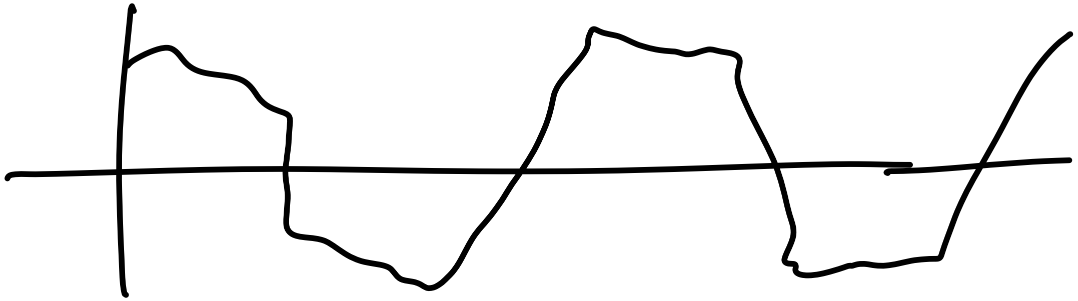
$$\sqrt{Re_\omega}\, z^* = \sqrt{\frac{\omega H^2}{\nu}} \left(\frac{z}{H}\right) = \left[\frac{z}{\sqrt{\nu/\omega}}\right]$$

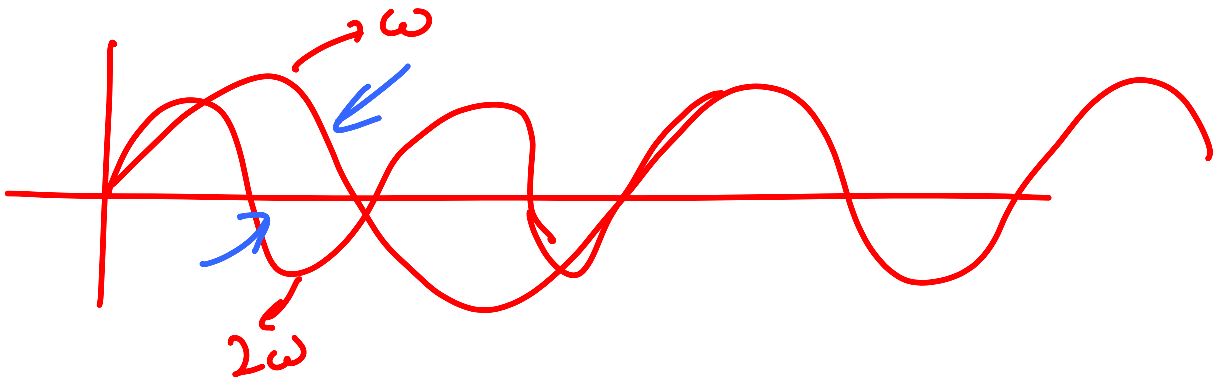
Penetration depth $= (\nu/\omega)^{1/2}$

$$Re_\omega = \frac{\omega H^2}{\nu} = \left[\frac{H}{(\nu/\omega)^{1/2}}\right]^2$$

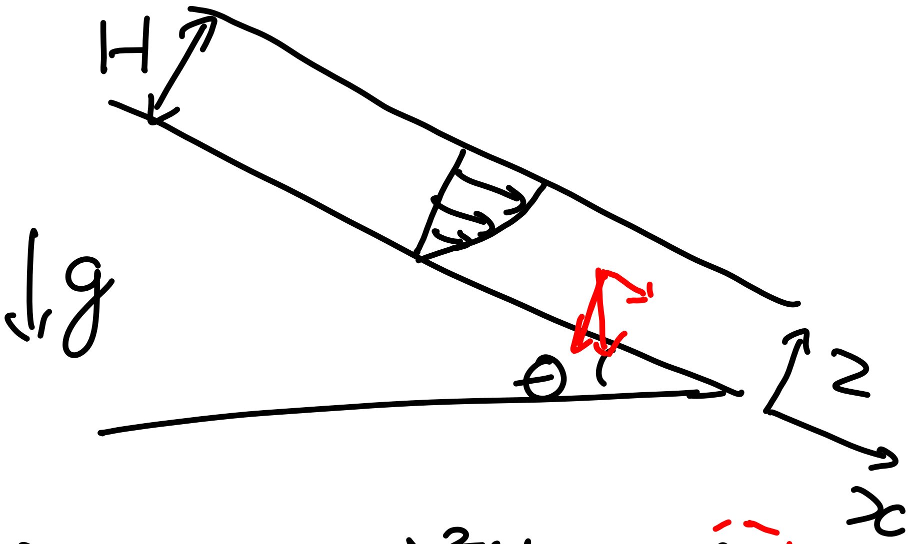
# Oscillatory flows:





Sources | Sinks within the field:




$$f_x = (g\sin\theta)\rho$$

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial z^2} + g\sin\theta$$

$$z^* = (z/H)$$

$$\frac{1}{g\sin\theta}\frac{\partial u_x}{\partial t} = \frac{\nu}{H^2 g\sin\theta}\frac{\partial^2 u_x}{\partial z^{*2}} + 1$$

$$u_x^* = \left(\frac{u_x \nu}{H^2 g\sin\theta}\right)$$

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial z^2} + \frac{f_x}{\rho}$$

At $z = 0$, $u_x = 0$

At $z = H$; $\tau_{xz} = 0$

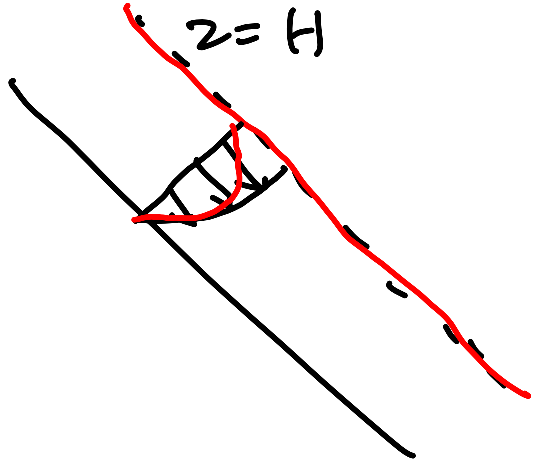
$$\mu \frac{\partial u_x}{\partial z} = 0$$

$$\frac{\partial u_x}{\partial z} = 0$$

$$\frac{\partial u_x^*}{\partial t^*} = \frac{\partial^2 u_x^*}{\partial z^{*2}} + 1$$

where $z^* = (z/H)$, $u_x^* = \left(\frac{u_x N}{H^2 g \sin\theta}\right)$, $t^* = \left(\frac{tN}{H^2}\right)$

Boundary conditions



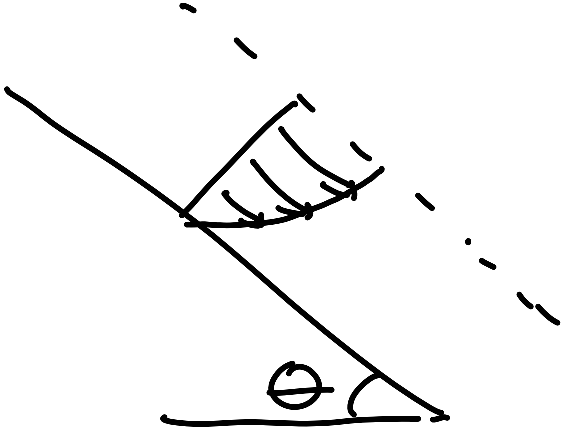

$u_x^* = 0$ at $z^* = 0$

$\frac{du_x^*}{dz^*} = 0$ at $z^* = 1$

Steady solution:

$$\frac{\partial^2 u_x^*}{\partial z^{*2}} + 1 = 0 \qquad u_x^* = z^* - \frac{z^{*2}}{2}$$

$$u_x = \frac{u_x^*(H^2 g \sin\theta)}{N} = \frac{g \sin\theta}{N}\left(zH - \frac{z^2}{2}\right)$$




$$\frac{\partial u_x^*}{\partial t^*} = \frac{\partial^2 u_x^*}{\partial z^{*2}} + 1$$

B.C. $u_x^* = 0$ at $z^* = 0$

$$\frac{du_x^*}{dz} = 0 \text{ at } z^* = 1$$

Initial condition

At $t^* = 0$, $u_x^* = 0$ for all $z^*$

$$u_x^* = u_{xs}^* + u_{xt}^*$$

$$\frac{\partial^2 u_{xs}^*}{\partial z^{*2}} + 1 = 0$$

$$u_{xs}^* = z^* - \frac{z^{*2}}{2}$$

$$\frac{\partial u_{xt}^*}{\partial t^*} = \frac{\partial^2 u_{xt}^*}{\partial z^{*2}}$$

BC $u_{xt}^* = 0$ at $z^* = 0$

$$\frac{du_{xt}^*}{dz^*} = 0 \text{ at } z^* = 1$$

IC $u_{xt}^* = -u_{xs}^*$ at $t^* = 0$

$$\frac{\partial u_{xt}^*}{\partial t^*} = \frac{\partial^2 u_{xt}^*}{\partial z^{*2}}$$

BC: $u_{xt}^* = 0$ at $z^* = 0$

$$\frac{du_{xt}^*}{dz^*} = 0 \text{ at } z^* = 1$$

IC: $u_{xt}^* = -u_{xs}^*$

$= -(z^* - z^{*2}/2)$

at $t^* = 0$

$$u_{x_t}^{*} = \Theta(t)\, Z(z^*)$$

$$Z(z^*) \frac{\partial \Theta}{\partial t} = \Theta \frac{\partial^2 Z}{\partial z^{*2}}$$

$$\frac{1}{\Theta} \frac{\partial \Theta}{\partial t} = \frac{1}{Z} \frac{\partial^2 Z}{\partial z^{*2}}$$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^{*2}} = -\beta_n^2$$

$$Z = A \sin(\beta_n z^*) + B \cos(\beta_n z^*)$$

At $z^* = 0$, $Z = 0 \Rightarrow B = 0$

At $z^* = 1$, $\frac{dZ}{dz^*} = 0$

$$\beta_n = \left(\pi/2\right), \left(3\pi/2\right), \left(5\pi/2\right) \ldots$$

$$= \frac{(2n+1)\pi}{2}$$

$$Z = A \sin\left(\frac{(2n+1)\pi z^*}{2}\right)$$

$$\frac{1}{\Theta}\frac{d\Theta}{dt^*} = -\beta_n^2 = -\left(\frac{(2n+1)\pi}{2}\right)^2$$

$$\Theta = e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t^*}$$

$$u_{xt}^* = \Theta Z$$

$$= \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi z^*}{2}\right) e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t^*}$$

$$S_n = \sin\left(\frac{(2n+1)\pi z^*}{2}\right)$$

$$\langle S_n, S_m \rangle = \int_0^1 dz^* \, S_n S_m = \frac{\delta_{mn}}{2}$$

At $t^* = 0$

$$u_{xt}^* = \sum A_n \sin\left(\frac{(2n+1)\pi z^*}{2}\right)$$

$$= \sum A_n S_n = -(z^* - z^{*2}/2)$$

$$\sum A_n \langle S_n, S_m \rangle = -\langle (z^* - z^{*2}/2), S_m \rangle$$

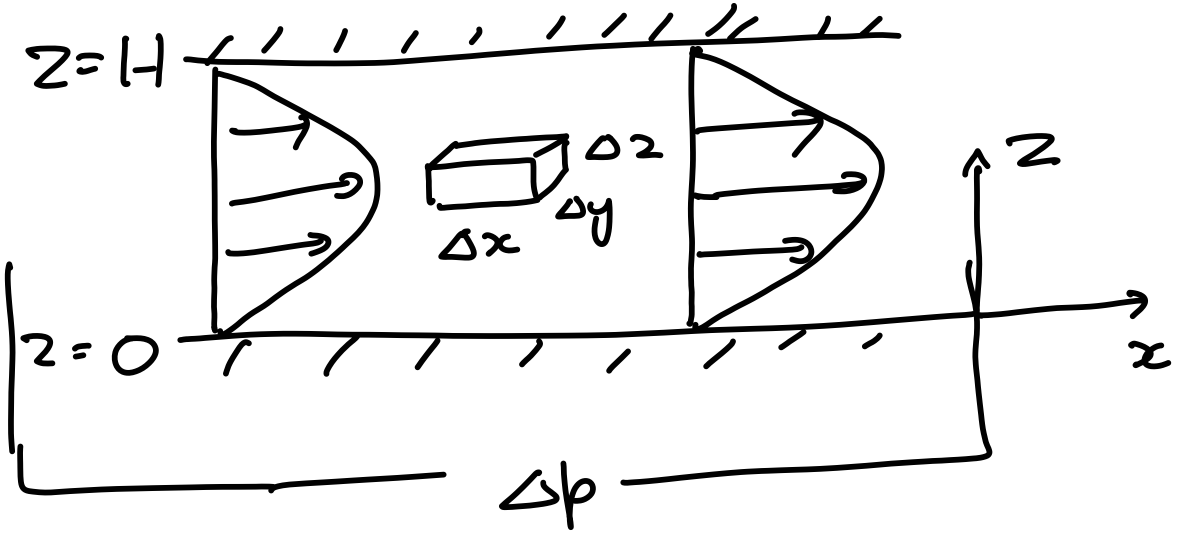
$$\sum A_n \frac{\delta_{mn}}{2} = -\int dz^* (z^* - z^{*2}/2) \sin\left(\frac{(2m+1)\pi z^*}{2}\right)$$

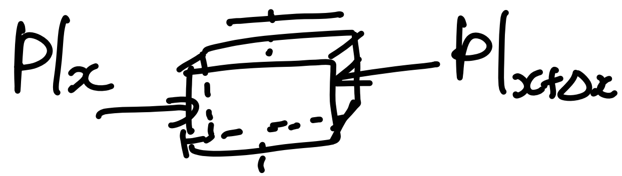
$$\frac{A_m}{2} = -\frac{1}{\pi^3 \left(\frac{2m+1}{2}\right)^3}$$

$$A_m = \frac{-2}{\pi^3 \left(\frac{2m+1}{2}\right)^3}$$

$$u_{xt}^* = \sum_{n=0}^{\infty} -\frac{2}{\pi^3 \left(\frac{2n+1}{2}\right)^3} \sin\left(\frac{(2n+1)\pi z^*}{2}\right) e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t^*}$$

# Pressure driven flow in a channel:




$$\tau_{xz} = \mu \frac{du_x}{dz}$$

: Force in $x$ direction at surface with normal in $z$ direction

$$\begin{pmatrix}\text{Rate of} \\ \text{change of} \\ \text{momentum}\end{pmatrix} = \begin{pmatrix}\text{Sum of} \\ \text{forces}\end{pmatrix}$$

$$\rho \frac{u_x(x,y,z,t+\Delta t) - u_x(x,y,z,t)}{\Delta t} \Delta x \Delta y \Delta z =$$

$$(\tau_{xz}|_{z+\Delta z}) \Delta x \Delta y - \tau_{xz}|_z \Delta x \Delta y$$

$$+ (p|_x \Delta y \Delta z) - p|_{x+\Delta x} (\Delta y \Delta z)$$

Divide by $\Delta x \Delta y \Delta z$

$$\rho\frac{\left(u_x|_{t+\Delta t}-u_x|_t\right)}{\Delta t}=\frac{p|_x-p|_{x+\Delta x}}{\Delta x}+\frac{\tau_{xz}|_{z+\Delta z}-\tau_{xz}|_z}{\Delta z}.$$

$$\rho\frac{\partial u_x}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}$$

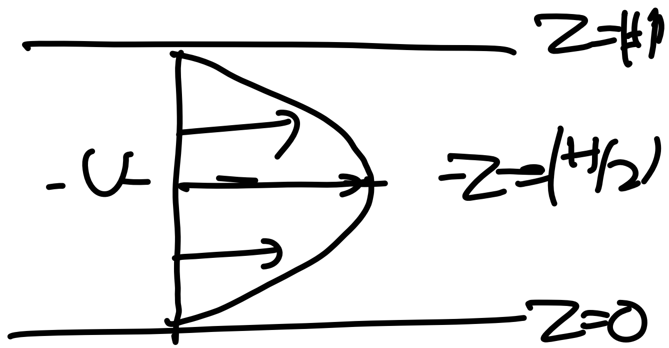
$$\tau_{xz}=\mu\frac{\partial u_x}{\partial z}$$

$$\rho\frac{\partial u_x}{\partial t}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u_x}{\partial z^2}$$

$$\frac{\partial u_x}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u_x}{\partial z^2}$$

At steady state,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial z^2} = 0$$




B.C:

$u_x^* = 0$ at $z = 0$

$u_x^* = 0$ at $z = H$

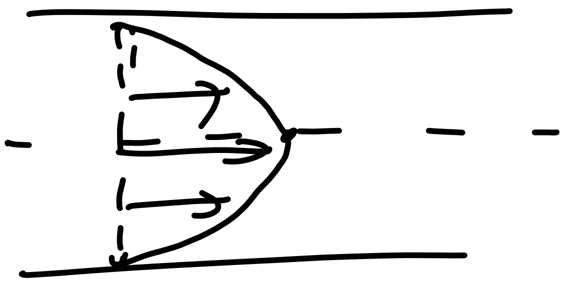
$z^* = (z/H)$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{H^2}\frac{\partial^2 u_x}{\partial z^{*2}} = 0$$

$$-1 + \left(\frac{\mu}{H^2}\right)\left(\frac{\partial p}{\partial x}\right)^{-1}\frac{\partial^2 u_x}{\partial z^{*2}} = 0$$

$$u_x^* = \left(\frac{\mu u_x}{H^2}\right)\left(\frac{\partial p}{\partial x}\right)^{-1} \qquad u_x \sim \left(\frac{\partial p}{\partial x}\right)\left(\frac{H^2}{\mu}\right)$$

$$\frac{\partial^2 u_x^*}{\partial z^{*2}} - 1 = 0$$



B.C.

$u_x^* = 0$ at $z^* = 0$

$u_x^* = 0$ at $z^* = 1$

$$u_x^* = \frac{z^{*2}}{2} + C_1 z^* + C_2$$

$$u_x^* = \left(\frac{z^{*2}}{2} - \frac{z^*}{2}\right)$$

$$u_x = \left(\frac{\partial p}{\partial x}\right)\left(\frac{H^2}{\mu}\right)\left(\frac{z^{*2}}{2} - \frac{z^*}{2}\right)$$

$$= -\frac{1}{2\mu}\left(\frac{dp}{dx}\right) z(z-H)$$

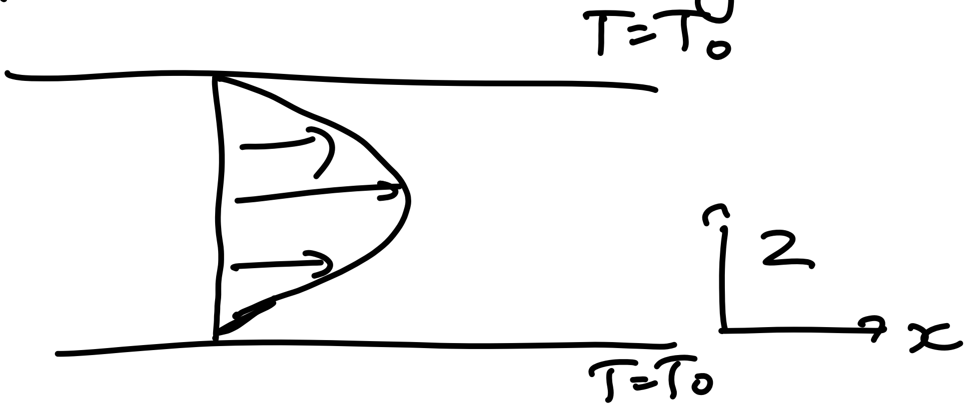
'Plane Poiseuille flow'

Maximum velocity at $z = H/2$

$$U_x = -\frac{1}{2\mu}\left(\frac{dp}{dx}\right)\frac{H^2}{4} = U$$

$$U_x = 4U\left(\frac{z}{H} - \left(\frac{z}{H}\right)^2\right)$$

Viscous heating in the channel




$$S_e = \tau_{xy}\frac{dU_x}{dy}$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} + \frac{S_e}{\rho C_p}$$

$T = T_0$ at $z = 0$

$= T_0$ at $z = H$

$z^* = (z/H)$

$T^* = \left(\frac{T - T_0}{T_0}\right)$

Steady state $\frac{\partial T}{\partial t} = 0$

$$k \frac{\partial^2 T}{\partial z^2} + S_e = 0$$

$$S_e = \tau_{xz}\left(\frac{du_x}{dz}\right) = \mu\left(\frac{du_x}{dz}\right)^2$$

$$u_x = 4U\left(\frac{z}{H} - \left(\frac{z}{H}\right)^2\right)$$

$$\Longrightarrow \frac{du_x}{dz} = \frac{4U}{H}\left[1 - \frac{2z}{H}\right]$$

$$S_e = \frac{16U^2}{H^2}\left[1-\frac{2z}{H}\right]^2 = \frac{16U^2}{H^2}(1-2z^*)^2$$

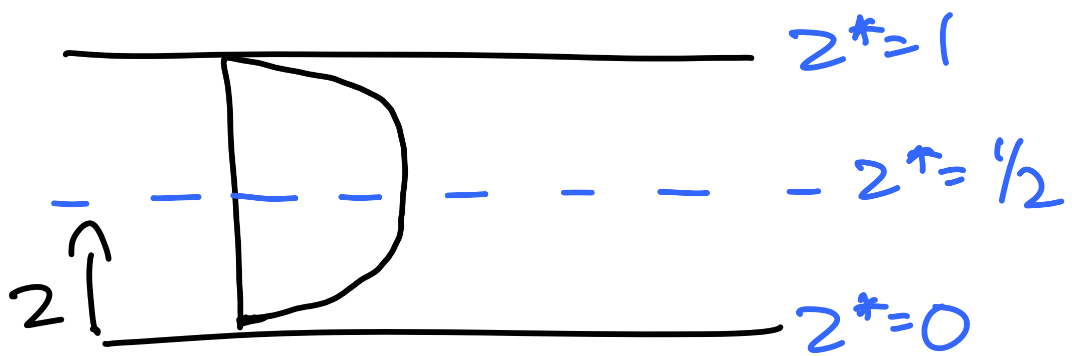
$$k\frac{\partial^2 T}{\partial z^2} + \frac{16U^2}{H^2}(1-2z^*)^2 = 0$$

$$\frac{kT_0}{H^2}\frac{d^2T^*}{dz^{*2}} + \frac{16\mu U^2}{H^2}(1-2z^*)^2 = 0$$

$$\frac{d^2T^*}{dz^{*2}} + 16\,Br\,(1-2z^*)^2 = 0$$

$$Br = \left(\frac{\mu U^2}{kT_0}\right)$$

$$\boxed{T^* = Br\left(\frac{8z^*(1-z^*)(1-2z^*+2z^{*2})}{3}\right)}$$




$$q_z = -k\frac{dT}{dz} = -\frac{kT_0}{H}\frac{dT^*}{dz^*}$$

$$= \frac{-8kT_0}{3H}\left(1-2z^*\right)^3 Br$$

$$= \frac{-8\mu U^2}{3H}\left(1-2z^*\right)^3$$

$$q_z = \frac{8\mu U^2}{3H} \ll$$ Flux due to temperature difference

$$q_z = \frac{k\,\Delta T}{\Delta z} \qquad \frac{\mu U^2}{3H} \ll \frac{k\,\Delta T}{h_w}$$

$$\Delta T \gg \left(\frac{\mu U^2 h_w}{H k}\right)$$

---

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial z^2} + S$$

$S = -kc$ (If c is concentration of a reactant)

$= +kc$ (if c is the concentration of a product)

Steady state $\quad D \dfrac{\partial^2 c}{\partial z^2} - kc = 0$

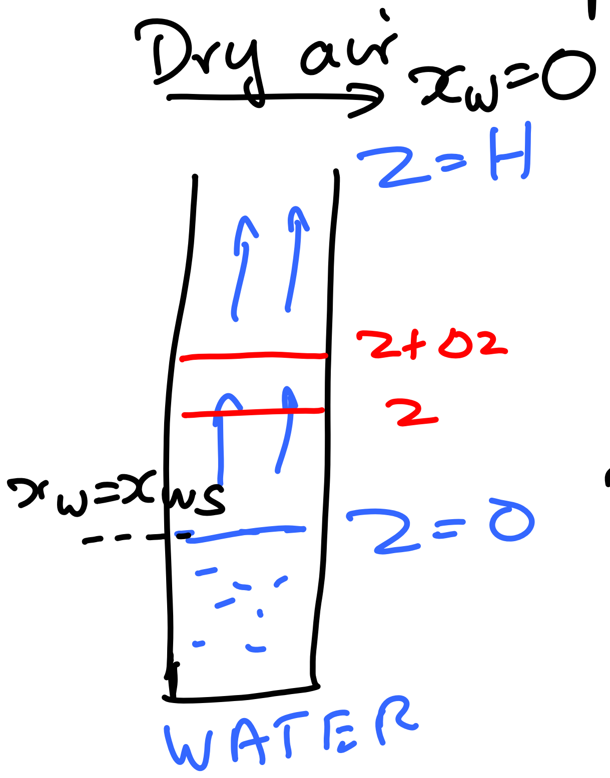
$$\frac{\partial^2 C}{\partial z^2} - \frac{k}{D} C = 0$$

$$C^* = (C/C_0) \; ; \; z^* = z(k/D)^{1/2}$$

Penetration depth

$$\frac{\partial^2 C^*}{\partial z^{*2}} - C^* = 0$$

# Multicomponent diffusion:




$$j_w = -D\frac{dc_w}{dz} + x_w(j_w + j_{air})$$

$$= -Dc\frac{dx_w}{dz} + x_w(j_w + \cancel{j_{air}})$$

Total mean flow

$$= (j_w + \cancel{j_{air}})$$

$$(1-x_w)\, j_w = -Dc\frac{dx_w}{dz}$$

$$j_w = \frac{-Dc}{1-x_w}\frac{dx_w}{dz}$$

At steady state, $j_w|_{z+\Delta z} - j_w|_z = 0$

$$\frac{d f_w}{dz} = 0$$

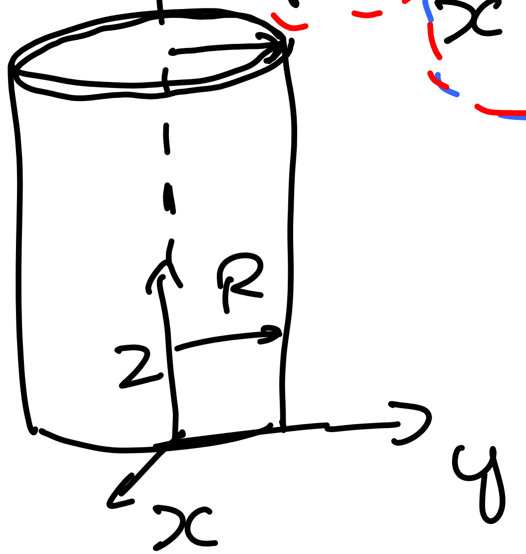
$$\frac{d}{dz}\left(\frac{1}{1-x_w}\frac{dx_w}{dz}\right) = 0$$

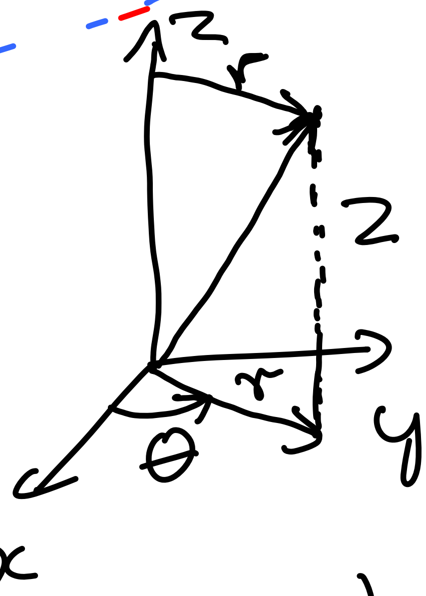
$$-\log(1-x_w) = A_1 z + A_2$$

$$\frac{(1-x_w)}{(1-x_{ws})} = \left(\frac{1}{1-x_{ws}}\right)^{z/H}$$

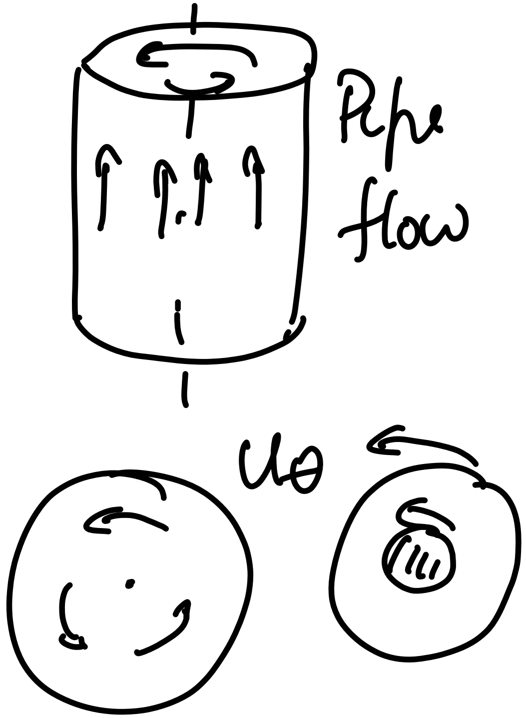
# Transport in cylindrical coordinates:

$r=R \qquad x^2+y^2=R^2$



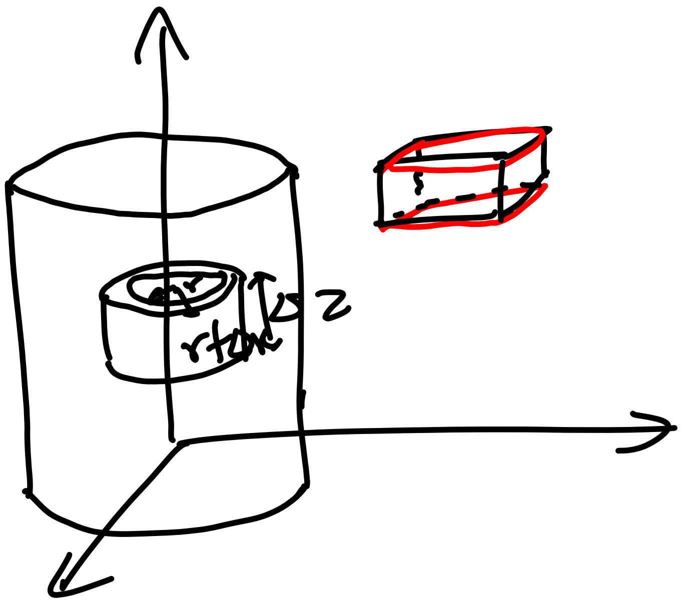

$(r, \theta, z)$

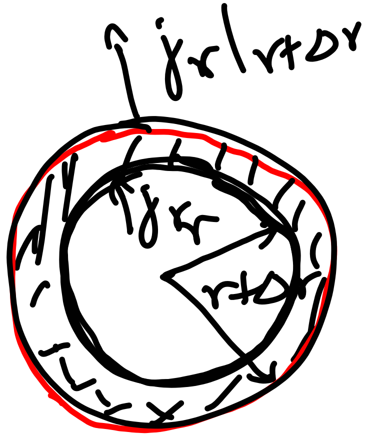



$$r = \sqrt{x^2+y^2}$$

$$z = z$$

$$\tan\theta = (y/x)$$




Mass balance:

$$\begin{pmatrix}\text{Accumulation of} \\ \text{mass in time } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Input of} \\ \text{mass}\end{pmatrix} - \begin{pmatrix}\text{Output} \\ \text{of mass}\end{pmatrix} + \text{Sources}$$

$$\begin{pmatrix}\text{Accumulation of} \\ \text{mass in time } \Delta t\end{pmatrix} = \left[c(r, z, t+\Delta t) - c(r, z, t)\right] 2\pi r \Delta r \Delta z$$

$$\begin{pmatrix}\text{Input of mass}\\ \text{at } r\end{pmatrix} = (j_r 2\pi r \Delta z)\big|_r \Delta t$$

$$\begin{pmatrix}\text{Output of}\\ \text{mass at } r+\Delta r\end{pmatrix} = (j_r 2\pi r \Delta z)\big|_{r+\Delta r} \Delta t$$

$$\begin{pmatrix}\text{Source of}\\ \text{mass}\end{pmatrix} = S(2\pi r \Delta r \Delta z)\Delta t$$

$$[c(r,z,t+\Delta t) - c(r,z,t)]\, 2\pi r \Delta r \Delta z$$
$$= (j_r 2\pi r \Delta z)\big|_r \Delta t - (j_r 2\pi r \Delta z)\big|_{r+\Delta r} \Delta t$$
$$+ S(2\pi r \Delta r \Delta z \Delta t)$$

Divide by $2\pi r \Delta r \Delta z \Delta t$

$$\frac{c(r,z,t+\Delta t)-c(r,z,t)}{\Delta t} =$$

$$\frac{1}{r}\frac{1}{\Delta r}\left[(r j_r)\big|_r - (r j_r)\big|_{r+\Delta r}\right] + S$$

$$\frac{\partial c}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(r j_r) + S$$

$$j_r = -D\frac{\partial c}{\partial r}$$

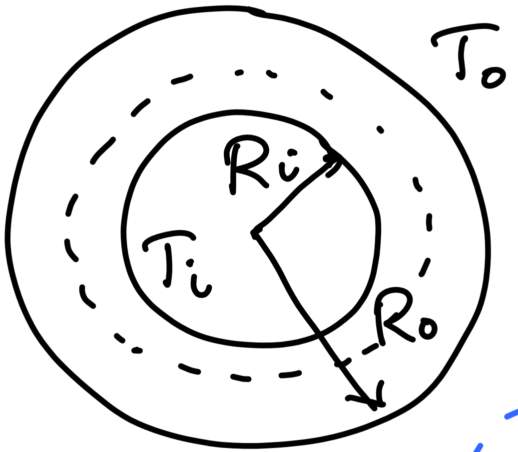
$$\frac{\partial c}{\partial t} = D\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)\right) + S$$

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right) + \frac{S_e}{\rho C_p}$$

$$\frac{\partial u_\theta}{\partial t} = \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)\right) + \frac{f_\theta}{\rho} - \nu\frac{u_\theta}{r^2}$$

---

Steady diffusion:




$$\alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right) = 0$$

$$r^* = (r/R_i)$$

$$T^* = \left(\frac{T - T_i}{T_o - T_i}\right)$$

At $r^* = 1$ ; $T^* = 0$

At $r^* = (R_o/R_i)$ ; $T^* = 1$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right)=0$$

$$r^*\frac{\partial T^*}{\partial r^*}=C_1$$

$$\frac{\partial T^*}{\partial r^*}=\frac{C_1}{r^*}\implies T^*=C_1\log(r^*)+C_2$$

$$T^*=\frac{\log(r^*)}{\log(R_o/R_i)}$$

$$\frac{T-T_i}{T_o-T_i}=\frac{\log(r/R_i)}{\log(R_o/R_i)}$$

$$q_r=-k\frac{\partial T}{\partial r}=-\frac{k(T_o-T_i)}{R_i}\frac{\partial T^*}{\partial r^*}$$

$$= \frac{-k(T_o - T_i)}{R_i\, r^* \log(R_o/R_i)} = \frac{-k(T_o - T_i)}{r \log(R_o/R_i)}$$

$$Q = (2\pi \cancel{r} L)\left[\frac{-k(T_o - T_i)}{\cancel{r} \log(R_o/R_i)}\right]$$

$$= \frac{-k(T_o - T_i)(2\pi L)}{\log(R_o/R_i)}$$

$$Q = \frac{-k(T_o - T_i)\, A_L}{(R_o - R_i)}$$

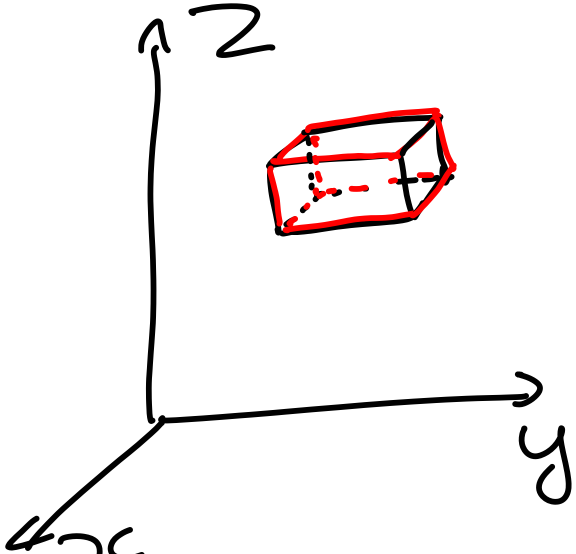
$$A_L = \frac{(2\pi L)(R_o - R_i)}{\log(R_o/R_i)} = 2\pi L\, r_L$$

$$r_L = \frac{R_o - R_i}{\log(R_o/R_i)}$$

Cylindrical co-ordinate system:

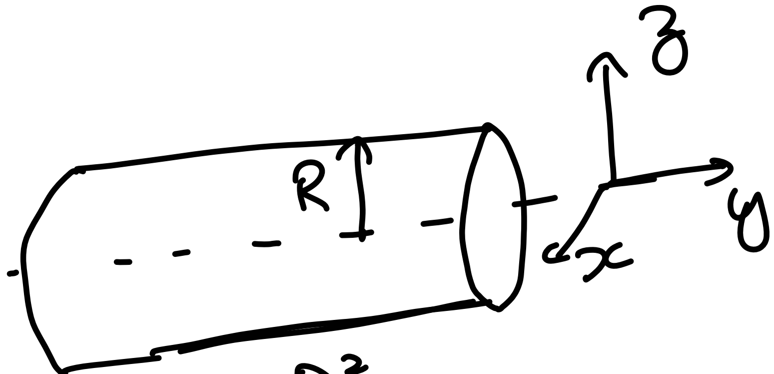
Curvilinear co-ordinate systems

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$



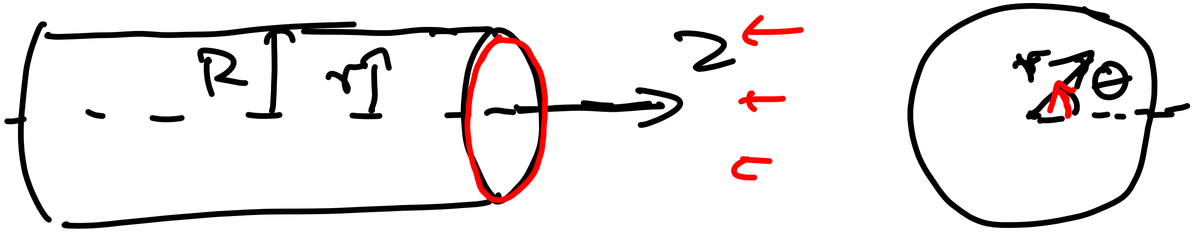

$T^* = 0$ at $z^* = 1$

$T^* = 1$ at $z^* = 0$



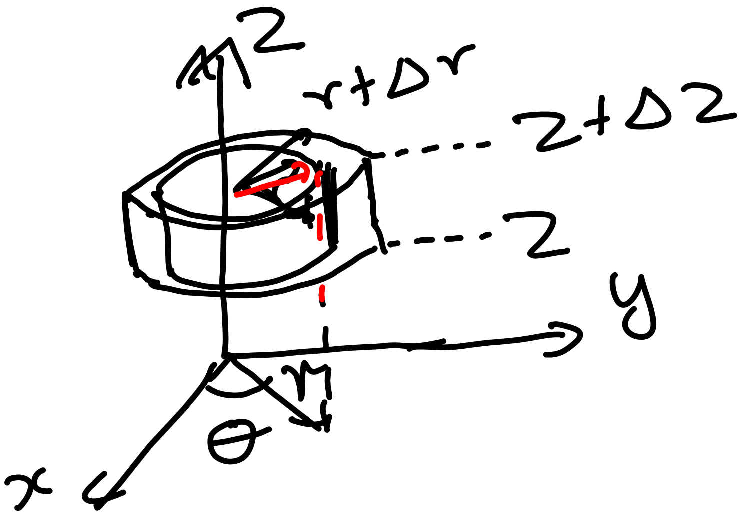

$x^2 + z^2 = R^2$

$T^* = 1$ at $x^2 + y^2 = R^2$




$r = R$

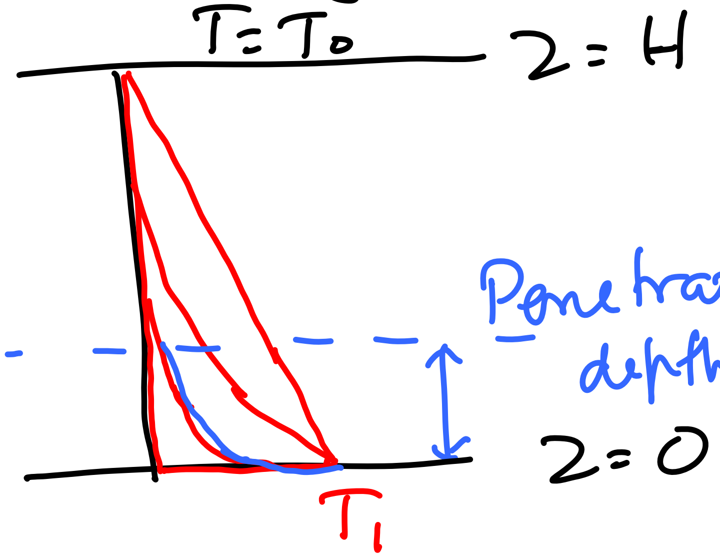
Surface of constant r




$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \tan\theta = \left(\frac{y}{x}\right)$

Unsteady diffusion in cylindrical co-ordinates:



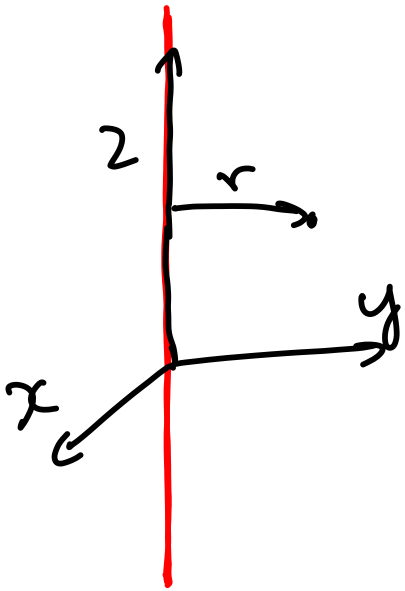

$\eta = \dfrac{z}{\sqrt{\alpha t}}$

At $t = 0$, $T = T_1$ at $z = 0$

$T = T_0$ for $z > 0$

$T = T_0$ as $z \to \infty$

$T = T_1$ at $z = 0$




$T = T_0$

as $r \to \infty$

At $t=0$, $T=T_0$ everywhere

B.C. $T = T_0$ as $r \to \infty$

$$\cancel{q_r = -k\frac{\partial T}{\partial r}}$$

At $r \to 0$

$$q_r(2\pi r L) = Q$$

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right)$$

$$T^* = \left(\frac{T - T_0}{T_0}\right)$$

$$\frac{\partial T^*}{\partial t} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^*}{\partial r}\right)\right)$$

Boundary conditions

$T^* = 0$ as $r \to \infty$ for all $t$

$(q_r\, 2\pi r L) = Q$ as $r \to 0$

Initial condition

$T^* = 0$ at $t = 0$ for $r > 0$

$$\frac{\partial T^*}{\partial t} = \alpha\left[\frac{\partial^2 T^*}{\partial r^2} + \frac{1}{r}\frac{\partial T^*}{\partial r}\right]$$

$$\xi_1 = \frac{r}{\sqrt{\alpha t}}$$

$$\frac{\partial T^*}{\partial t} = \left(\frac{\partial \xi_1}{\partial t}\right)\left(\frac{\partial T}{\partial \xi_1}\right)$$

$$= \frac{-r}{2\sqrt{\alpha}\, t^{3/2}}\left(\frac{\partial T}{\partial \xi_1}\right)$$

$$= -\frac{\xi_1}{2t}\frac{\partial T}{\partial \xi_1}$$

$$\frac{\partial T^*}{\partial r} = \left(\frac{\partial \xi_1}{\partial r}\right)\left(\frac{\partial T}{\partial \xi_1}\right)$$

$$= \frac{1}{\sqrt{\alpha t}}\left(\frac{\partial T}{\partial \xi_1}\right)$$

$$\frac{\partial^2 T^*}{\partial r^2} = \frac{1}{\alpha t}\left(\frac{\partial^2 T}{\partial \xi_1^2}\right)$$

$$\frac{1}{r}\frac{\partial T^*}{\partial r} = \frac{1}{r}\frac{1}{\sqrt{\alpha t}}\frac{\partial T^*}{\partial \xi_1}$$

$$= \frac{1}{\alpha t}\left(\frac{1}{\xi_1}\frac{\partial T^*}{\partial \xi_1}\right)$$

$$-\frac{\xi_1}{2\cancel{t}}\frac{\partial T^*}{\partial \xi_1} = \frac{\cancel{\alpha}}{\cancel{\alpha}\cancel{t}}\left(\frac{\partial^2 T^*}{\partial \xi_1^2} + \frac{1}{\xi_1}\frac{\partial T^*}{\partial \xi_1}\right)$$

$$\left(\frac{\partial^2 T^*}{\partial \eta^2}\right) + \left(\frac{1}{\eta} + \frac{\eta}{2}\right)\left(\frac{\partial T}{\partial \eta}\right) = 0 \qquad \frac{\partial^2 T^*}{\partial \eta^2} + \frac{\eta}{2}\frac{\partial T}{\partial \eta} = 0$$

Boundary conditions:

$T^* = 0$ as $r \to \infty$ or $\eta \to \infty$

$2\pi r L q_r = Q$ as $r \to 0$

Initial condition

$T^* = 0$ at $t = 0$ or $\eta \to \infty$

$$\frac{\partial T^*}{\partial \eta} = \frac{C}{\eta} e^{-\eta^2/4}$$

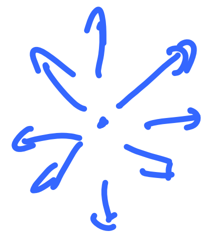
$$T^* = \int_{\infty}^{\eta} \frac{C}{\eta'} e^{-\eta'^2/4}\, d\eta'$$

$$q_r = -k \frac{\partial T}{\partial r}$$

$$= -k \frac{\partial \eta}{\partial r} \frac{\partial T}{\partial \eta}$$

$$= -k \frac{1}{\sqrt{\alpha t}} \frac{C}{\eta} e^{-\eta^2/4}$$

$$= -\frac{k}{r} C e^{-\eta^2/4}$$

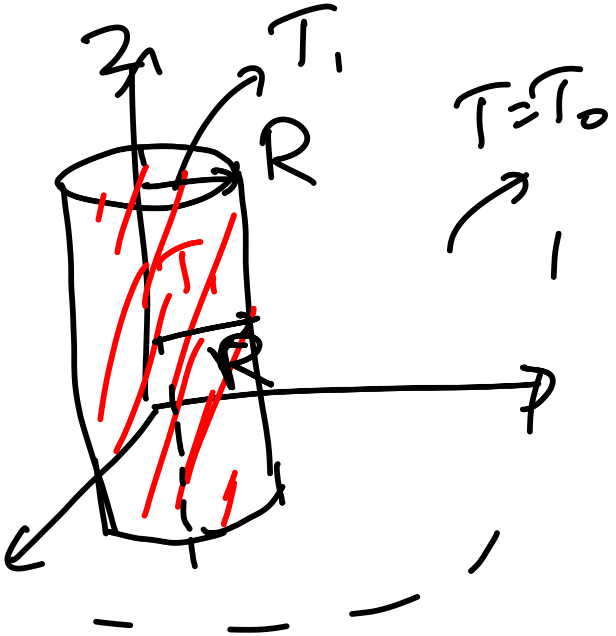


$$Q = 2\pi r L q_r = -k C (2\pi L) e^{-\eta^2/4}$$

for $r \to 0$ or $\eta \to 0$

$$C = \frac{-Q}{2\pi k L} \qquad r/\sqrt{\alpha t}$$

$$T^* = -\frac{Q}{2\pi k L} \int_{\infty} d\eta' \frac{1}{\eta'} e^{-\eta'^2/4}$$




Boundary conditions:

$T = T_0$ at $r = R$

$$\frac{\partial T}{\partial r} = 0 \quad r = 0 \text{ 'Symmetry'}$$

Initial condition

$T = T_1$ for all $r < R$
at $t = 0$

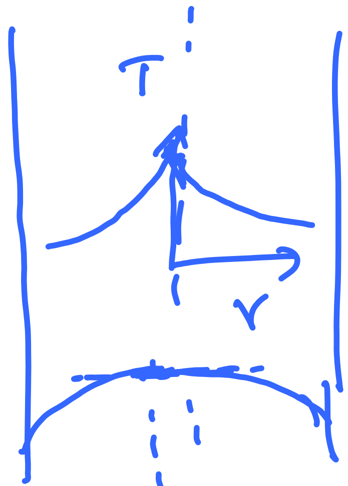
$$r^* = (r/R) \qquad t^* = \frac{t}{(R^2/\alpha)}$$

$$T^* = \left(\frac{T - T_0}{T_1 - T_0}\right)$$

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)$$

$$\frac{\partial T^*}{\partial t} = \frac{\alpha}{R^2} \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right)$$

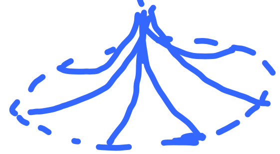
$$\frac{\partial T^*}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right)$$




Boundary conditions

$T = T_0$ at $r = R \Rightarrow T^* = 0$ at $r^* = 1$

$\frac{\partial T}{\partial r} = 0$ at $r = 0 \Rightarrow \frac{\partial T^*}{\partial r^*} = 0$ at $r^* = 0$



$\frac{\partial T^*}{\partial r^*} = 0$

Initial condition:

$T = T_1$ at $t = 0$ for $r < R$

$T^* = 1$ at $t^* = 0$ for $r^* < 1$

$$T^* = R(r^*)\,\Theta(t^*)$$

$$\frac{\partial}{\partial t^*}(R\Theta) = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial}{\partial r^*}(R\Theta)\right)$$

$$R\frac{\partial \Theta}{\partial t^*} = \Theta\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial R}{\partial r^*}\right)$$

Divide by $R\Theta$

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial t} = \frac{1}{R}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial R}{\partial r^*}\right)\right)$$

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial t} = -\beta^2$$

$$\frac{1}{R}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial R}{\partial r^*}\right) = -\beta^2$$

$$\frac{\partial^2 R}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial R}{\partial r^*} + \beta^2 R = 0$$

$$r^{*2}\frac{\partial^2 R}{\partial r^{*2}} + r^*\frac{\partial R}{\partial r^*} + \beta^2 r^{*2} R = 0$$

$$r^+ = \beta r^*$$

$$r^{+2}\frac{\partial^2 R}{\partial r^{+2}} + r^+\left(\frac{\partial R}{\partial r^+}\right) + r^{+2} R = 0$$

'Bessel eqn.'

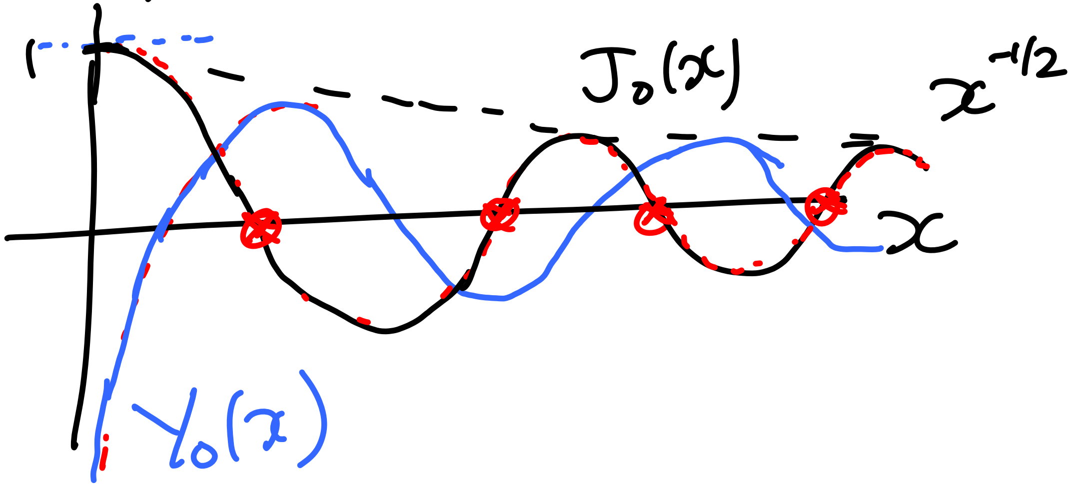
Bessel functions:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

$$y = A_1 J_n(x) + A_2 Y_n(x)$$

$$\frac{d^2y}{dx^2} + y = 0 \Rightarrow y\ A\sin x + B\cos x$$

$$R(r^+) = C_1 J_0(r^+) + \cancel{C_2} Y_0(r^+)$$




$C_2 = 0$ to satisfy BC at $r^* = 0$

$$R(r^+) = C_1 J_0(r^+)$$

$$R(r^*) = C_1 J_0(\beta r^*)$$

B.C $T^* = 0$ at $r^* = 1 \Rightarrow R(r^*) = 0$ at $r^* = 1$

$$R(r^*) = C_1 J_0(\beta r^*) \Longrightarrow C_1 J_0(\beta) = 0$$

Discrete set of $\beta$ at which $J_0(\beta) = 0$

$$\beta_1 = 2.40483$$
$$\beta_2 = 5.52008$$
$$\beta_3 = 8.65373$$
$$\beta_4 = 11.79150$$

$$R = C_1 J_0(\beta_n r^*)$$

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial t^*} = -\beta_n^2$$

$$\Rightarrow \Theta = e^{-\beta_n^2 t^*}$$

$$T^* = R\,\Theta = \sum_{n=1}^{\infty} C_n J_0(\beta_n r^*)\, e^{-\beta_n^2 t^*}$$

At $t^* = 0$, $T^* = 1$

$$\sum_{n=1}^{\infty} C_n J_0(\beta_n r^*) = 1$$

$$\int_0^1 r^* dr^* \, J_0(\beta_n r^*) J_0(\beta_m r^*)$$
$$= 0 \quad \text{for } n \neq m$$
$$= \frac{1}{2}\left(J_1(\beta_n)\right)^2 \quad \text{for } n = m$$

$$\langle S_n, S_m \rangle = \int_0^1 dz^* \sin(n\pi z^*) \sin(m\pi z^*)$$

$$= \tfrac{1}{2}\delta_{nm}$$

$$\langle J_n, J_m \rangle = \int_0^1 r^* dr^* J_0(\beta_m r^*) J_0(\beta_n r^*)$$

$$= \tfrac{1}{2}\left(J_1(\beta_n)\right)^2 \delta_{mn}$$

$$\sum_{n=1}^{\infty} C_n J_0(\beta_n r^*) = 1$$

$$\sum_{n=1}^{\infty} C_n \int r^* dr^* J_0(\beta_n r^*) J_0(\beta_m r^*) = \int_0^1 J_0(\beta_m r^*) r^* dr^*$$

$$\sum_{n=1}^{\infty} C_n \delta_{mn} \left(J_1(\beta_n)\right)^2 = \frac{J_1(\beta_m)}{\beta_m}$$

$$C_m = \frac{2}{\beta_m J_1(\beta_m)}$$

$$T^* = \sum C_n J_n e^{-\beta_n^2 t^*}$$

At $t^* = 0$, $T^* = 1$

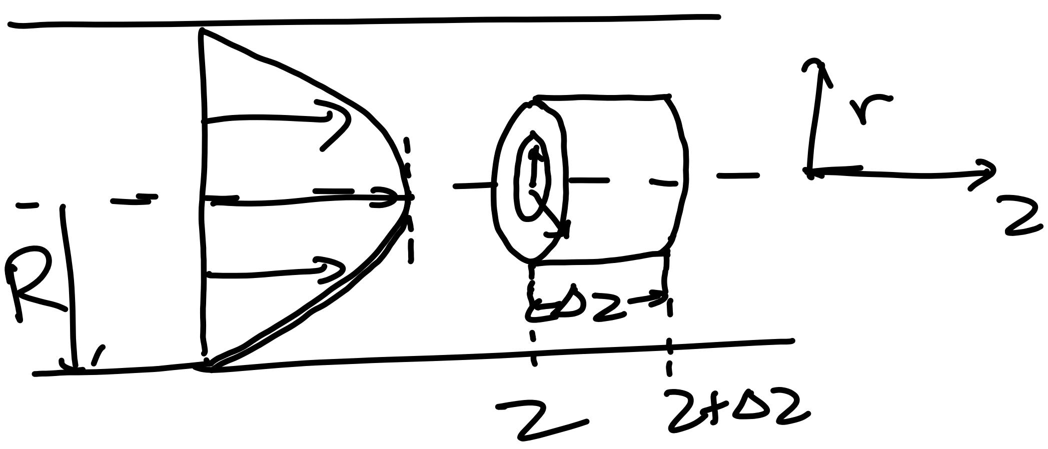
$$\sum_{n=1}^{\infty} C_n J_n = 1$$

$$\sum_{n=1}^{\infty} C_n \langle J_n, J_m \rangle = \langle 1, J_m \rangle$$

$$\sum_{n=1}^{\infty} C_n \, \tfrac{1}{2} \left(J_1(\beta_m)\right)^2 \delta_{mn} = \langle 1, J_m \rangle$$

$$C_m \left[\tfrac{1}{2} J_1(\beta_n)\right]^2 = \int r^* dr^* \, J_0(\beta_m r^*)$$

# Flow in a pipe:



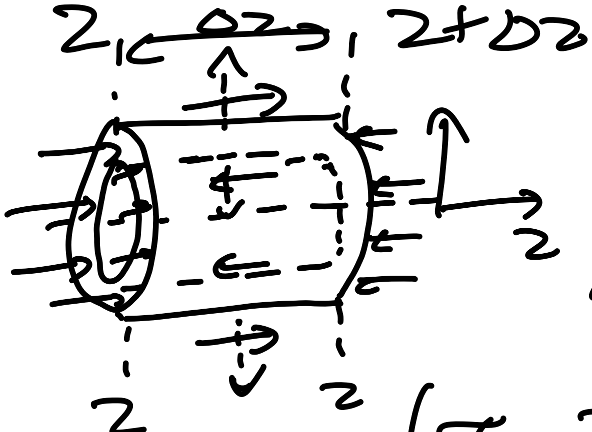


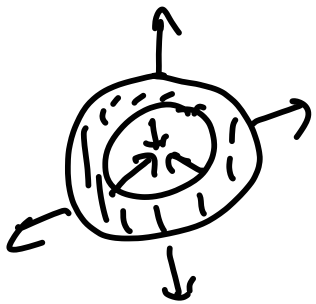
At $r = R$ $\quad u_z = 0$

At $r = 0$ $\quad \dfrac{\partial u_z}{\partial r} = 0$

$$\begin{pmatrix}\text{Rate of} \\ \text{change of} \\ \text{momentum}\end{pmatrix} = \begin{pmatrix}\text{Sum of} \\ \text{applied} \\ \text{forces}\end{pmatrix}$$

$$\frac{\left[\rho v_z(r, z, t+\Delta t) - \rho v_z(r, z, t)\right](2\pi r \Delta r \Delta z)}{\Delta t}$$



$$\text{Shear forces} = \left(\tau_{zr} 2\pi r \Delta z\right)\Big|_{r+\Delta r} - \left(\tau_{zr} 2\pi r \Delta z\right)\Big|_{r}$$

$\tau_{zr}$ = Force in z direction
at surface with unit
normal in r-direction

$$\text{Pressure forces} = \left(p\, 2\pi r \Delta r\right)\Big|_{z} - \left(p\, 2\pi r \Delta r\right)\Big|_{z+\Delta z}$$

$$\frac{\left[\rho u_z(r,z,t+\Delta t) - \rho u_z(r,z,t)\right] 2\pi r \Delta r \Delta z}{\Delta t}$$

$$= \left(\tau_{zr} 2\pi r \Delta z\right)\Big|_{r+\Delta r} - \left(\tau_{zr} 2\pi r \Delta z\right)\Big|_{r} + \left(p\, 2\pi r \Delta r\right)\Big|_{z} - \left(p\, 2\pi r \Delta r\right)\Big|_{z+\Delta z}$$

Divide by $2\pi r \Delta r \Delta z$

$$\frac{\rho u_z(r,z,t+\Delta t) - \rho u_z(r,z,t)}{\Delta t} =$$

$$\frac{1}{r}\frac{1}{\Delta r}\left[(\tau_{zr} r)\big|_{r+\Delta r} - (\tau_{zr} r)\big|_{r}\right]$$

$$+ \frac{\left(p|_z - p|_{z+\Delta z}\right)}{\Delta z}$$

$$\rho \frac{\partial u_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \tau_{zr}\right) - \frac{\partial p}{\partial z}$$

$$\tau_{zr} = \mu\left(\frac{\partial u_z}{\partial r}\right)$$

$$\rho \frac{\partial u_z}{\partial t} = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) - \frac{\partial p}{\partial z}$$

$$\frac{\partial u_z}{\partial t} = \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Steady state $\frac{\partial u_z}{\partial t} = 0$

$$\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) = \left(\frac{\partial p}{\partial z}\right)$$

$$\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) = \frac{1}{\mu}\left(\frac{\partial p}{\partial z}\right) r$$

$$r \frac{\partial u_z}{\partial r} = \frac{1}{\mu}\frac{\partial p}{\partial z}\frac{r^2}{2} + C_1$$

$$\frac{\partial u_z}{\partial r} = \frac{1}{\mu}\frac{\partial p}{\partial z}\frac{r^2}{2} + \cancel{\frac{C_1}{r}}$$

$$u_z = \frac{1}{4\mu}\frac{\partial p}{\partial z} r^2 + \cancel{C_1 \log r} + C_2$$

Boundary conditions

$u_z = 0$ at $r = R$

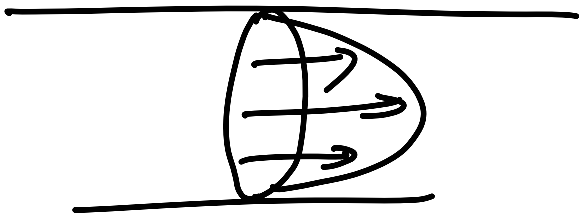
$\frac{\partial u_z}{\partial r} = 0$ at $r = 0$

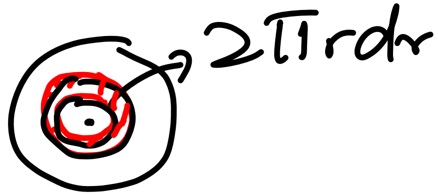
$$u_z = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(R^2 - r^2\right)$$

'Hagen-Poiseuille flow'

$$u_z = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$Q = \int_0^R u_z\, r\, dr\, 2\pi$$





$$Q = -\left(\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial x}\right)$$

$$\overline{U_z} = \frac{Q}{\pi R^2} = -\frac{\pi R^2}{8\mu}\frac{\partial p}{\partial x}$$

$$= \left(\frac{U_{z\,max}}{2}\right)$$

$$U_z(r) = U_{z\,max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\tau_{zr} = \mu\frac{\partial U_z}{\partial r} = -\frac{2U_{z\,max}\, r\mu}{R^2}$$

Wall shear stress

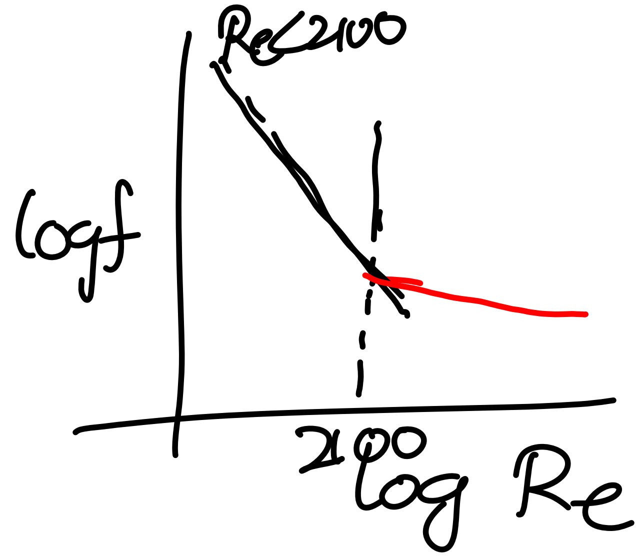
$$\tau_{zr}\Big|_{r=R} = -\frac{2U_{z\,max}\,\mu}{R}$$

$$f = \frac{\tau_{zr}}{\frac{1}{2}\rho\bar{u}^2} = \frac{-2u_{z\,max}\,\mu}{R\left(\frac{1}{2}\rho\bar{u}^2\right)}$$

$$= \frac{4\bar{u}\mu}{R\left(\frac{1}{2}\rho\bar{u}^2\right)} = \frac{8\mu}{\rho\bar{u}R}$$

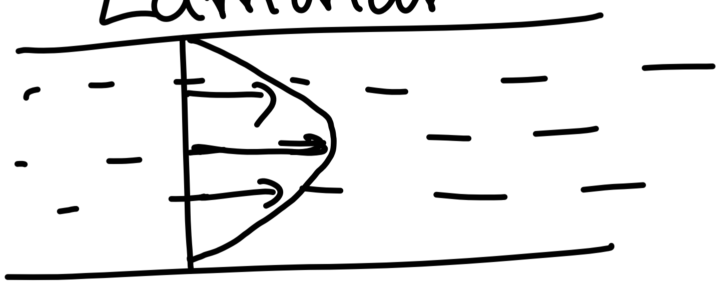
$$f = \frac{16\mu}{\rho\bar{u}D} = \frac{16}{Re}$$

$$Re = \left(\frac{\rho\bar{u}D}{\mu}\right) = \left(\frac{\rho u_z^{max}R}{\mu}\right)$$

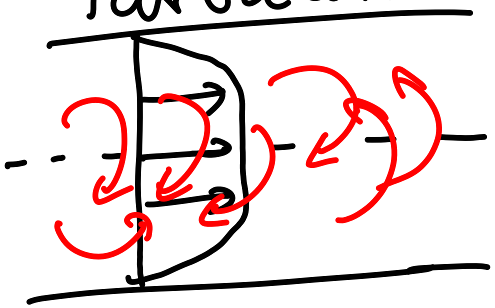



$$\log f = \log(16) - \log Re$$

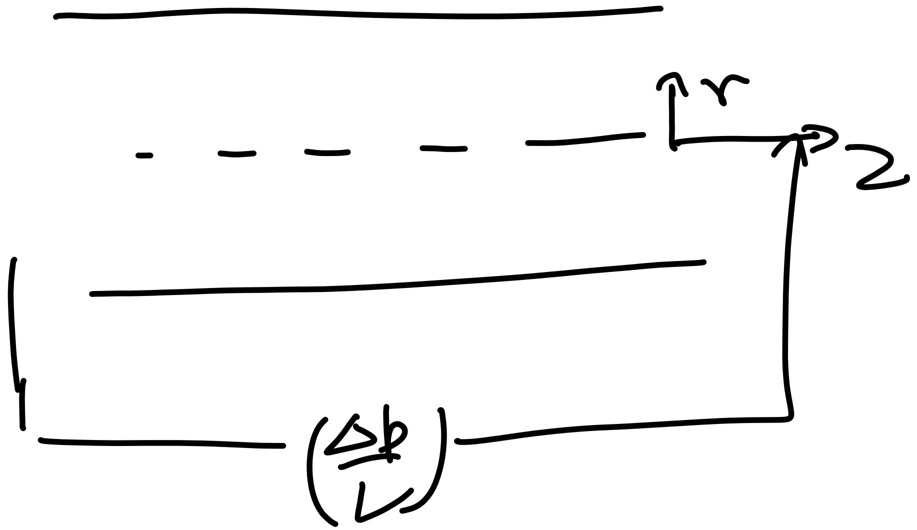
Laminar

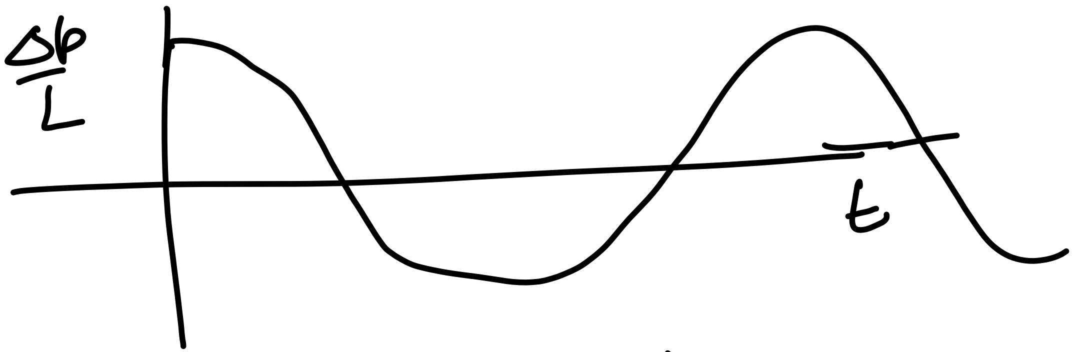


Turbulent



# Oscillatory flow in a pipe:




$$\frac{\Delta p}{L} = K \cos(\omega t)$$

$$\rho \frac{\partial u_z}{\partial t} = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) - \frac{\partial p}{\partial z}$$

$$\rho \frac{\partial u_z}{\partial t} = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) + K\cos(\omega t)$$

Boundary conditions:

$u_z = 0$ at $r = R$

$\frac{\partial u_z}{\partial r} = 0$ at $r = 0$

$r^* = (r/R) \qquad t^* = \omega t$

$$\rho \omega \frac{\partial u_z}{\partial t^*} = \frac{\mu}{R^2}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z}{\partial r^*}\right)\right) - K\cos(t^*)$$

$$\frac{\rho\omega}{k}\frac{\partial u_z}{\partial t^*} = \frac{\mu}{kR^2}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z}{\partial r^*}\right)\right) - \cos t^*$$

$$u_z^* = \left(\frac{\mu u_z}{kR^2}\right)$$

$$Re_\omega = \left(\frac{\rho\omega R^2}{\mu}\right)$$

$$\left(\frac{\rho\omega R^2}{\mu}\right)\frac{\partial u_z^*}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right) - \cos t^*$$

$$Re_\omega\frac{\partial u_z^*}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right) - \cos t^*$$

At $r^* = 0$, $\frac{\partial u_z^*}{\partial r^*} = 0$

At $r^* = 1$ $u_z^* = 0$

$$\cos(t^*) = \mathrm{Real}(e^{it^*})$$

$$Re_\omega \frac{\partial u_z^+}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r}\left(r^* \frac{\partial u_z^+}{\partial r^*}\right) - e^{it^*}$$

$$u_z^* = \mathrm{Real}(u_z^+)$$

$$\frac{\partial u_z^+}{\partial r} = 0 \quad \text{at} \quad r^* = 0$$

$$u_z^+ = 0 \quad \text{at} \quad r^* = 1$$

$$u_z^+ = \tilde{u}_z(r^*)\, e^{it^*}$$

$$Re_\omega\, \tilde{u}_z(r^*)\, i\, \cancel{e^{it^*}} = \cancel{e^{it^*}}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_z}{\partial r}\right)\right) - \cancel{e^{it^*}}$$

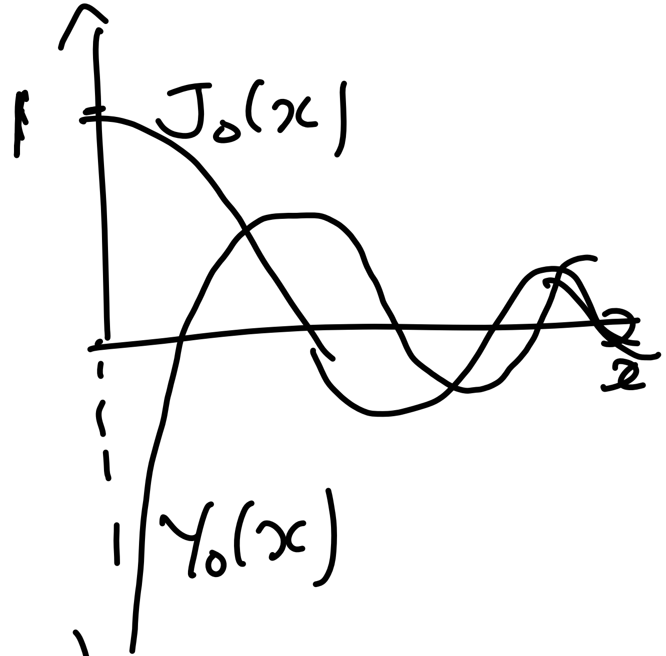
$$i Re_\omega\, \tilde{u}_z(r^*) = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z^*}{\partial r^*}\right) - 1$$

$$\frac{\partial^2 \tilde{u}_2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial \tilde{u}_2}{\partial r^*} - i\,Re_\omega\, \tilde{u}_2(r^*) = 1$$

$$\frac{\partial^2 \tilde{u}_{2g}}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial \tilde{u}_{2g}}{\partial r^*} - i\,Re_\omega\, \tilde{u}_{2g} = 0$$

$$r^{*2}\frac{\partial^2 \tilde{u}_{2g}}{\partial r^{*2}} + r^*\frac{\partial \tilde{u}_{2g}}{\partial r^*} - i\,Re_\omega\, r^{*2}\,\tilde{u}_{2g} = 0$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$




$$x = \left(\sqrt{-i\,Re_\omega}\; r^*\right)$$

$$\tilde{u}_{2g} = C_1 J_0\left(\sqrt{-i\,Re_\omega}\; r^*\right) + \cancel{C_2 Y_0\left(\sqrt{-i\,Re_\omega}\; r^*\right)}$$

$$-i Re_\omega \tilde{u}_{2p} = 1 \;;\; \tilde{u}_{2p} = +\frac{1}{i Re_\omega} = \frac{-i}{Re_\omega}$$

$$\tilde{u}_2 = \frac{i}{Re_\omega} + C_1 J_0\left(\sqrt{-i Re_\omega}\, r^*\right)$$

Boundary condition

$\tilde{u}_2 = 0$ at $r^* = 1$

$$\tilde{u}_2 = \frac{i}{Re_\omega}\left(1 - \frac{J_0\left(\sqrt{-i Re_\omega}\, r^*\right)}{J_0\left(\sqrt{-i Re_\omega}\right)}\right)$$

$$u_2^+ = \frac{i}{Re_\omega}\left(1 - \frac{J_0\left(\sqrt{-i Re_\omega}\, r^*\right)}{J_0\left(\sqrt{-i Re_\omega}\right)}\right) e^{it^*}$$

$$u_2^* = \mathrm{Real}\left(u_2^+\right)$$

Low Reynolds number

$Re_\omega << 1$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*} r^* \frac{\partial u_z^*}{\partial r^*} - \cos t^* = 0$$

$$u_z^* = -\frac{1}{4}(1-r^{*2})\cos t^*$$

$$u_z = u_z^* \left(\frac{K}{\mu R^2}\right) = -\frac{K}{4\mu}(R^2-r^2)\cos(\omega t)$$

$$Re_\omega = \left(\frac{8\omega R^2}{\mu}\right) = \left(\frac{\omega}{\nu/R^2}\right) \sim \left(\frac{t_{diff}}{t_{period}}\right)$$

$$\omega \sim \frac{2\pi}{t_{period}}$$

Limit $Re_\omega \ll 1$ $\quad Re_\omega \to 0$

$$\tilde{u}_2 = \tilde{u}_2^{(0)} + Re_\omega \tilde{u}_2^{(1)} + Re_\omega^2 \tilde{u}_2^{(2)} + \dots$$

$$Re_\omega i \tilde{u}_2 = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_2}{\partial r^*}\right) - 1$$

$$Re_\omega i \left[\tilde{u}_2^{(0)} + Re_\omega \tilde{u}_2^{(1)} + Re_\omega^2 \tilde{u}_2^{(2)}\right]$$

$$= \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial}{\partial r^*}\left(\tilde{u}_2^{(0)} + Re_\omega \tilde{u}_2^{(1)} + Re_\omega^2 \tilde{u}_2^{(2)}\right) - 1\right.$$

$$0 = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_2^{(0)}}{\partial r^*}\right) - 1 \qquad O(1)$$

$$+ Re_\omega i u_2^{(0)} \qquad + Re_\omega \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_2^{(1)}}{\partial r^*}\right) \qquad O(Re_\omega)$$

$$+\, Re_\omega^2\, i\, u_2^{(1)} \qquad + Re_\omega^2 \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_2^{(2)}}{\partial r^*}\right) \qquad O(Re_\omega^2)$$

$$Re_\omega \ll 1 \implies Re_\omega^2 \ll Re_\omega \ldots$$

$$0 = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_2^{(0)}}{\partial r^*}\right) - 1$$

$$i\, \tilde{u}_2^{(0)} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_2^{(1)}}{\partial r^*}\right)$$

$$i\, \tilde{u}_2^{(1)} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_2^{(2)}}{\partial r^*}\right)$$

$$\tilde{u}_2^{(0)} + Re_\omega \tilde{u}_2^{(1)} + Re_\omega^2 \tilde{u}_2^{(2)} = 0$$

at $r^* = 1$

$$\frac{d}{dr^*}\left(\tilde{u}_2^{(0)} + Re_\omega \tilde{u}_2^{(1)} + Re_\omega^2 \tilde{u}_2^{(2)}\right) = 0$$

at $r^* = 0$

$$\tilde{u}_2^{(0)} = 0 \;;\; \tilde{u}_2^{(1)} = 0 \;;\; \tilde{u}_2^{(2)} = 0 \text{ at } r^* = 1$$

$$\frac{\partial \tilde{u}_2^{(0)}}{\partial r^*} = 0 \;;\; \frac{\partial \tilde{u}_2^{(1)}}{\partial r^*} = 0 \;;\; \frac{\partial \tilde{u}_2^{(2)}}{\partial r^*} = 0 \text{ at } r^* = 1$$

$$\tilde{u}_2^{(0)} = \frac{-1}{4}\left(1 - r^{*2}\right)$$

$$\tilde{u}_2^{(1)} = \frac{i\left(3 - 4r^{*2} + r^{*4}\right)}{64}$$

$$\tilde{u}_2^{(2)} = \frac{\left(19 - 27r^{*2} + 9r^{*4} - r^{*6}\right)}{2304}$$

$$u_2^* = -\frac{\left(1 - r^{*2}\right)\cos(t^*)}{4} - \frac{Re_\omega \sin(t^*)\left(3 - 4r^{*2} + r^{*4}\right)}{64}$$

$$+ \frac{Re_\omega^2\left(19 - 27r^{*2} + 9r^{*4} - r^6\right)\cos(t^*)}{2304}$$

$$+ O\left(Re_\omega^3\right)$$ Regular perturbation expansion

$$\rho \frac{\partial u_z}{\partial t} = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right) - K\cos(\omega t) \qquad Re_\omega >> 1$$

$$r^* = (r/R) \ ; \quad t^* = \omega t$$

$$\rho\omega \frac{\partial u_z}{\partial t^*} = \frac{\mu}{R^2}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z}{\partial r^*}\right) - K\cos(t^*)$$

$$\frac{\rho\omega}{K}\frac{\partial u_z}{\partial t^*} = \frac{\mu}{R^2 K}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z}{\partial r^*}\right) - \cos(t^*)$$

$$u_z^* = \left(\frac{u_z \rho\omega}{K}\right)$$

$$\frac{\partial u_z^*}{\partial t^*} = \left(\frac{\mu}{\rho\omega R^2}\right)\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial u_z^*}{\partial r^*}\right) - \cos(t^*)$$

$$\frac{\partial u_z^*}{\partial t^*} = \frac{1}{Re_\omega} \frac{1}{r^*} \frac{\partial}{\partial r^*}\left( r^* \frac{\partial u_z^*}{\partial r^*} \right) - \cos(t^*)$$

Limit $Re_\omega >> 1$

$$\frac{\partial u_z^*}{\partial t^*} = -\cos(t^*) \Longrightarrow u_z^* = -\sin(t^*)$$

Boundary conditions:

$\frac{\partial u_z^*}{\partial r^*} = 0$ at $r^* = 0$

$u_z^* = 0$ at $r^* = 1$

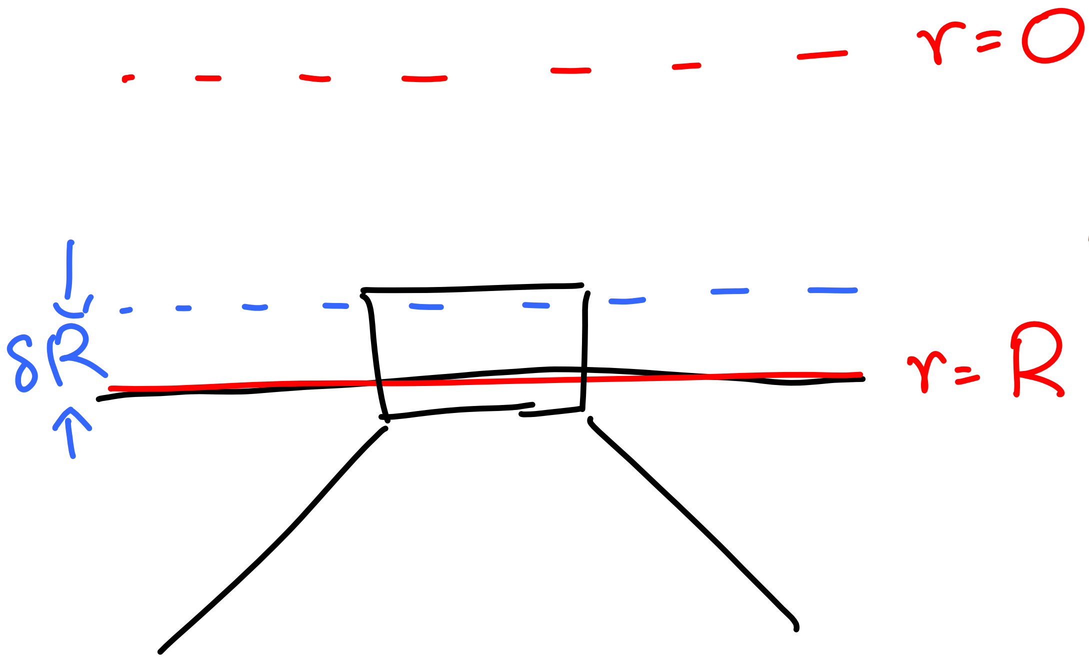
$Re_\omega >> 1$

$\left(\frac{\omega R^2}{\nu}\right) >> 1$

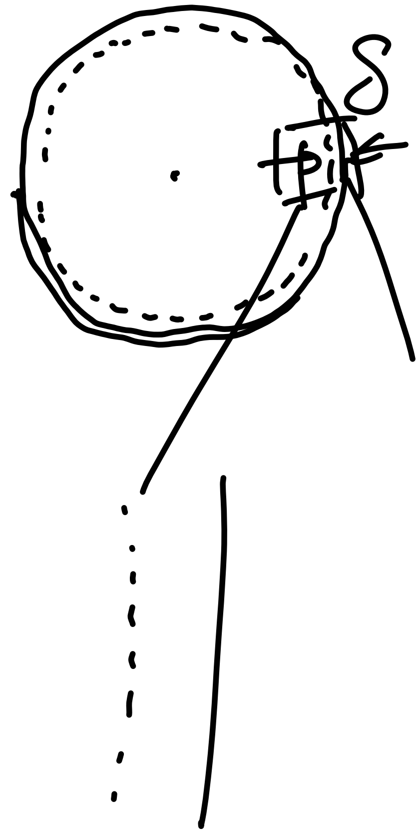
$\frac{R^2}{\nu} >> \omega^{-1}$

Distance $= \left(\frac{\nu}{\omega}\right)^{1/2} = \delta R$

$$\delta = \left(\frac{\nu}{R^2\omega}\right)^{1/2} = Re_\omega^{-1/2}$$




'Boundary layer'





'Inner co-ordinate'

$$y = \frac{(R-r)}{\delta R} = \frac{1}{\delta}\left(1 - r^*\right)$$

$$r^* = (1 - \delta y)$$

$$\frac{\partial u_z^*}{\partial t^*} = \frac{1}{Re_\omega}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right)\right) - \cos(t^*)$$

$$\frac{\partial u_z^*}{\partial t^*} = \frac{1}{Re_\omega}\left(\frac{1}{(1-\cancel{\delta y})}\frac{1}{\delta}\frac{\partial}{\partial y}\left((1-\cancel{\delta y})\frac{1}{\delta}\frac{\partial u_z^*}{\partial y}\right)\right) - \cos t^*$$

$$\frac{\partial u_z^*}{\partial t^*} = \frac{1}{Re_\omega \delta^2}\frac{\partial^2 u_z^*}{\partial y^2} - \cos t^*$$

$$\delta \sim Re_\omega^{-1/2} \qquad \delta = c\,Re_\omega^{-1/2}$$

$$\frac{\partial u_z^*}{\partial t^*} = \frac{1}{c^2}\frac{\partial^2 u_z^*}{\partial y^2} - \cos t^*$$

$$u_2^* = \text{Real}\left[\tilde{u}_2 e^{it^*}\right]$$

$$i\tilde{u}_2 = \frac{1}{C^2}\frac{d^2\tilde{u}_2}{dy^2} - 1$$

$$\tilde{u}_{2p} = -\frac{1}{i} = i$$

$$\tilde{u}_{2g} = C_1 e^{+\sqrt{i}Cy} + C_2 e^{-\sqrt{i}Cy}$$

Boundary conditions.

$$\frac{\partial \tilde{u}_2}{\partial r^*} = 0 \quad \text{at} \quad r^* = 0 \Rightarrow y = (1/\delta)$$

as $y \to \infty$

$$\tilde{u}_2 = 0 \quad \text{at} \quad r^* = 1 \Rightarrow y = 0$$

$$r^* = (1 - \delta y)$$

$$\tilde{u}_2 = i\left(1 - e^{-\sqrt{i}\, c y}\right)$$

$$\tilde{u}_2 = i\left[1 - e^{-\left(\sqrt{i}\,\frac{c(1-r^*)}{\delta}\right)}\right]$$

$$= i\left[1 - e^{-\left(\frac{\sqrt{i}\,\cancel{c}\,(1-r^*)}{\cancel{c}\, Re_\omega^{-1/2}}\right)}\right]$$

$$= i\left[1 - e^{-\left(\sqrt{i\, Re_\omega}\,(1-r^*)\right)}\right]$$

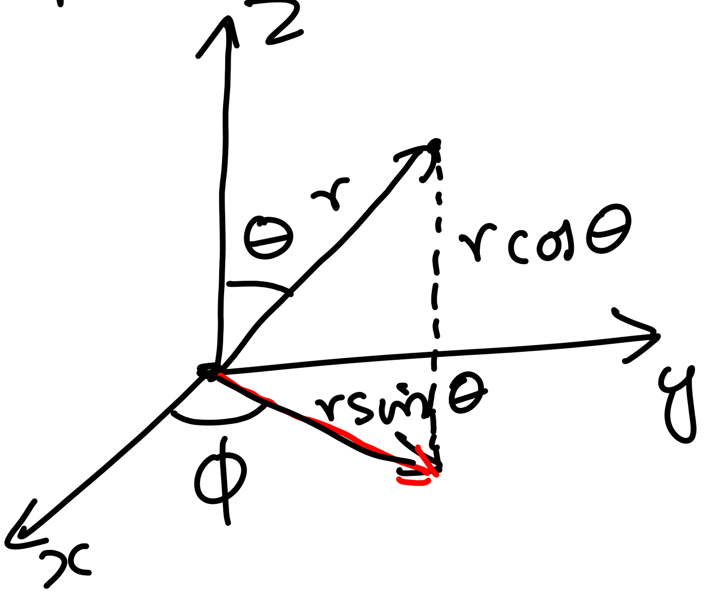
$$u_2^* = \mathrm{Real}\left[\tilde{u}_2 e^{i t^*}\right]$$

$$= -\sin t^*\left[1 - \exp\left[\frac{-Re_\omega^{1/2}(1-r^*)}{\sqrt{2}}\right]\cos\left(\frac{-Re_\omega^{1/2}(1-r^*)}{\sqrt{2}}\right)\right.$$

$$\left. + \cos t^* \sin\left(\frac{-Re_\omega^{1/2}(1-r^*)}{\sqrt{2}}\right)\exp\left[\frac{-Re_\omega^{1/2}(1-r^*)}{\sqrt{2}}\right]\right]$$

# 'Singular perturbation expansion'

## Spherical co-ordinate system:




$x^2+y^2+z^2=R^2$

$r=\sqrt{x^2+y^2+z^2}$

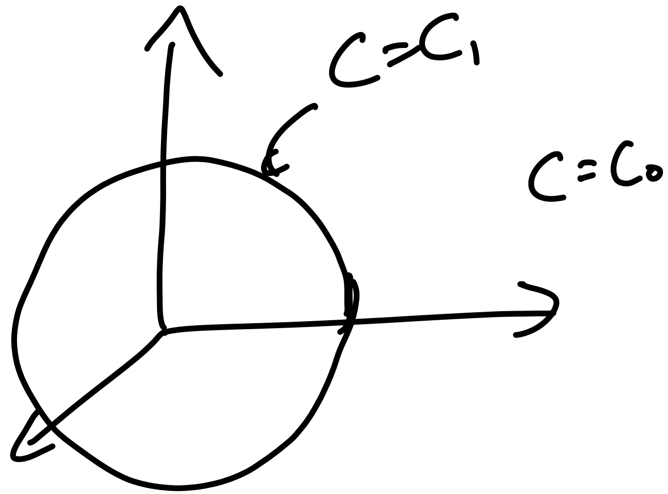
Azimuthal angle $\theta$

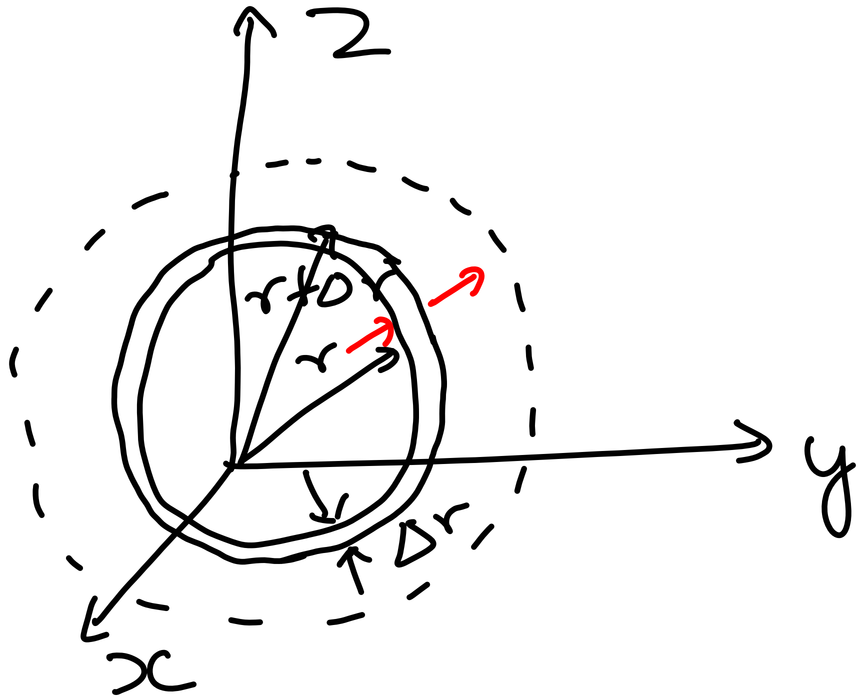
$z=r\cos\theta$

$x=r\sin\theta\cos\phi$

$y=r\sin\theta\sin\phi$

Meridional angle $\phi$






$$\begin{pmatrix}\text{Change in} \\ \text{mass} \\ \text{in time } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Mass} \\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{Mass} \\ \text{out}\end{pmatrix} + \begin{pmatrix}\text{Sources}\end{pmatrix}$$

$$\big(c(r, t+\Delta t) - c(r,t)\big) \times 4\pi r^2 \Delta r$$

$$(\text{Mass in}) = \left(j_r 4\pi r^2\right)\Big|_r \Delta t$$

$$(\text{Mass out}) = \left(j_r 4\pi r^2\right)\Big|_{r+\Delta r} \Delta t$$

$$(\text{Source}) = S\left(4\pi r^2 \Delta r\right)\Delta t$$

$$\left(c(r,t+\Delta t) - c(r,t)\right)\left(4\pi r^2 \Delta r\right) = \Delta t\left(j_r\left(4\pi r^2\right)\Big|_r - j_r\left(4\pi r^2\right)\Big|_{r+\Delta r}\right)$$

$$+ S\left(4\pi r^2 \Delta r \Delta t\right)$$

$$\frac{c(r,t+\Delta t) - c(r,t)}{\Delta t} = \frac{1}{r^2 \Delta r}\left[\left(j_r r^2\right)\Big|_r - \left(j_r r^2\right)\Big|_{r+\Delta r}\right] + S$$

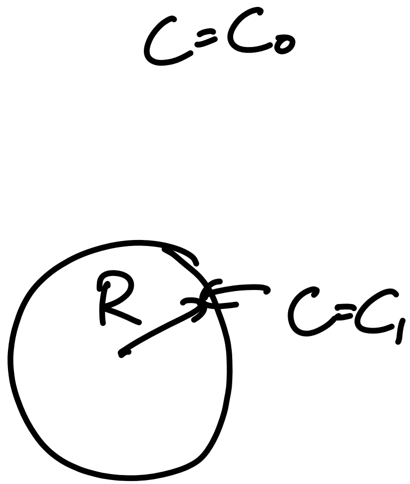
Limit $\Delta r \to 0$ & $\Delta t \to 0$

$$\frac{\partial c}{\partial t} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 j_r\right) + S$$

$$j_r = -D\frac{\partial c}{\partial r}$$

$$\frac{\partial c}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + S$$

$$\frac{\partial T}{\partial t} = \alpha\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{S_e}{\rho C_p}$$




Steady state, no sources:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) = 0$$

Boundary conditions:

$C = C_1$ at $r = R$

$C = C_0$ as $r \to \infty$

$$c^* = \frac{c - C_0}{C_1 - C_0} \qquad r^* = \left(\frac{r}{R}\right)$$

$$\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial c^*}{\partial r^*}\right) = 0$$

$c^* = 1$ at $r^* = 1$ $\quad A = 1$

$c^* = 0$ as $r^* \to \infty$ $\quad B = 0$

$$C^* = \frac{A}{r^*} + B$$

$$C^* = \frac{1}{r^*}$$

$$C - C_0 = \left(\frac{(C_1 - C_0) R}{r}\right)$$

$$j_r = -D\left(\frac{\partial C}{\partial r}\right)$$

$$= -D\left[\frac{-(C_1 - C_0) R}{r^2}\right]$$

$$= \frac{D(C_1 - C_0) R}{r^2}$$

$$J = 4\pi r^2 j_r$$

$$= 4\pi D R (C_1 - C_0)$$

$$(C - C_0) = \frac{J}{4\pi D r}$$

$$T - T_0 = \frac{(T_1 - T_0) R}{r}$$

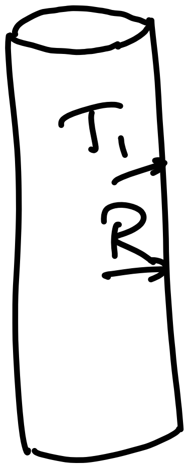
$$q_r = \frac{k(T_1 - T_0) R}{r^2}$$

$$Q = 4\pi k R (T_1 - T_0)$$

$$T_1 - T_0 = \frac{Q}{4\pi k r}$$

Limit $R \to 0$ 'point particle limit'

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$




$T_0$

$T = T_1$ at $r = R$

$T = T_0$ as $r \to \infty$

$$T^* = \left(\frac{T - T_0}{T_1 - T_0}\right) \quad \& \quad r^* = \left(\frac{r}{R}\right)$$

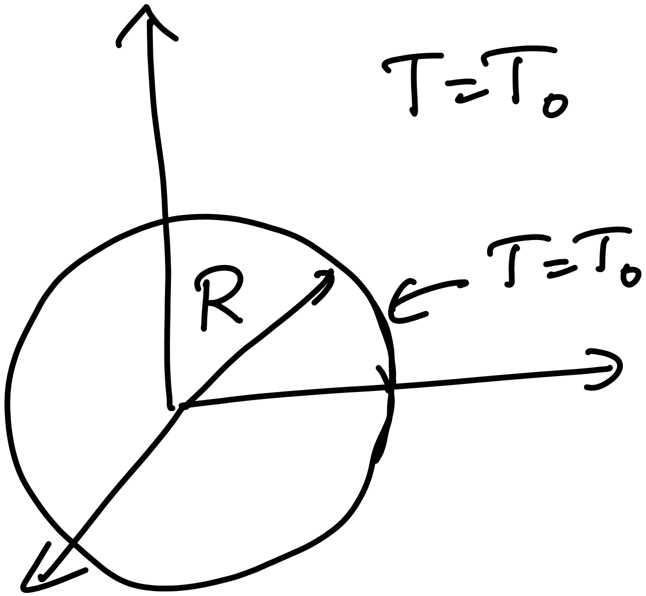
$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) = 0$$

$T^* = 1$ at $r^* = 1$

$T^* = 0$ as $r^* \to \infty$

$$T^* = A\log(r^*) + B$$

# Unsteady diffusion in spherical co-ordinates.




Boundary condition:

$T = T_0$ at $r = R$

Initial condition

$T = T_1$ at $t = 0$ for $r < R$

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \right)$$

$$T^* = \left( \frac{T - T_0}{T_1 - T_0} \right) \qquad r^* = \left( \frac{r}{R} \right) \qquad t^* = \left( \frac{t\alpha}{R^2} \right)$$

$$\boxed{\frac{\partial T^*}{\partial t^*} = \frac{1}{r^{*2}} \frac{\partial}{\partial r^*} \left( r^{*2} \frac{\partial T^*}{\partial r^*} \right)}$$

At $r^* = 1$, $T^* = 0$ BC 1

At $t^* = 0$, $T^* = 1$ for $r^* < 1$ IC

At $r^* = 0$, $\frac{\partial T^*}{\partial r^*} = 0$ BC 2

$$T^* = F(r^*)\,\Theta(t^*)$$

$$F(r^*)\frac{\partial \Theta}{\partial t^*} = \Theta(t^*)\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial F}{\partial r^*}\right)$$

Divide by $F(r^*)\,\Theta(t^*)$

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial t^*} = \frac{1}{F(r^*)}\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial F}{\partial r^*}\right)$$

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial t} = -\beta^2$$

$$\frac{1}{F(r^*)}\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial F}{\partial r^*}\right) = -\beta^2$$

$$\frac{\partial^2 F}{\partial r^{*2}} + \frac{2}{r^*}\frac{\partial F}{\partial r^*} + \beta^2 F = 0$$

$$r^{*2}\frac{\partial^2 F}{\partial r^{*2}} + 2r^*\frac{\partial F}{\partial r^*} + \beta^2 r^{*2} F = 0$$

$$r^+ = \beta r^*$$

$$r^{+2}\frac{\partial^2 F}{\partial r^{+2}} + 2r^+\frac{\partial F}{\partial r^+} + r^{+2} F = 0$$

$$F = \frac{A' \sin(r^+)}{r^+} + \frac{B' \cos(r^+)}{r^+}$$

$$= \frac{A \sin(\beta r^*)}{r^*} + \frac{B \cos(\beta r^*)}{r^*}$$

$B = 0$ for $\frac{\partial T^*}{\partial r^*} = 0$ at $r^* = 0$

$$F = \frac{A \sin(\beta r^*)}{r^*} \qquad T^* = 0 \text{ at } r^* = 1$$

Only if $\beta_n = (n\pi)$

$$F = \frac{A \sin(n\pi r^*)}{r^*}$$

$$\frac{1}{\Theta} \frac{\partial \Theta}{\partial t^*} = -\beta_n^2 = -n^2\pi^2$$

$$\Theta = e^{-n^2\pi^2 t^*}$$

$$T^* = \sum_{n=1}^{\infty} A_n \frac{\sin(n\pi r^*)}{r^*} e^{-n^2\pi^2 t^*}$$

$$\psi_n = \frac{\sin(n\pi r^*)}{r^*}$$

$$\langle \psi_n, \psi_m \rangle = \int r^{*2} dr^* \left(\frac{\sin(n\pi r^*)}{r^*}\right)\left(\frac{\sin(m\pi r^*)}{r^*}\right)$$

$$= \frac{1}{2}\delta_{mn}$$

Initial condition:

At $t^* = 0$, $T^* = 1$ for all $r^* < 1$

$$T^* = \sum_{n=1}^{\infty} A_n \left(\frac{\sin(n\pi r^*)}{r^*}\right) e^{-n^2\pi^2 t^*}$$

At $t^* = 0$,

$$T^* = \sum_{n=1}^{\infty} A_n \left(\frac{\sin(n\pi r^*)}{r^*}\right) = 1$$

Multiply by $\left(\frac{\sin(m\pi r^*)}{r^*}\right) r^{*2} dr^*$

& integrate from 0 to 1

$$\sum_{n=1}^{\infty} A_n \int_0^1 r^{*2} dr^* \left(\frac{\sin(n\pi r^*)}{r^*}\right)\left(\frac{\sin(m\pi r^*)}{r^*}\right)$$

$$= \int_0^1 r^{*2} dr^* \left(1 \times \frac{\sin(m\pi r^*)}{r^*}\right)$$

$$\sum_{n=1}^{\infty} A_n \left(\frac{\delta_{mn}}{2}\right) = \int_0^1 r^* dr^* \sin(m\pi r^*)$$

$$\frac{A_m}{2} = \left(\frac{1}{m\pi}\right)^2 \Rightarrow A_m = \frac{2}{(m\pi)^2}$$

Bessel equation $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

$$y=C_1J_n(x)+C_2Y_n(x)$$

$$\langle\psi_n,\psi_m\rangle=\int x\,dx\,(\psi_n(x)\,\psi_m(x))$$

$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2-n(n+1))y=0$$

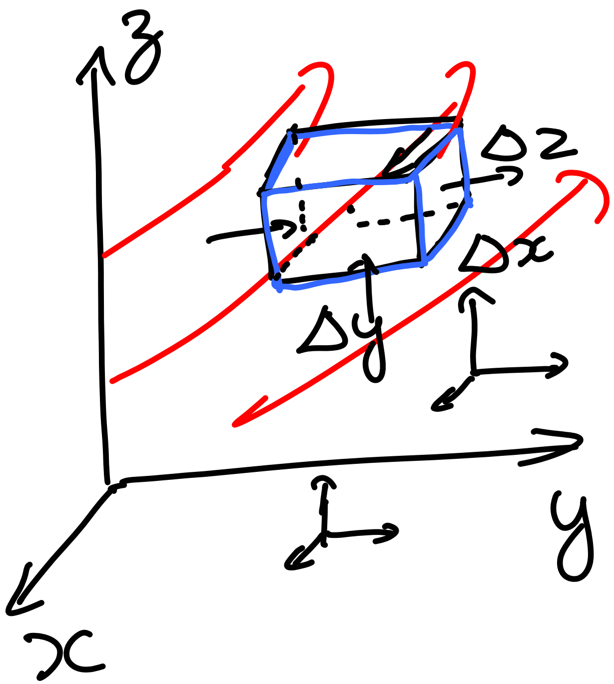
$$y=C_1j_n(x)+C_2y_n(x)$$

$$\langle\psi_n,\psi_m\rangle=\int x^2\,dx\,\psi_n(x)\,\psi_m(x)$$

$j_0(x)=\frac{\sin x}{x}$ & $y_0(x)=\left(\frac{\cos x}{x}\right)$

# Conservation Equations for Mass and Energy:

## Cartesian co-ordinate system:




Accumulation of mass in time $\Delta t$

$$= \left(c(x,y,z,t+\Delta t) - c(x,y,z,t)\right)\Delta x\,\Delta y\,\Delta z$$

$$\begin{pmatrix}\text{Accumulation of} \\ \text{mass in time } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Mass in}\end{pmatrix} - \begin{pmatrix}\text{Mass out}\end{pmatrix} + \begin{pmatrix}\text{Production} \\ \text{in volume}\end{pmatrix}$$

Accumulation of mass $= \left(C(x,y,z,t+\Delta t) - C(x,y,z,t)\right)\Delta x \Delta y \Delta z$

Mass in at $\left(z-\frac{\Delta z}{2}\right) = j_z\Big|_{\left(z-\frac{\Delta z}{2}\right)} \Delta x \Delta y \Delta t$

Mass in at $\left(y-\frac{\Delta y}{2}\right) = j_y\Big|_{y-\frac{\Delta y}{2}} \Delta x \Delta z \Delta t$

Mass in at $\left(x-\frac{\Delta x}{2}\right) = j_x\Big|_{x-\Delta x/2} \Delta y \Delta z \Delta t$

Mass out at $\left(z+\frac{\Delta z}{2}\right) = j_z\Big|_{z+\Delta z/2} \Delta x \Delta y \Delta t$

Mass out at $\left(y+\frac{\Delta y}{2}\right) = j_y\Big|_{y+\Delta y/2} \Delta x \Delta z \Delta t$

Mass out at $\left(x+\frac{\Delta x}{2}\right) = j_x\Big|_{x+\frac{\Delta x}{2}} \Delta y \Delta z \Delta t$

---

Convection:

Mass in at $\left(z-\frac{\Delta z}{2}\right) = C u_z\Big|_{z-\frac{\Delta z}{2}} \Delta x \Delta y \Delta t$

Mass in at $\left(y-\frac{\Delta y}{2}\right) = C u_y\Big|_{y-\frac{\Delta y}{2}} \Delta x \Delta z \Delta t$

Mass in at $\left(x-\frac{\Delta x}{2}\right) = CU_x\big|_{x-\Delta x/2}\,\Delta y\,\Delta z\,\Delta t$

Mass out at $\left(z+\frac{\Delta z}{2}\right) = CU_z\big|_{z+\frac{\Delta z}{2}}\,\Delta x\,\Delta y\,\Delta t$

Mass out at $\left(y+\frac{\Delta y}{2}\right) = CU_y\big|_{y+\frac{\Delta y}{2}}\,\Delta x\,\Delta z\,\Delta t$

Mass out at $\left(x+\frac{\Delta x}{2}\right) = CU_x\big|_{x+\frac{\Delta x}{2}}\,\Delta y\,\Delta z\,\Delta t$

Production of mass $= S\,(\Delta x\,\Delta y\,\Delta z)\,\Delta t$

$$\left(c(x,y,z,t+\Delta t) - c(x,y,z,t)\right)\Delta x \Delta y \Delta z =$$

$$\left( (cu_x)\big|_{x-\frac{\Delta x}{2}} - (cu_x)\big|_{x+\Delta x/2} \right) \Delta y \Delta z \Delta t$$

$$+ \left( (cu_y)\big|_{y-\frac{\Delta y}{2}} - (cu_y)\big|_{y+\frac{\Delta y}{2}} \right) \Delta x \Delta z \Delta t$$

$$+ \left( (cu_z)\big|_{z-\frac{\Delta z}{2}} - (cu_z)\big|_{z+\frac{\Delta z}{2}} \right) \Delta x \Delta y \Delta t$$

$$+ \left( j_x\big|_{x-\frac{\Delta x}{2}} - j_x\big|_{x+\frac{\Delta x}{2}} \right) \Delta y \Delta z \Delta t$$

$$+ \left( j_y\big|_{y-\frac{\Delta y}{2}} - j_y\big|_{y+\frac{\Delta y}{2}} \right) \Delta x \Delta z \Delta t$$

$$+ \left( j_z\big|_{z-\frac{\Delta z}{2}} - j_z\big|_{z+\frac{\Delta z}{2}} \right) \Delta x \Delta y \Delta t$$

$$+ S \, \Delta x \Delta y \Delta z \Delta t$$

Divide by $\Delta x \Delta y \Delta z \Delta t$

$$\frac{c|_{t+\Delta t} - c|_t}{\Delta t} = \frac{\left(cu_x|_{x-\frac{\Delta x}{2}} - cu_x|_{x+\Delta x/2}\right)}{\Delta x} + \frac{\left(j_x|_{x-\frac{\Delta x}{2}} - j_x|_{x+\frac{\Delta x}{2}}\right)}{\Delta x}$$

$$+ \frac{\left(cu_y|_{y-\frac{\Delta y}{2}} - cu_y|_{y+\frac{\Delta y}{2}}\right)}{\Delta y} + \frac{j_y|_{y-\frac{\Delta y}{2}} - j_y|_{y+\frac{\Delta y}{2}}}{\Delta y}$$

$$+ \frac{\left(cu_z|_{z-\frac{\Delta z}{2}} - cu_z|_{z+\frac{\Delta z}{2}}\right)}{\Delta z} + \frac{\left(j_z|_{z-\frac{\Delta z}{2}} - j_z|_{z+\frac{\Delta z}{2}}\right)}{\Delta z}$$

$$\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x}(cu_x) - \frac{\partial j_x}{\partial x}$$

$$- \frac{\partial}{\partial y}(cu_y) - \frac{\partial j_y}{\partial y}$$

$$- \frac{\partial}{\partial z}(cu_z) - \frac{\partial j_z}{\partial z}$$

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(c u_x) + \frac{\partial}{\partial y}(c u_y) + \frac{\partial}{\partial z}(c u_z) = -\frac{\partial j_x}{\partial x} - \frac{\partial j_y}{\partial y} - \frac{\partial j_z}{\partial z} + S$$

$$\underline{u} = u_x \underline{e_x} + u_y \underline{e_y} + u_z \underline{e_z}$$

$$\underline{j} = j_x \underline{e_x} + j_y \underline{e_y} + j_z \underline{e_z}$$

$$\nabla = \left(\underline{e_x} \frac{\partial}{\partial x} + \underline{e_y} \frac{\partial}{\partial y} + \underline{e_z} \frac{\partial}{\partial z}\right)$$

$$\nabla \cdot \underline{j} = \left(\underline{e_x} \frac{\partial}{\partial x} + \underline{e_y} \frac{\partial}{\partial y} + \underline{e_z} \frac{\partial}{\partial z}\right)\left(j_x \underline{e_x} + j_y \underline{e_y} + j_z \underline{e_z}\right)$$

$$= \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$$

$$\nabla \cdot (c\,\underline{u}) = \frac{\partial}{\partial x}(c u_x) + \frac{\partial}{\partial y}(c u_y) + \frac{\partial}{\partial z}(c u_z)$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\,\underline{u}) = -\nabla \cdot \underline{j} + S$$

$\nabla \cdot \underline{j} =$ Divergence $(\underline{j})$

$$j_x = -D\frac{\partial c}{\partial x} \qquad j_y = -D\frac{\partial c}{\partial y} \qquad j_z = -D\frac{\partial c}{\partial z}$$

$$\underline{j} = j_x\underline{e_x} + j_y\underline{e_y} + j_z\underline{e_z}$$

$$= -D\left[\underline{e_x}\frac{\partial c}{\partial x} + \underline{e_y}\frac{\partial c}{\partial y} + \underline{e_z}\frac{\partial c}{\partial z}\right]$$

$$= -D\nabla c$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}c) = -\nabla\cdot(-D\nabla c)$$

$$= D\nabla^2 c$$

$$\nabla^2 = \nabla\cdot\nabla = \left(\underline{e_x}\frac{\partial}{\partial x} + \underline{e_y}\frac{\partial}{\partial y} + \underline{e_z}\frac{\partial}{\partial z}\right)\cdot\left(\underline{e_x}\frac{\partial}{\partial x} + \underline{e_y}\frac{\partial}{\partial y} + \underline{e_z}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(u_x c) + \frac{\partial}{\partial y}(u_y c) + \frac{\partial}{\partial z}(u_z c) = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) + S$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}c) = D\nabla^2 c + S$$

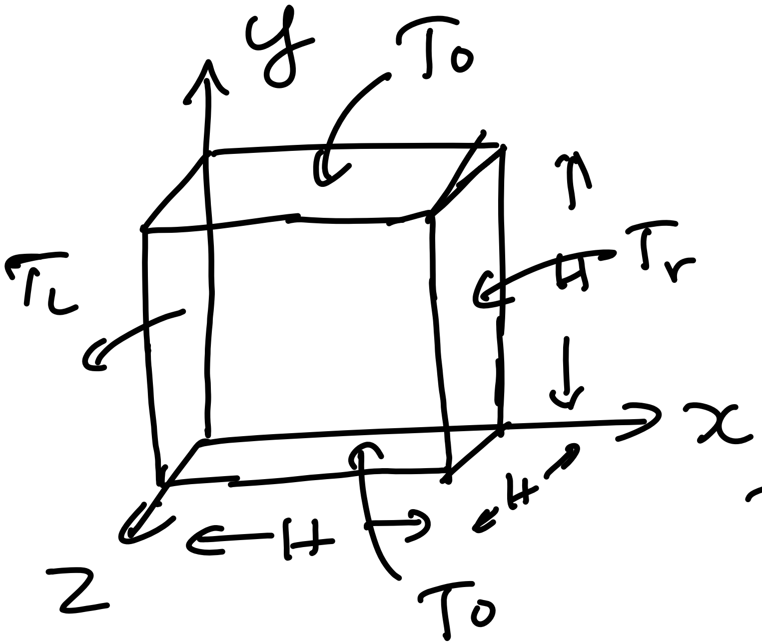
$q = -k\nabla T$

$j = -D\nabla c$

$$\rho C_p\left(\frac{\partial T}{\partial t} + \nabla\cdot(\underline{u}T)\right) = k\nabla^2 T + S_e$$

$$\boxed{\left(\frac{\partial T}{\partial t} + \nabla\cdot(\underline{u}T)\right) = \alpha\nabla^2 T + \frac{S_e}{\rho C_p}}$$

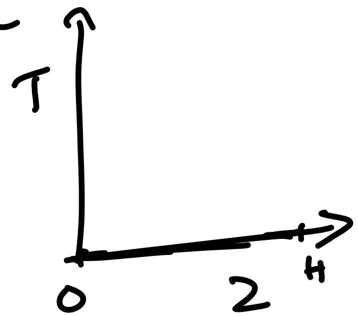
Conduction in a cube:



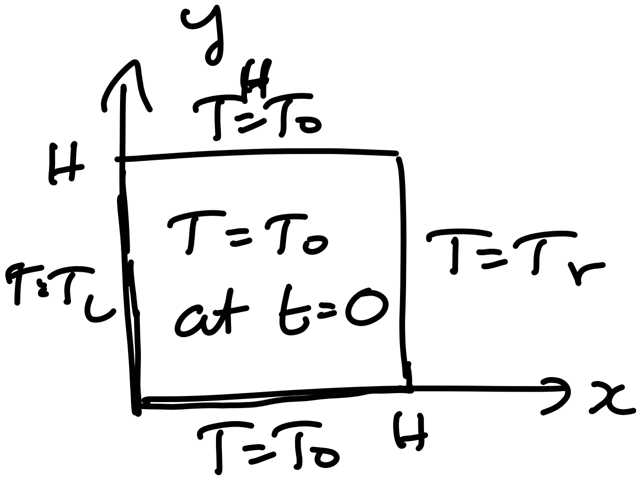

Front & back- insulated

$$\Rightarrow k\frac{\partial T}{\partial z} = 0$$

At $t=0$, $T=T_0$ everywhere



$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right)$$




$$x^* = (x/H), \quad y^* = (y/H)$$

$$T^* = \left(\frac{T - T_0}{T_0}\right)$$

$$t^* = \left(\frac{t\alpha}{H^2}\right)$$

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}$$

B.C. $T^* = 0$ at $y^* = 0$

$T^* = 0$ at $y^* = 1$

$T^* = \left(\frac{T_L - T_0}{T_0}\right) = T_L^*$ at $x^* = 0$

$= \left(\frac{T_r - T_0}{T_0}\right) = T_r^*$ at $x^* = 1$

I.C. $T^* = 0$ for all $0 < x^* < 1$ & $0 < y^* < 1$ at $t^* = 0$

$$T^* = T_t^* + T_s^*$$

$$\frac{\partial^2 T_s^*}{\partial x^{*2}} + \frac{\partial^2 T_s^{*2}}{\partial y^{*2}} = 0$$

At $y^* = 0$ & $y^* = 1$, $T_s^* = 0$

At $x^* = 0$, $T_s^* = T_c^*$

$x^* = 1$, $T_s^* = T_c^*$

$$T_s^* = X(x^*)\,Y(y^*)$$

$$Y(y^*)\frac{\partial^2 X}{\partial x^{*2}} + X(x^*)\frac{\partial^2 Y}{\partial y^{*2}} = 0$$

Divide by $XY$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^{*2}} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^{*2}} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^{*2}} = \beta_n^2 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^{*2}} = -\beta_n^2$$

$$Y = A\sin(\beta_n y^*) + B\cos(\beta_n y^*)$$

At $y^* = 0$, $Y = 0 \Rightarrow B = 0$

At $y^* = 1$, $Y = 0 \Rightarrow \beta_n = n\pi$

$$Y_n = \sin(n\pi y^*)$$

$$X = Ce^{+n\pi x^*} + De^{-(n\pi x^*)}$$

$$T_s^* = \sum_{n=1}^{\infty}\left(C_n e^{n\pi x^*} + D_n e^{-(n\pi x^*)}\right)\sin(n\pi y^*)$$

Boundary conditions in $x$-direction

At $x^* = 0$, $T_s^* = T_c^*$

$$\sum_{n=1}^{\infty} (C_n + D_n) \sin(n\pi y^*) = T_c^*$$

At $x^* = L$, $T_s^* = T_r^*$

$$\sum_{n=0}^{\infty} (C_n e^{n\pi} + D_n e^{-n\pi}) \sin(n\pi y^*) = T_r^*$$

Multiply both sides by $\sin(m\pi y^*)$ & integrate.

$$\sum_{n=1}^{\infty} (C_n + D_n)\left(\delta_{mn}/2\right) = \int_0^1 dy^* \, T_c^* \sin(m\pi y^*)$$

$$\sum_{n=1}^{\infty} (C_n e^{n\pi} + D_n e^{-n\pi})\left(\delta_{mn}/2\right) = \int_0^1 dy^* \, T_r^* \sin(m\pi y^*)$$

$$\frac{1}{2}(C_m + D_m) = \frac{2}{m\pi} T_c^*$$

$$\frac{1}{2}(C_m e^{m\pi} + D_m e^{-m\pi}) = \frac{2}{m\pi} T_r^*$$

$$C_m = \frac{4}{m\pi}\left(\frac{T_r^* - e^{-m\pi}T_l^*}{1 - e^{-m\pi}}\right)$$

$$D_m = \frac{4}{m\pi}\left(\frac{T_l^* - e^{-m\pi}T_r^*}{1 - e^{-m\pi}}\right)$$

$$T_s^* = \sum_{n=1}^{\infty}\left(C_n e^{n\pi x^*} + D_n e^{-n\pi x^*}\right)\sin(m\pi y^*)$$

Transcient temperature profile:

$$T_t^* = T^* - T_s^*$$

$$\frac{\partial T^*}{\partial t} = \frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}$$

$$0 = \frac{\partial^2 T_s^*}{\partial x^{*2}} + \frac{\partial^2 T_s^*}{\partial y^{*2}}$$

$$\frac{\partial T_t^*}{\partial t^*} = \frac{\partial^2 T_t^*}{\partial x^{*2}} + \frac{\partial^2 T_t^*}{\partial y^{*2}}$$

Boundary conditions:

At $y^*=0$, $T^*=0$, $T_s^*=0 \Rightarrow T_t^*=0$

$y^*=1$ $T^*=0$, $T_s^*=0 \Rightarrow T_t^*=0$

$x^*=0$ $T^*=T_L^*$, $T_s^*=T_L^* \Rightarrow T_t^*=0$

$x^*=1$ $T^*=T_r^*$, $T_s^*=T_r^* \Rightarrow T_t^*=0$

At $t^*=0$, $T^*=0$; $T_s^*=T_s^* \Rightarrow T_t^*=-T_s^*$

Separation of variables $T_t^*=X(x^*)Y(y^*)\Theta(t^*)$

$$\frac{1}{\Theta}\frac{\partial\Theta}{\partial t^*}=\frac{1}{X}\frac{\partial^2 X}{\partial x^{*2}}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^{*2}}$$

$$X_n=\sin(n\pi x^*)$$

$$Y_m=\sin(m\pi y^*)$$

$$\frac{1}{\Theta}\frac{\partial\Theta}{\partial t}=-(n^2+m^2)\pi^2$$

$$\Theta = A\, e^{-(n^2+m^2)\pi^2 t^*}$$

$$T_t^* = \Theta X Y$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} e^{-(n^2+m^2)\pi^2 t^*} \sin(n\pi x^*)\sin(m\pi y^*)$$

Initial condition:

At $t^* = 0$, $T_t^* = -T_s^*$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin(n\pi x^*)\sin(m\pi y^*) = -T_s^*(x^*, y^*)$$

Multiply by $\sin(p\pi x^*)\sin(q\pi y^*)$ & integrate over $0 < x^* < 1$ & $0 < y^* < 1$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\left(\frac{\delta_{np}}{2}\right)\left(\frac{\delta_{mq}}{2}\right) = -\int_0^1 dx^* \int_0^1 dy^*\, T_s^*(x^*, y^*)\sin(p\pi x^*)\sin(q\pi y^*)$$

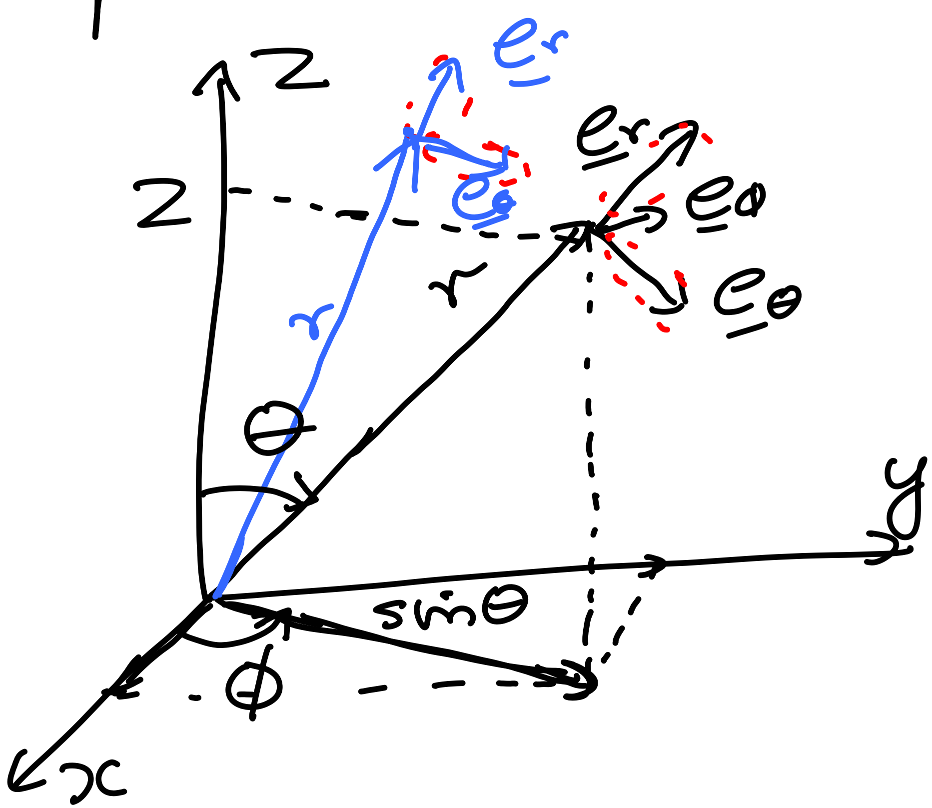
$$\frac{A_{pq}}{4} = -\int_0^1 dx^* \int_0^1 dy^*\, T_s^*(x^*, y^*)\sin(p\pi x^*)\sin(q\pi y^*)$$

$$\int_0^1 dx^* \int_0^1 dy^* \sin(p\pi x^*) \sin(q\pi y^*) \sum_{n=1}^{\infty} \left(C_n e^{n\pi x^*} + D_n e^{-n\pi x^*}\right) \sin(n\pi y^*)$$

$$= \int_0^1 dx^* \sin(p\pi x^*) \left(C_n e^{n\pi x^*} + D_n e^{-n\pi x^*}\right)\left(\frac{\delta_{nq}}{2}\right)$$

$$A_{pq} = -\int_0^1 dx^* \sin(p\pi x^*) \frac{\left(C_q e^{q\pi x^*} + D_q e^{-q\pi x^*}\right)}{2}$$

# Spherical co-ordinate system:




$$r = \sqrt{x^2+y^2+z^2}$$

$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{r}$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$0 \leq \theta \leq \pi$$

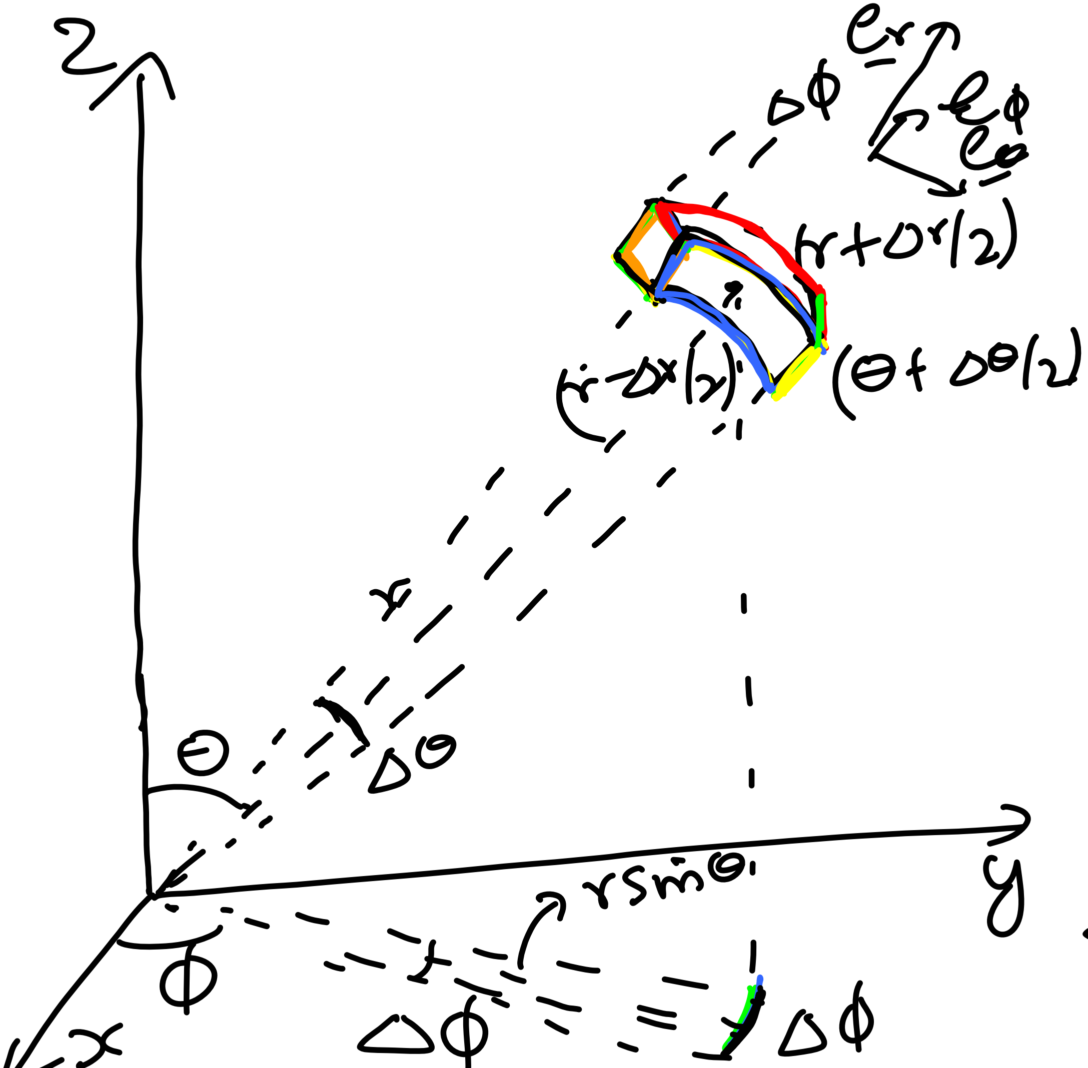
$$0 \leq \phi \leq 2\pi$$

.

.

.

.




Centered at $(r, \theta, \phi)$

Surface areas:

Surface at $(r+\Delta r/2)$

$$= (r\Delta\theta)(r\sin\theta\Delta\phi)\Big|_{r+\Delta r/2}$$

Surface at $(r-\Delta r/2)$

$$= (r\Delta\theta)(r\sin\theta\,\Delta\phi)\Big|_{r-\Delta r/2}$$

Surface at $(\phi+\Delta\phi/2)$

$$= (\Delta r)\, r\Delta\theta$$

Surface at $(\phi-\Delta\phi/2)$

$$= (\Delta r)(r\Delta\theta)$$

Surface at $(\theta-\Delta\theta/2)$

$$= (\Delta r)(r\sin\theta\ \Delta\phi)\Big|_{\theta-\Delta\theta/2}$$

Surface at $(\theta+\Delta\theta/2)$

$$= (\Delta r)(r\sin\theta\Delta\phi)\Big|_{\theta+\Delta\theta/2}$$

$$\begin{pmatrix}\text{Change in}\\ \text{mass in}\\ \text{time } \Delta t\end{pmatrix} = (\text{Mass in}) - (\text{Mass out}) + (\text{Sources})$$

$$\begin{pmatrix}\text{Change in}\\ \text{mass in } \Delta t\end{pmatrix} = \left(c(r,\theta,\phi,t+\Delta t) - c(r,\theta,\phi,t)\right) \times (\Delta r)(r\Delta\theta)(r\sin\theta\,\Delta\phi)$$

$$= c(r,\theta,\phi,t+\Delta t) - c(r,\theta,\phi,t)\, r^2 \Delta r \sin\theta\, \Delta\theta\, \Delta\phi$$

$$\begin{pmatrix}\text{Mass in at}\\ r - \Delta r/2\end{pmatrix} = j_r\,(r\Delta\theta)(r\sin\theta\,\Delta\phi)\Delta t\Big|_{(r-\Delta r/2)}$$

$$\begin{pmatrix}\text{Mass out}\\ \text{at } r + \Delta r/2\end{pmatrix} = j_r\,(r\Delta\theta)(r\sin\theta\,\Delta\phi)\Delta t\Big|_{(r+\Delta r/2)}$$

$$\begin{pmatrix}\text{Mass in at}\\ \theta - \frac{\Delta\theta}{2}\end{pmatrix} = j_\theta\,(\Delta r)(r\sin\theta\,\Delta\phi)\Big|_{\theta-\Delta\theta/2}^{\Delta t}$$

$$\begin{pmatrix}\text{Mass in out}\\ \theta+\Delta\theta/2\end{pmatrix} = j_\theta(\Delta r)(r\sin\theta\,\Delta\phi)\Big|_{\theta+\frac{\Delta\theta}{2}}\Delta t$$

$$\begin{pmatrix}\text{Mass in at}\\ \phi-\Delta\phi/2\end{pmatrix} = j_\phi(\Delta r)(r\Delta\theta)\Big|_{\phi-\Delta\phi/2}\Delta t$$

$$\begin{pmatrix}\text{Mass out at}\\ \phi+\Delta\phi/2\end{pmatrix} = j_\phi(\Delta r)(r\Delta\theta)\Big|_{\phi-\Delta\phi/2}\Delta t$$

$$(\text{Source}) = S(\Delta r)(r\Delta\theta)(r\sin\theta\,\Delta\phi)\,\Delta t$$

$$\big(c(r,\theta,\phi,t+\Delta t) - c(r,\theta,\phi;t)\big)(\Delta r)(r\Delta\theta)(r\sin\theta\,\Delta\phi)$$
$$= j_r(r\Delta\theta)(r\sin\theta\,\Delta\phi)\Big|_{r-\Delta r/2} - j_r(r\Delta\theta)(r\sin\theta\,\Delta\phi)\Big|_{r+\Delta r/2}$$
$$+ j_\theta(\Delta r)(r\sin\theta\,\Delta\phi)\Big|_{\theta-\Delta\theta/2} - j_\theta(\Delta r)(r\sin\theta\,\Delta\phi)\Big|_{\theta+\Delta\theta/2}$$
$$+ j_\phi(\Delta r)(r\Delta\theta)\Big|_{\phi-\Delta\phi/2} - j_\phi(\Delta r)(r\Delta\theta)\Big|_{\phi+\Delta\phi/2}$$

$$+ (cu_r)(r\Delta\theta)(r\sin\theta\Delta\phi)\big|_{r-\Delta r/2} - (cu_r)(r\Delta\theta)(r\sin\theta\Delta\phi)\big|_{r+\Delta r/2}$$

$$+ (cu_\theta)(\Delta r)(r\sin\theta\Delta\phi)\big|_{\theta-\Delta\theta/2} - (cu_\theta)(\Delta r)(r\sin\theta\Delta\phi)\big|_{\theta+\Delta\theta/2}$$

$$+ (cu_\phi)(\Delta r)(r\Delta\theta)\big|_{\phi+\Delta\phi/2} - (cu_\phi)(\Delta r)(r\Delta\theta)\big|_{\phi+\Delta\phi/2}$$

$$+ S\,\Delta r\; r\Delta\theta\; r\sin\theta\Delta\phi\;\Delta t$$

Divide by $(\Delta r)(r\Delta\theta)(r\sin\theta\Delta\phi)(\Delta t)$

$$\frac{c(r,\theta,\phi,t+\Delta t) - c(r,\theta,\phi,t)}{\Delta t} = \frac{1}{r^2\Delta r}\left( j_r r^2\big|_{r-\frac{\Delta r}{2}} - j_r r^2\big|_{r+\Delta r/2}\right]$$

$$+ \frac{1}{r\sin\theta\Delta\theta}\left[ j_\theta \sin\theta\big|_{\theta-\Delta\theta/2} - j_\theta\sin\theta\big|_{\theta+\Delta\theta/2}\right]$$

$$+ \frac{1}{r\sin\theta\Delta\phi}\left[ j_\phi\big|_{\phi-\frac{\Delta\phi}{2}} - j_\phi\big|_{\phi+\frac{\Delta\phi}{2}}\right]$$

$$+ \frac{1}{r^2\Delta r}\left[ cu_r r^2\big|_{r-\frac{\Delta r}{2}} - cu_r r^2\big|_{r+\Delta r/2}\right]$$

$\vdots$

$$+\frac{1}{r\sin\theta\,\Delta\theta}\left(cu_\theta\sin\theta|_{\theta-\frac{\Delta\theta}{2}}-cu_\theta\sin\theta|_{\theta+\Delta\theta/2}\right)$$

$$+\frac{1}{r\sin\theta\,\Delta\phi}\left(cu_\phi|_{\phi-\frac{\Delta\phi}{2}}-cu_\phi|_{\phi+\frac{\Delta\phi}{2}}\right)$$

$$+S$$

$$\frac{\partial c}{\partial t}=-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 j_r)-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, j_\theta)-\frac{1}{r\sin\theta}\frac{\partial j_\phi}{\partial\phi}$$

$$-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 cu_r)-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, cu_\theta)-\frac{1}{r\sin\theta}\frac{\partial(cu_\phi)}{\partial\phi}$$

$$+S$$

$$\frac{\partial c}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 cu_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, cu_\theta)+\frac{1}{r\sin\theta}\frac{\partial(cu_\phi)}{\partial\phi}$$

$$=-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 j_r)-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, j_\theta)+\frac{1}{r\sin\theta}\frac{\partial j_\phi}{\partial\phi}+S$$

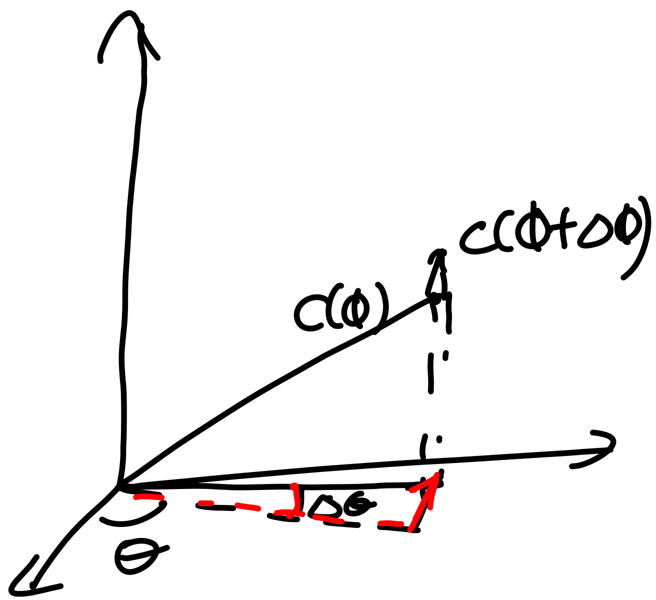
$$j_r = -D\frac{\Delta c}{\Delta r} = -D\frac{\partial c}{\partial r}$$

$$j_\theta = -D\left(\frac{c(\theta+\Delta\theta) - c(\theta)}{r\Delta\theta}\right)$$

$$= -\frac{D}{r}\frac{\partial c}{\partial \theta}$$

$$j_\phi = \frac{-D\left(c(\phi+\Delta\phi) - c(\phi)\right)}{r\sin\theta\,\Delta\phi}$$

$$= -\frac{D}{r\sin\theta}\frac{\partial c}{\partial \phi}$$




$$\frac{\partial c}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 c u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, c u_\theta) + \frac{1}{r\sin\theta}\frac{\partial(c u_\phi)}{\partial\phi}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 j_r) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, j_\theta) - \frac{1}{r\sin\theta}\frac{\partial j_\phi}{\partial\phi}$$

$$+ S$$

$$\frac{\partial c}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 c u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, c u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial(c u_\phi)}{\partial\phi}$$

$$= D\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 c}{\partial\phi^2}\right] + S$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}\, c) = D\nabla^2 c + S$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}\, c) = -\nabla\cdot j + S$$

$$j = -D\nabla c$$

$$\nabla c = \underline{e}_r\frac{\partial c}{\partial r} + \frac{\underline{e}_\theta}{r}\frac{\partial c}{\partial\theta} + \frac{\underline{e}_\phi}{r\sin\theta}\frac{\partial c}{\partial\phi}$$

$$j = j_r\,\underline{e}_r + j_\theta\,\underline{e}_\theta + j_\phi\,\underline{e}_\phi$$

$$\nabla\cdot(c\mathbf{u}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 c u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, c u_\theta) + \frac{1}{r\sin\theta}\frac{\partial(c u_\phi)}{\partial\phi}$$

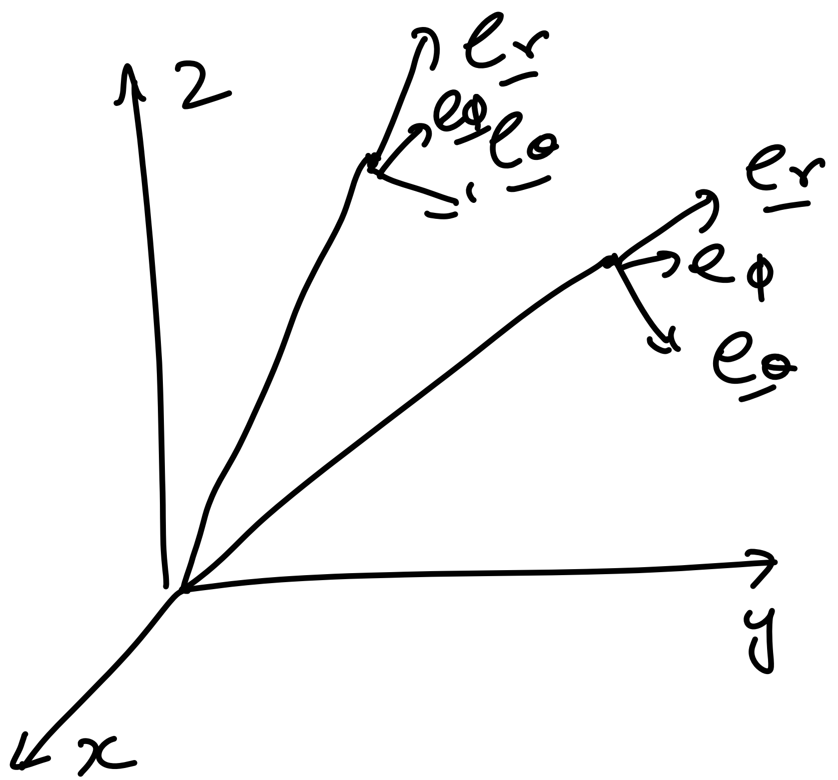
$$\nabla^2 c = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 c}{\partial\phi^2}$$

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\phi\mathbf{e}_\phi\right)$$

$$= \left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\cdot\left(A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$




$$\nabla\cdot\underline{A} = \left(\underline{e}_r\frac{\partial}{\partial r} + \underline{e}_\theta\frac{\partial}{\partial\theta} + \underline{e}_\phi\frac{\partial}{\partial\phi}\right)\cdot\left(A_r\,\underline{e}_r + A_\theta\,\underline{e}_\theta + A_\phi\,\underline{e}_\phi\right)$$

$$\nabla^2 = \nabla\cdot\nabla$$

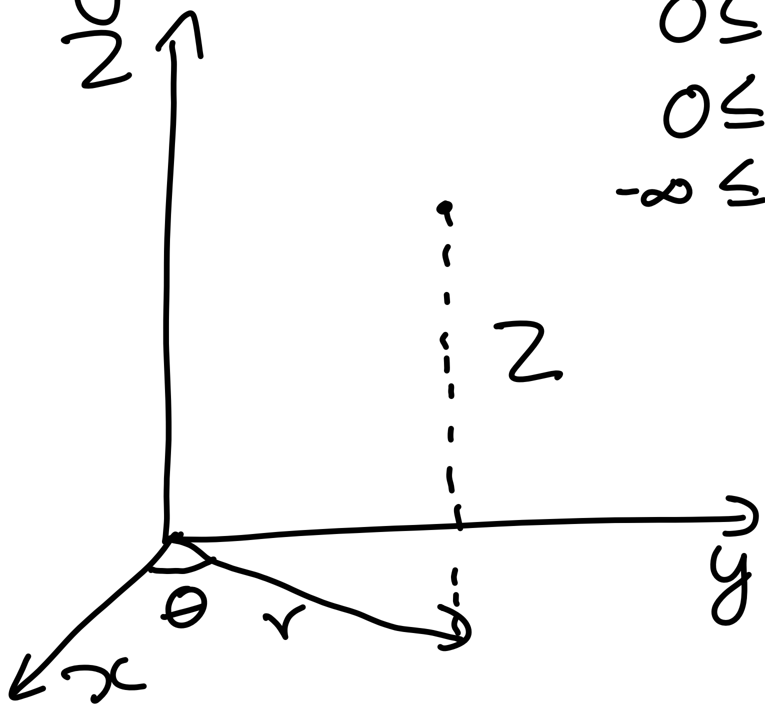
$$= \left(\underline{e}_r\frac{\partial}{\partial r} + \frac{\underline{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\underline{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\underline{e}_r\frac{\partial}{\partial r} + \frac{\underline{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\underline{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\frac{\partial T}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, T u_\theta) + \frac{1}{r\sin\theta}\frac{\partial (T u_\phi)}{\partial \phi}$$

$$= \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\right] + \frac{S_e}{\rho C_p}$$

Cylindrical co-ordinate system:




$0 \le \theta \le 2\pi$ $\quad (r, \theta, z)$

$0 \le r \le \infty$

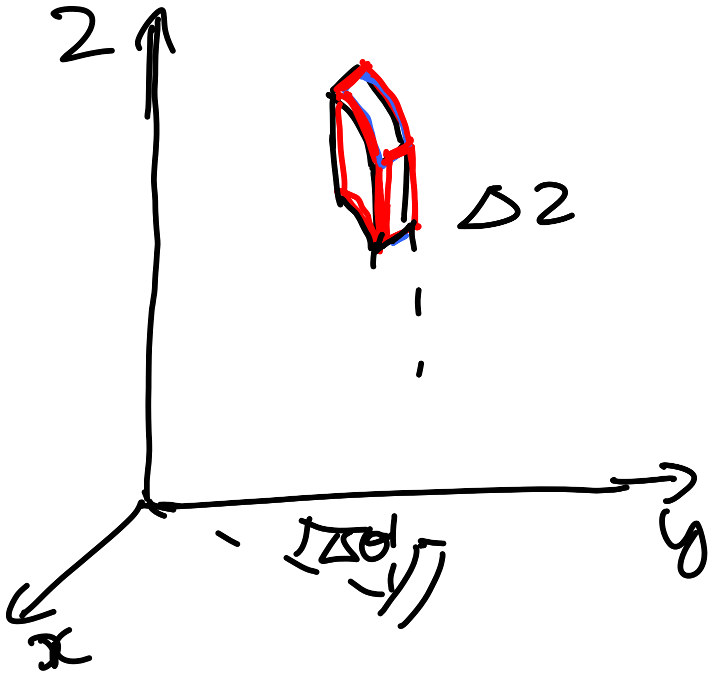
$-\infty \le z \le \infty$ $\quad r = \sqrt{x^2+y^2}$

$z = z$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

$\tan\theta = (y/x)$




$$\frac{\partial c}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r c u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(c u_\theta) + \frac{\partial}{\partial z}(c u_z)$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}(r j_r) - \frac{1}{r}\frac{\partial}{\partial \theta}(j_\theta) - \frac{\partial}{\partial z}(j_z) + S$$

$$\underline{j} = -D\left[\underline{e}_r \frac{\partial c}{\partial r} + \frac{\underline{e}_\theta}{r}\frac{\partial c}{\partial \theta} + \underline{e}_z \frac{\partial c}{\partial z}\right]$$

$$\nabla = \underline{e}_r \frac{\partial}{\partial r} + \frac{\underline{e}_\theta}{r}\frac{\partial}{\partial \theta} + \underline{e}_z \frac{\partial}{\partial z}$$

$$\nabla \cdot j = \frac{1}{r}\frac{\partial}{\partial r}(r j_r) + \frac{1}{r}\frac{\partial j_\theta}{\partial \theta} + \frac{\partial j_z}{\partial z}$$

$$\frac{\partial c}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r c u_r) + \frac{1}{r}\frac{\partial (c u_\theta)}{\partial \theta} + \frac{\partial (c u_z)}{\partial z}$$

$$= D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 c}{\partial \theta^2} + \frac{\partial^2 c}{\partial z^2}\right]$$

$$j = -D\left[\underline{e}_r \frac{\partial c}{\partial r} + \frac{e_\theta}{r}\frac{\partial c}{\partial \theta} + \underline{e}_z \frac{\partial c}{\partial z}\right]$$

$$\frac{\partial c}{\partial t} + \underline{u} \cdot \nabla c = D \nabla^2 c + S$$

$$\nabla^2 = \left(\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}\, c) = D\nabla^2 c + S$$ 'Convection-diffusion eqn'.

$Pe \ll 1$ $\quad D\nabla^2 c + S = 0$

$Pe \gg 1$ $\quad \frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}\, c) = 0$

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{u}\, c) = D\nabla^2 c + S$$

$c^* = (c/c_0)$ $\quad \underline{u}^* = (\underline{u}/U)$ ; $r^* = (r/L)$ $\quad t^* = \left(\frac{tU}{L}\right)$

$$\frac{\partial c^*}{\partial t} + \frac{U}{L}\nabla^*\cdot(\underline{u}^* c^*) = \frac{D}{L^2}\nabla^{*2} c^* + S$$

$\nabla^* = \frac{1}{L}\nabla$ $\quad \nabla^* = \left(\underline{e}_x \frac{\partial}{\partial x^*} + \underline{e}_y \frac{\partial}{\partial y^*} + \underline{e}_z \frac{\partial}{\partial z^*}\right)$

$$Pe\left(\frac{\partial c^*}{\partial t^*} + \nabla^* \cdot (u^* c)\right) = \nabla^{*2} c^* + \left(\frac{SL^2}{D}\right)$$

$$S^* = \left(SL^2/D\right)$$

$$Pe = \left(\frac{UL}{D}\right)$$

$$D\nabla^2 c + S = 0$$ Diffusion equation.

Diffusion equation:

$$\nabla^2 C = 0 \qquad \nabla^2 T = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial C}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial C}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 C}{\partial\phi^2} = 0$$

$$C(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

$$\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$r^2\sin^2\theta\left[\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right] + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

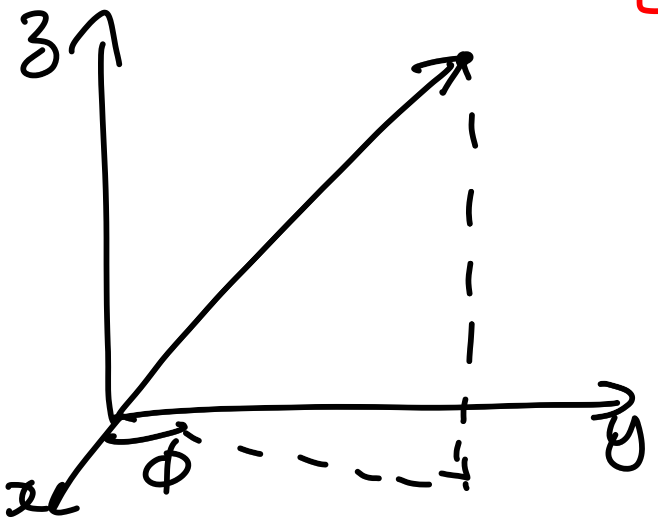
$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2$$

$\cancel{\text{If } C = +m^2,\ \Phi = Ae^{m\phi} + Be^{-m\phi}}$

$\boxed{\text{If } C = -m^2,}$ $\Phi = A\sin(m\phi) + B\cos(m\phi)$

$$\Phi(\phi + 2\pi) = \Phi(\phi)$$

$m =$ Integer




$$r^2\sin\theta\left[\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right] - m^2 = 0$$

$$\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right] + \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin\theta}\right] = 0$$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = C$$

Case where $m=0$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=C$$

$$\frac{\partial^2\Theta}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial\Theta}{\partial\theta}-C\Theta=0$$

$$\cos\theta = x$$

$$\frac{\partial}{\partial x}=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

$$\frac{1}{\Theta}\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial\Theta}{\partial x}\right)=C$$

$$(1-x^2)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}-C\Theta=0$$

Legendre equation:

Has convergent solutions only for $C=-p(p+1)$ and $n$ is an integer.

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + p(p+1)\Theta = 0$$

$$\Theta = \sum_{n=0}^{\infty} c_n x^n \qquad x = \cos\theta$$

$$\frac{d\Theta}{dx} = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$\frac{d^2\Theta}{dx^2} = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$$

$$\left(\sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}\right) - \left(\sum_{n=0}^{\infty} c_n (n(n-1)) x^n\right) - 2\sum_{n=0}^{\infty} c_n n x^n + p(p+1)\sum c_n x^n = 0$$

$$\sum_{n=2}^{\infty}\left[c_{n+2}\, n(n+1) x^n\right] - \sum_{n=0}^{\infty} c_n n(n+1) x^n + p(p+1)\sum c_n x^n = 0$$

$$(n+2)(n+1)c_{n+2} - n(n+1)c_n + p(p+1)c_n = 0$$

$$C_{n+2} = \frac{[n(n+1) - p(p+1)]C_n}{(n+2)(n+1)}$$

In the limit $n >> 1$ ; $C_{n+2} \cong C_n$

$$n(n+1) - p(p+1) = 0$$

$$C = -p(p+1)$$

$$\Theta = P_n(\cos\theta)$$

where $P_n$ = Legendre polynomial.

$$(1-x^2)\frac{\partial^2\Theta}{\partial x^2} - 2x\frac{\partial\Theta}{\partial x} + n(n+1)\Theta = 0$$

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\int_0^{\pi} \sin\theta \, d\theta \; P_n(\cos\theta) P_m(\cos\theta) = \frac{2n}{2n+1} \delta_{nm}$$

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} - \frac{m^2}{1-x^2} = -n(n+1)$$

$$\Theta = P_n^m(\cos\theta) \qquad \int \sin\theta \, d\theta \, P_n^m(\cos\theta) P_p^m(\cos\theta) = \left(\frac{2n}{2n+1}\right)\left(\frac{(n+m)!}{(n-m)!}\right)\delta_{np}$$

$$|m| \leq n$$

$$\Theta\Phi = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\theta,\phi)$$

$$Y_n^m(\theta,\phi) = P_n^m(\cos\theta)\begin{pmatrix}\sin(m\phi) \\ \cos(m\phi)\end{pmatrix}$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \; Y_n^m(\theta,\phi) \, Y_p^q(\theta,\phi) = \frac{2n}{2n+1}\left(\frac{(n+m)!}{(n-m)!}\right)\delta_{np}\,\delta_{mq}$$

$$\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{n(n+1)}{r^2}=0$$

$$r^2\frac{\partial^2 R}{\partial r^2}+2r\frac{\partial R}{\partial r}-n(n+1)R=0$$

$$R=r^\alpha$$

$$\alpha(\alpha-1)+2\alpha-n(n+1)=0$$

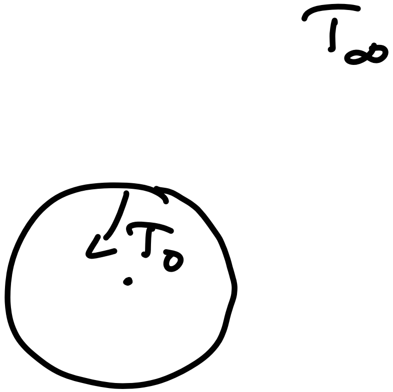
$$\alpha=n,\ -(n+1)$$

$$R=A_n r^n+B_n r^{-(n+1)}$$

$$\Theta=P_n^m(\cos\theta)\ ;\quad \Phi=\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}$$

$$\Theta\Phi=Y_n^m(\theta,\phi)$$

$$C=\sum_{n=0}^{\infty}\sum_{m=-n}^{+n}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)Y_n^m(\theta,\phi)$$



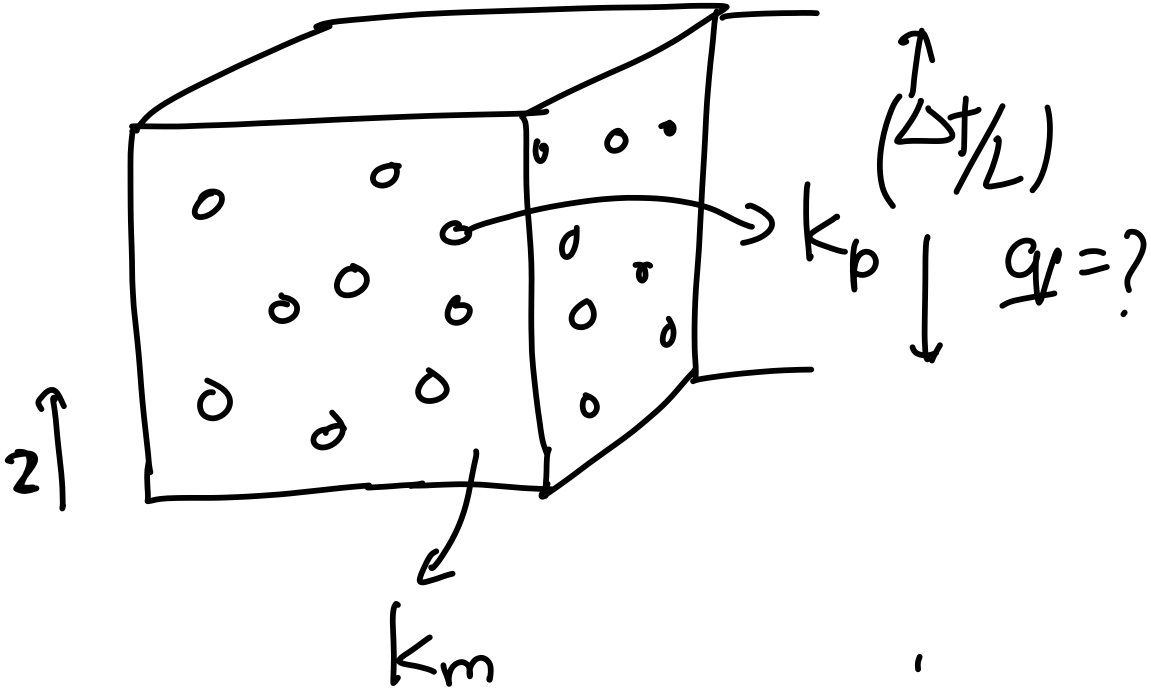

$$T = T_\infty + \frac{(T_0 - T_\infty)R}{r}$$

$n = 0$ & $m = 0$

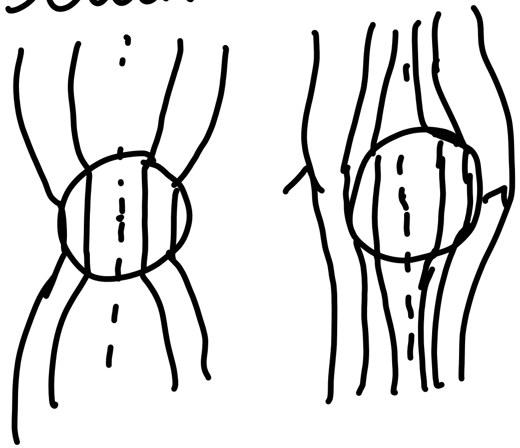
$$T = A_0 + \frac{B_0}{r}$$

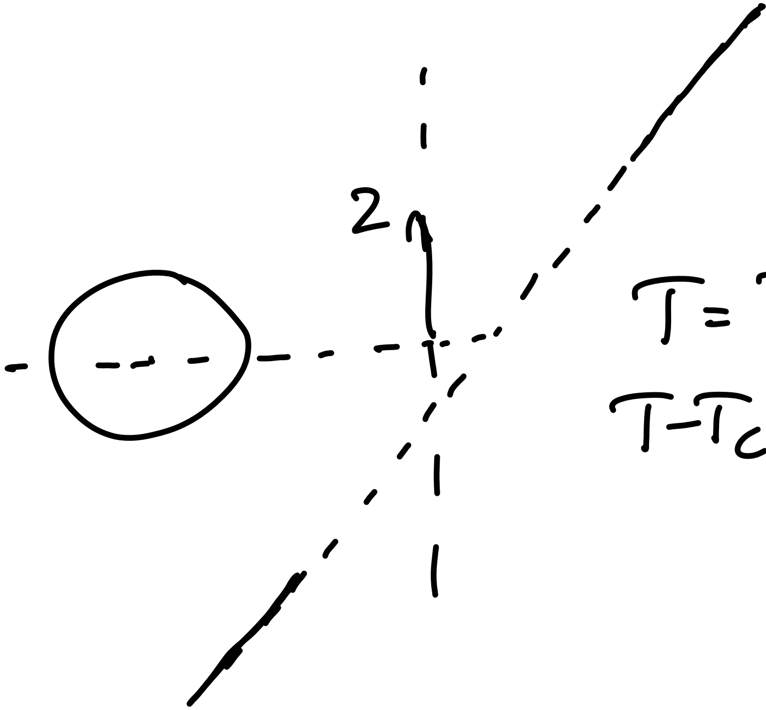
$$T = T_\infty + \frac{Q}{4\pi k r}$$

# Effective conductivity of a composite:




'Dilute'






$$T = T_c + T' z$$

$$T - T_c = T' z$$

$$\langle q_z \rangle = -k_{eff} \left\langle \frac{dT}{dz} \right\rangle = -k_{eff} T'$$

$$\langle q_z \rangle = \frac{1}{V} \int dV \, q_z$$

$$= \frac{1}{V} \left[ \int_{particles} dV \, q_z + \int_{matrix} dV \, q_z \right]$$

For particles, $q_z = -k_p \frac{\partial T}{\partial z}$

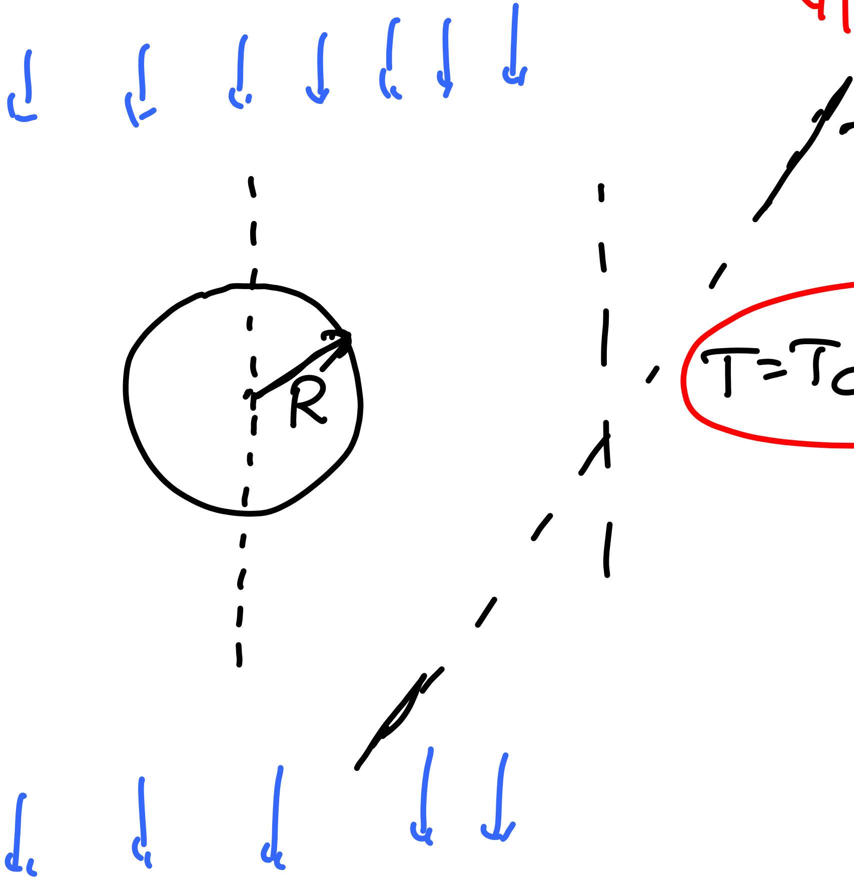
For matrix $\quad q_z = -k_m \frac{\partial T}{\partial z}$

$$\langle q_z \rangle = \frac{1}{V_{particle}} \int dV \left( -k_p \frac{\partial T}{\partial z} \right) + \frac{1}{V_{matrix}} \int dV \left( -k_m \frac{\partial T}{\partial z} \right)$$

$$= \frac{1}{V}\int_{\text{total volume}} dV\left(-k_m \frac{dT}{dz}\right) + \frac{1}{V_{\text{particle}}}\int dV\left(-(k_p-k_m)\frac{dT}{dz}\right)$$

$$= -k_m\left\langle \frac{dT}{dz}\right\rangle + \frac{1}{V_{\text{particle}}}\int dV\left(-(k_p-k_m)\right)\frac{dT}{dz}$$

$$= -k_m T' + \frac{N}{V_{\text{particle}}}\int dV\left(-(k_p-k_m)\right)\frac{dT}{dz}$$




$T'z$

$T = T_c + T'z$

$\nabla^2 T_p = 0$

$\nabla^2 T_m = 0$

At $r = R$,

$T_p = T_m$

$-q_r|_p = -q_r|_m$

As $r \to \infty$, $T = T'z$

$$T_p = \sum_{n=0}^{\infty}\left(A_{pn} r^n + \frac{B_{pn}}{r^{n+1}}\right) P_n(\cos\theta) \qquad T'_2 = T' r P_1(\cos\theta)$$

$$T_m = \sum_{n=0}^{\infty}\left(A_{mn} r^n + \frac{B_{mn}}{r^{n+1}}\right) P_n(\cos\theta)$$

At $r = R$, $T_p = T_m$

$$\sum_{n=0}^{\infty}\left(A_{pn} R^n + \frac{B_{pn}}{R^n}\right) P_n(\cos\theta) = \sum_{n=0}^{\infty}\left(A_{mn} R^n + \frac{B_{mn}}{R^n}\right) P_n(\cos\theta)$$

$$A_{pn} R^n + \cancel{\frac{B_{pn}}{R^n}} = A_{mn} R^n + \frac{B_{mn}}{R^n}$$

$$q_r = -k_p \frac{\partial T_p}{\partial r}\bigg|_{r=R} = -k_m \frac{\partial T_m}{\partial r}\bigg|_{r=R}$$

$$k_p\left[A_{pn}(nR^{n-1}) - \cancel{\frac{B_{pn}(n+1)}{R^{n+2}}}\right] = k_m\left[A_{mn}\,nR^{n-1} - \frac{B_{mn}(n+1)}{R^{n+2}}\right]$$

At $r=0$, $\frac{\partial T_p}{\partial r} = 0 \Rightarrow B_{pn} = 0$ for all $n$

As $r \to \infty$, $T = T'z = T'r\cos\theta = T'r P_1(\cos\theta)$

$$\sum_{n=0}^{\infty}\left(A_{mn}r^n + \frac{B_{mn}}{r^{n+1}}\right)P_n(\cos\theta) = T'r\cos\theta$$

$$= T'r P_1(\cos\theta)$$

$$= T'r\,\delta_{m1}$$

$$\Rightarrow A_{m1} = T' \text{ \& } A_{mn} = 0 \text{ for } n \neq 1$$

For $n=1$,

$$A_{p1} R = \underbrace{A_{m1} R}_{T} + \frac{B_{m1}}{R^2}$$

$$k_p A_{p1} = k_m A_{m1} - \frac{2 B_{m1}}{R^3}$$

For $n>1$

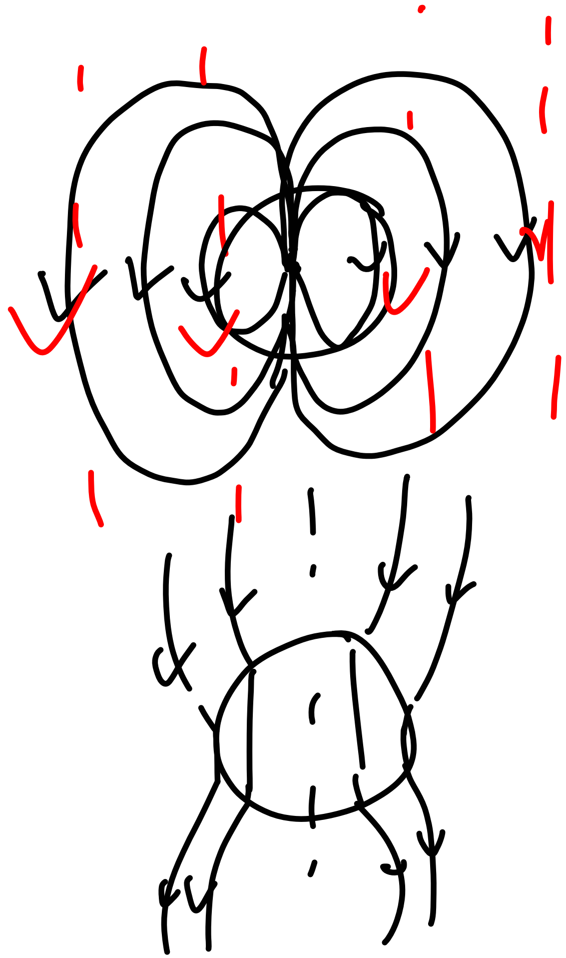
$$A_{pn} R^n = \frac{B_{mn}}{R^{n+1}}$$

$$k_p A_{pn}\, n (R^{n-1}) = -\frac{k_m B_{mn} (n+1)}{R^{n+2}}$$

$A_{pn} = 0$ & $B_{mn} = 0$ for $n>1$

Matrix

$$T = T' r P_1^0(\cos\theta) + \frac{B_{m1}}{r^2} P_1^0(\cos\theta)$$



$$A_{p1} = \frac{3T'}{(2 + k_p/k_m)} \qquad B_{m1} = \frac{(1 - k_p/k_m) R^3 T'}{(2 + k_p/k_m)}$$

$$T_p = \frac{3T' R r P_1(\cos\theta)}{2 + k_R} = \frac{3T' R z}{2 + k_R}$$

$$T_m = T' r P_1(\cos\theta) + \frac{(1 - k_R) R^3 T' P_1(\cos\theta)}{2 + k_R}$$

where $k_R = (k_p/k_m)$

$$\langle q_z \rangle = -k_m T' + \frac{N}{V} \int dV \left( -(k_p - k_m) \right) \frac{\partial T}{\partial z}$$

$$= \left[ -k_m T' + \frac{N}{V} \int dV \left[ -(k_p - k_m) \right] \left( \frac{3T' R}{2 + k_R} \right) \right]$$

$$= -\left[ k_m T' + \frac{N V_p}{V} (k_p - k_m)\left(\frac{3T'}{2+k_R}\right)\right]$$

$$= -\underbrace{\left[ k_m + \phi_v \frac{(k_p - k_m) 3}{2+k_R}\right]}T'$$

$$k_{eff} = k_m \left[ 1 + \phi_v \frac{3(k_R - 1)}{2+k_R}\right]$$

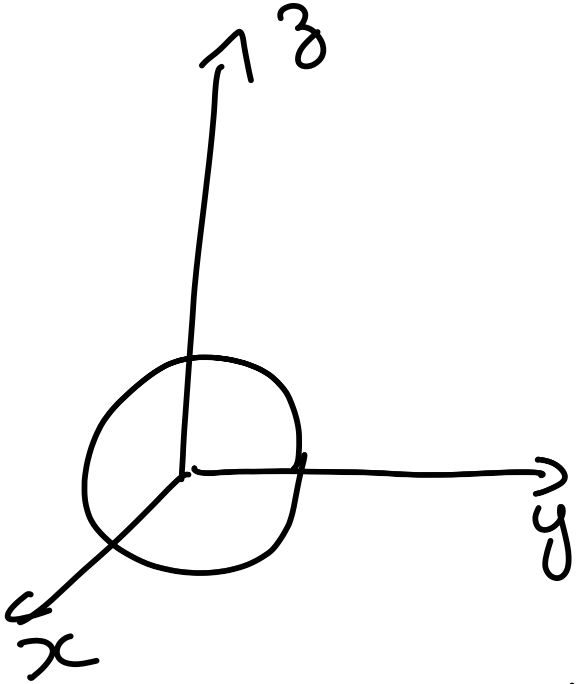
where $k_R = (k_p / k_m)$

Forcing term $= T' z = T' r \cos\theta$
$= T' r \, Y_{10}(\theta, \phi)$

$Y_{10} = P_1^0(\cos\theta)$

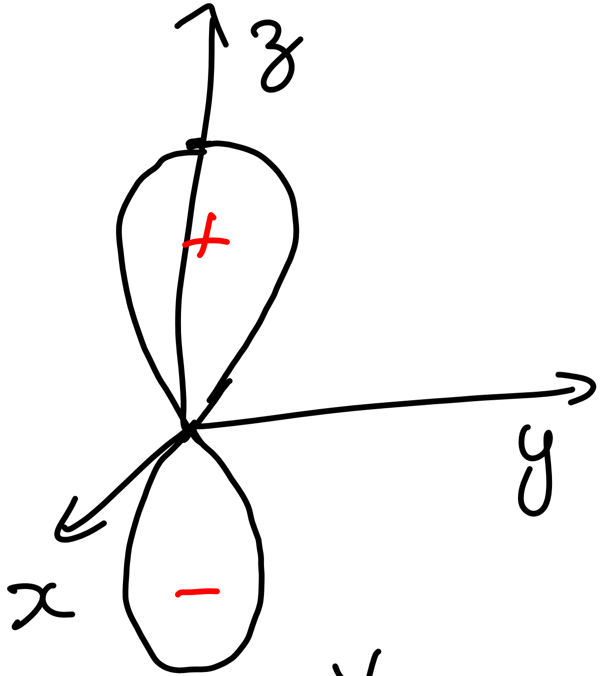
Symmetry - $Y_{10}(\theta, \phi)$

$Y_{00} = 1$



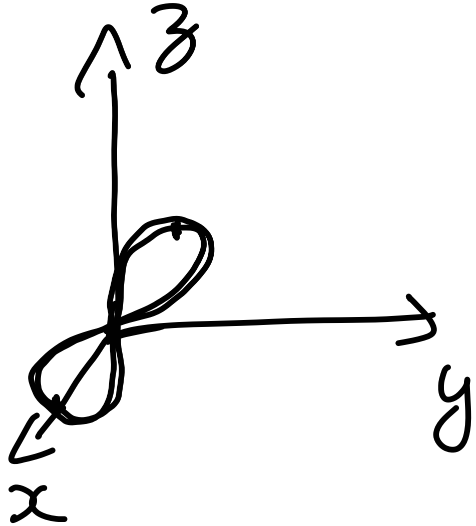

$Y_{10} = P_1^0(\cos\theta)$

$= \cos\theta$



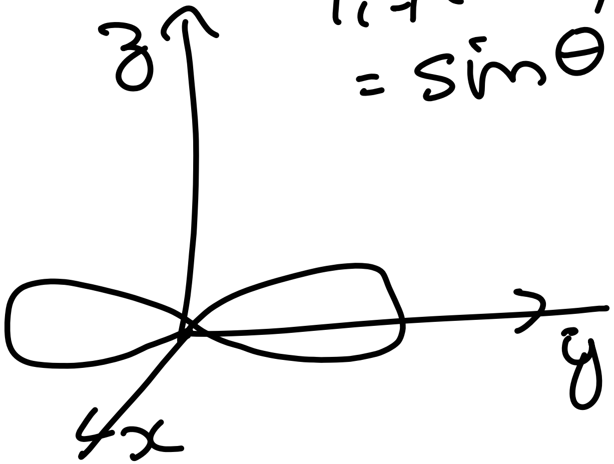

$Y_{11} = P_1^1(\cos\theta)\cos\phi$

$= \sin\theta \cos\phi$

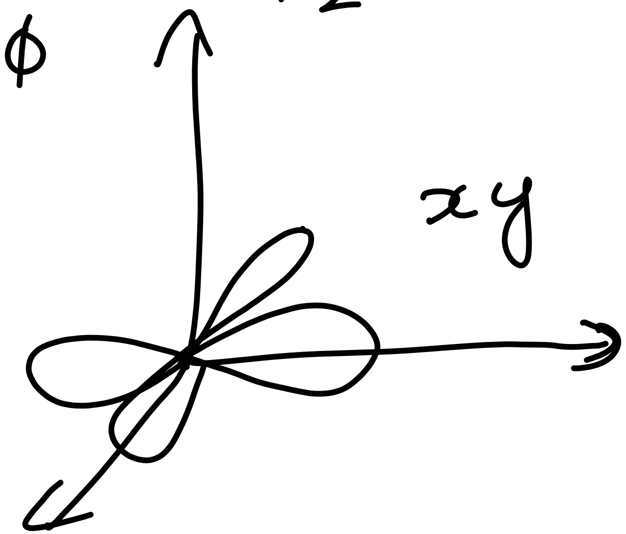


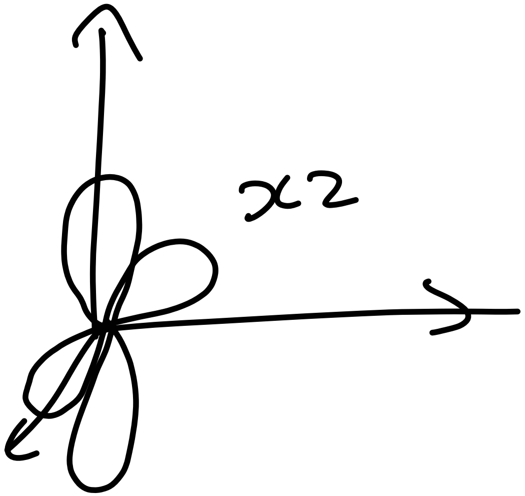

$Y_{1-1}(\theta,\phi)$

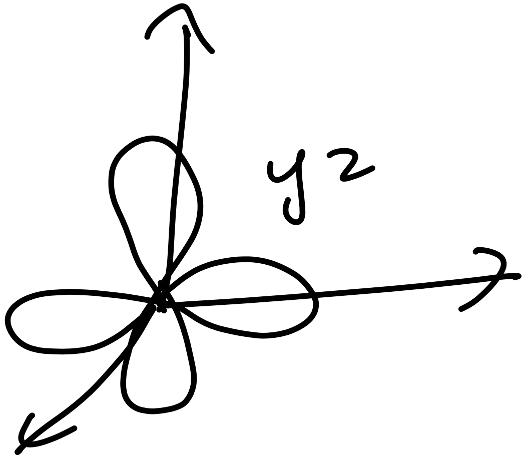
$= \sin\theta \sin\phi$

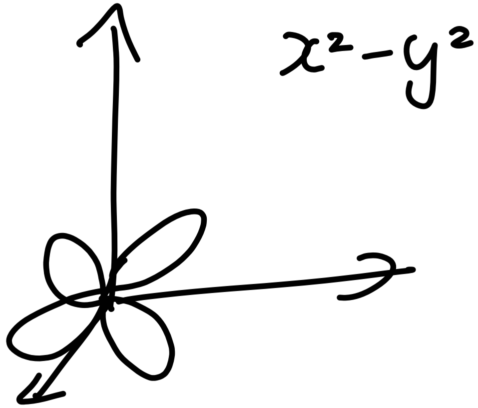


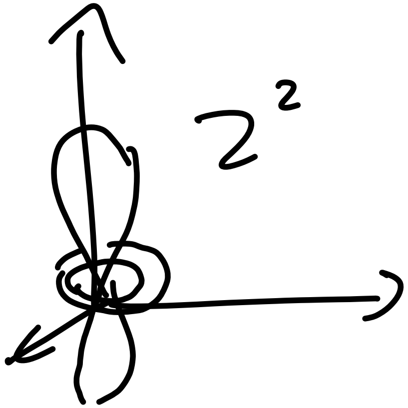

$Y_2$





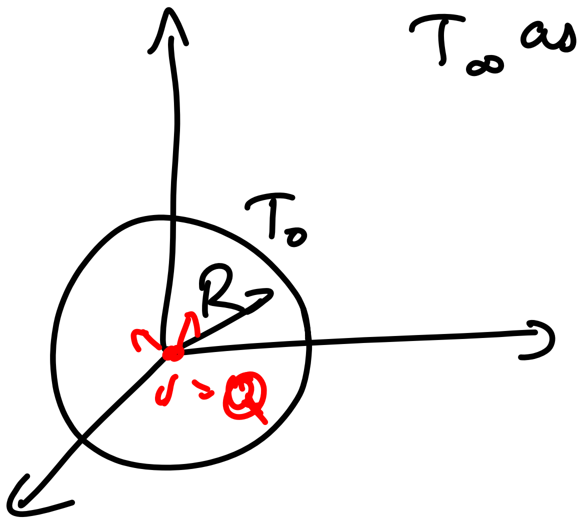




$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi = E\psi$$

Source, dipole, . . . .




$T_\infty$ as $r \to \infty$

$$T - T_\infty = \frac{(T_0 - T_\infty)R}{r}$$

$$q_r = -k\frac{\partial T}{\partial r} = +\frac{k(T_0 - T_\infty)R}{r^2}$$

$$Q = 4\pi r^2 q_r = k(T_0 - T_\infty)(4\pi R)$$

$$T - T_\infty = \frac{Q}{4\pi k r}$$

'Point source' $R \to 0$

$$k\nabla^2 T + S_e = 0$$

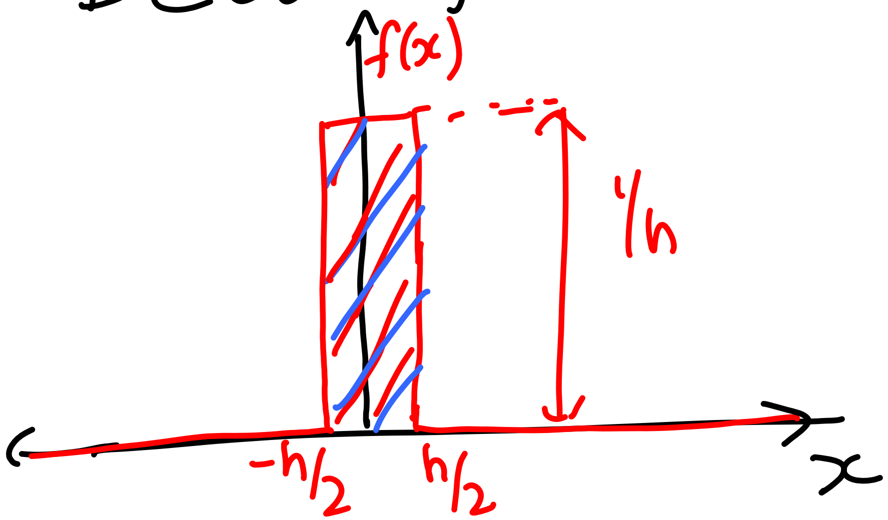
Delta function:

$$k\nabla^2 T + Q\delta(\underline{x}) = 0$$

$$Q\delta(\underline{x}) = 0 \text{ for } \underline{x} \neq 0$$

$$\int dV \, Q\,\delta(\underline{x}) = Q$$

Delta function:




Area = 1

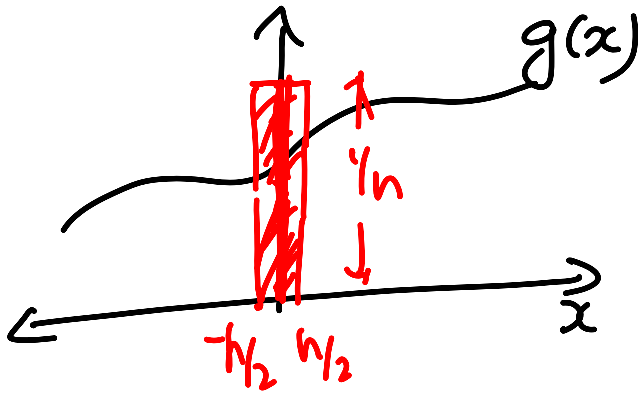
$$\int_{-h/2}^{h/2} dx\, f(x) = 1$$

$$\int_{-\infty}^{\infty} dx\, f(x) = 1$$

$$\delta(x) = \lim_{h \to 0} \big(f(x)\big)$$

$$\delta(x) = 0 \text{ for } x \neq 0$$

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1$$




$$\int_{-\infty}^{\infty} dx\, \delta(x)\, g(x) = g(0)$$

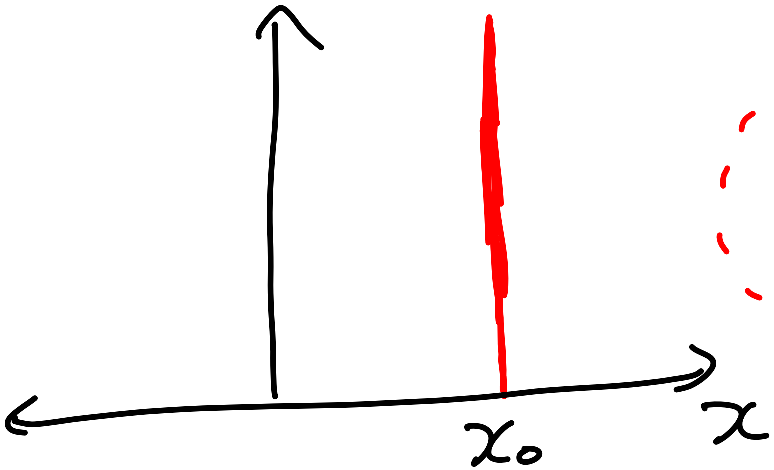
$$\int_{-\infty}^{\infty} dx\, f(x)\, g(x) = \int_{-h/2}^{h/2} dx\, (1/h)\, g(x)$$

$$= \int_{-h/2}^{h/2} dx \left(\frac{1}{h}\right)\left[ g(0) + x \left.\frac{dg}{dx}\right|_{x=0} + \frac{x^2}{2}\left.\frac{d^2g}{dx^2}\right|_{x=0} + \cdots \right]$$

$$= \int_{-h/2}^{h/2} dx\, \frac{1}{h}\left(g(0)\right) + \frac{1}{h}\left.\frac{dg}{dx}\right|_{x=0} \int_{-h/2}^{h/2} dx\, x$$

$$+ \frac{1}{h}\left.\frac{d^2g}{dx^2}\right|_{x=0} \int_{-h/2}^{h/2} dx\, x^2$$

$$= g(0)$$




$\delta(x-x_0) \neq 0$ any for $x = x_0$

$$\int_{-\infty}^{\infty} dx \, \delta(x-x_0) = 1$$

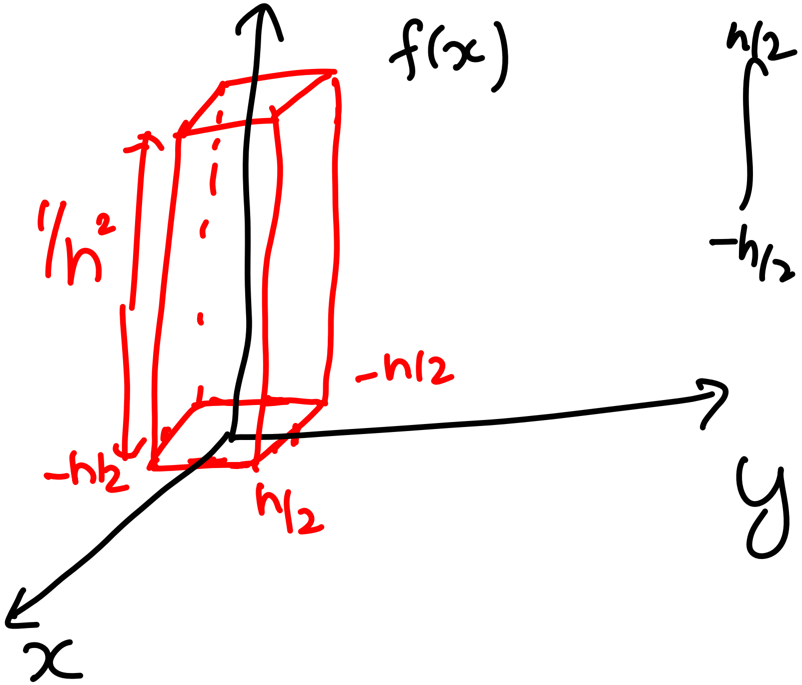
$$\int_{-\infty}^{\infty} dx \, \delta(x-x_0) \, g(x) = g(x_0)$$

Function $f(x,y) = 1/h^2$ for $-h/2 < x < h/2$

& $-h/2 < y < h/2$

$= 0$ otherwise

$$\delta(x,y) = \lim_{h \to 0} f(x,y)$$




$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} dx\, dy\, f(x,y) = 1$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x,y) = 1$$

$$\lim_{h\to 0} f(x,y) = \delta(x,y)$$

Two-dimensional delta function:

$$\boxed{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \delta(x,y) = 1}$$

$\delta(x, y) = 0$ for $x \neq 0$ or $y \neq 0$

$\neq 0$ only for $x = 0$ & $y = 0$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \delta(x, y)\, g(x, y) = g(0,0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0)\, g(x, y) = g(x_0, y_0)$$

Three dimensional delta function:

$f(x, y, z) = 1/h^3$ for $-h/2 < x < h/2$

& $-h/2 < y < h/2$

& $-h/2 < z < h/2$

$= 0$ otherwise

$$\delta(x, y, z) = \lim_{h \to 0} f(x, y, z)$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \delta(x,y,z) = 1$$

$$\delta(x,y,z) = 0 \quad \text{for} \quad x \neq 0 \text{ or } y \neq 0 \text{ or } z \neq 0$$

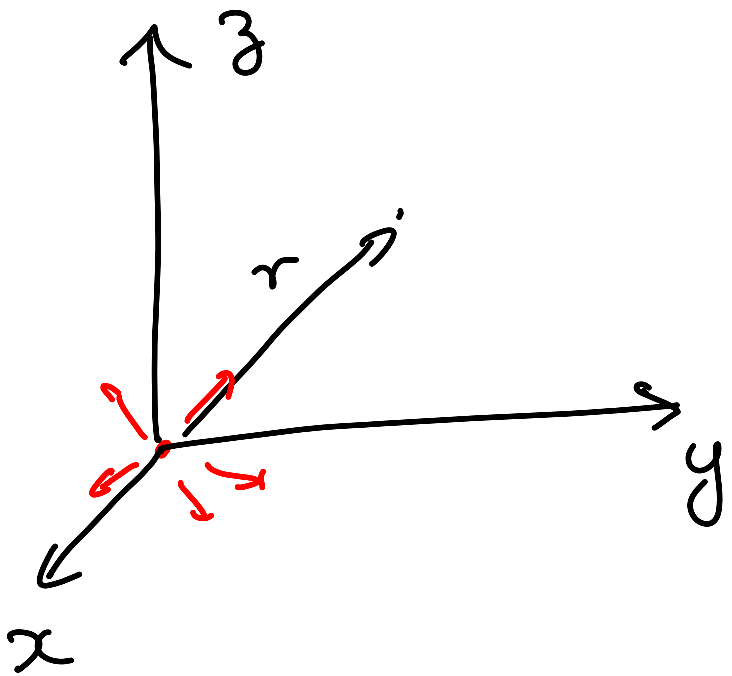
$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \delta(x,y,z)\, g(x,y,z) = g(0,0,0)$$

$$\delta(\underline{x}) = \delta(x,y,z)$$

$$\int dV \, \delta(\underline{x}) = 1$$

$$\delta(\underline{x}) = 0 \quad \text{for} \quad \underline{x} \neq 0$$

$$\int dV \, \delta(\underline{x})\, g(\underline{x}) = g(0)$$




Heat/unit time = Q

$$T = \frac{Q}{4\pi k r}$$

$$k\nabla^2 T + Q\,\delta(\underline{x}) = 0$$

For $\underline{x} \neq 0$ $\quad k\nabla^2 T = 0$

$$k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right) = 0$$

$$T = \frac{A}{r}$$

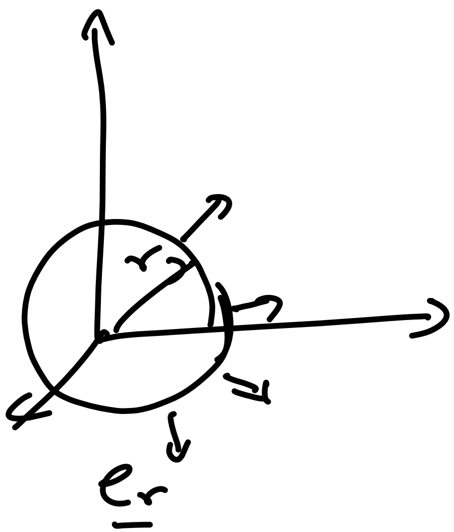
$$k\nabla^2 T = -Q\,\delta(\underline{x})$$

$$\int dV\, k\nabla^2 T = -\int dV\, Q\,\delta(\underline{x})$$

$$= -Q\int dV\,\delta(\underline{x}) = -Q$$

$$\int dV\, k\nabla^2 T = \int dV\, \nabla\cdot(k\nabla T)$$

$$= \int ds\, \underline{n}\cdot(k\nabla T)$$




$$k\nabla T = k\,\underline{e}_r \frac{\partial T}{\partial r}$$

$$= -k\,\underline{e}_r \frac{A}{r^2}$$

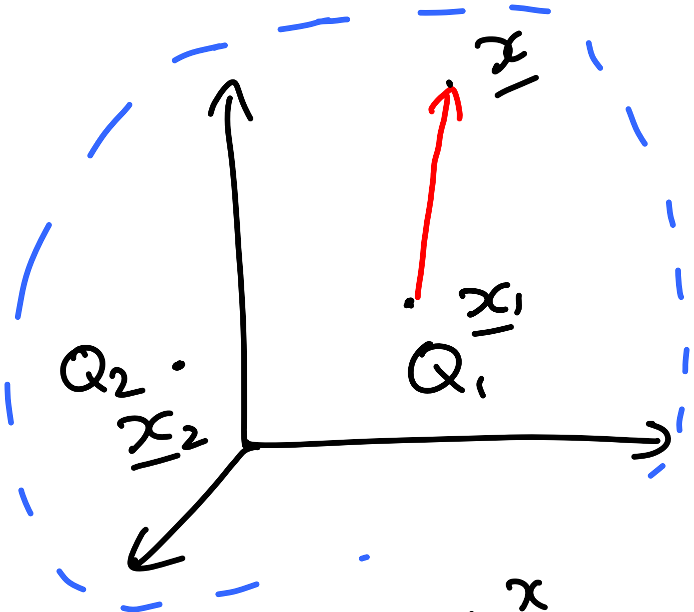
$$\underline{n}\cdot k\nabla T = \frac{-kA}{r^2}$$

$$\int ds\, \underline{n}\cdot k\nabla T = 4\pi r^2\left(\frac{-kA}{r^2}\right) = -4\pi kA$$

$$-4\pi kA = -Q \Rightarrow A = \frac{Q}{4\pi k}$$

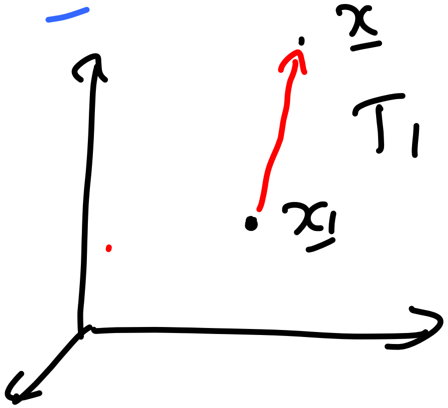
$$k\nabla^2 T + Q\,\delta(\underline{x}) = 0 \qquad k\nabla^2 T + S_e = 0$$

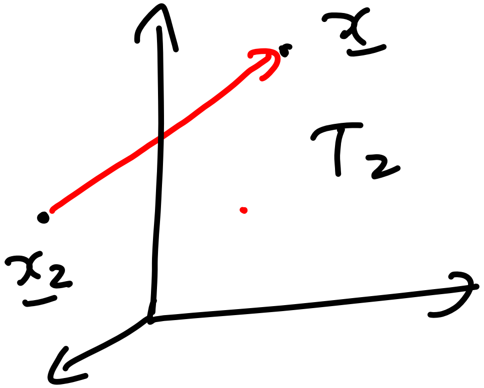
$$T = \frac{Q}{4\pi k r}$$




$$T(\underline{x}) = \frac{Q_1}{4\pi k|\underline{x}-\underline{x}_1|} + \frac{Q_2}{4\pi k|\underline{x}-\underline{x}_2|}$$

$$|\underline{x}-\underline{x}_1| = \left((x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\right)^{1/2}$$

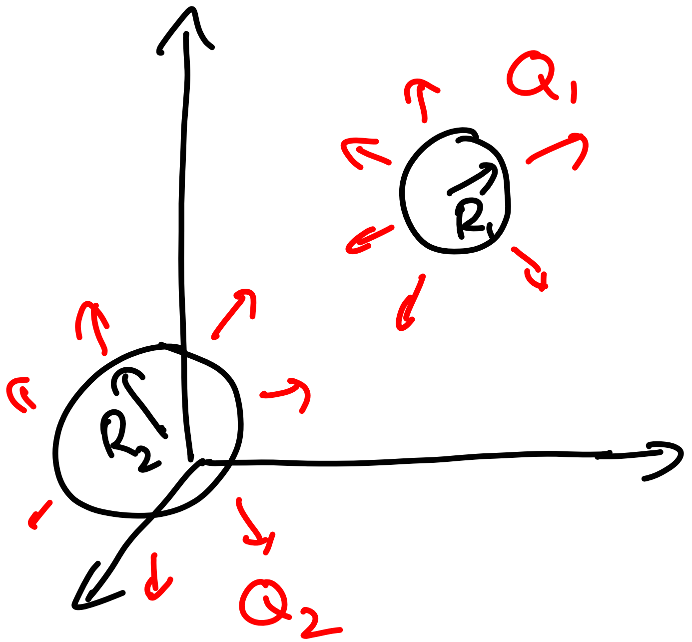


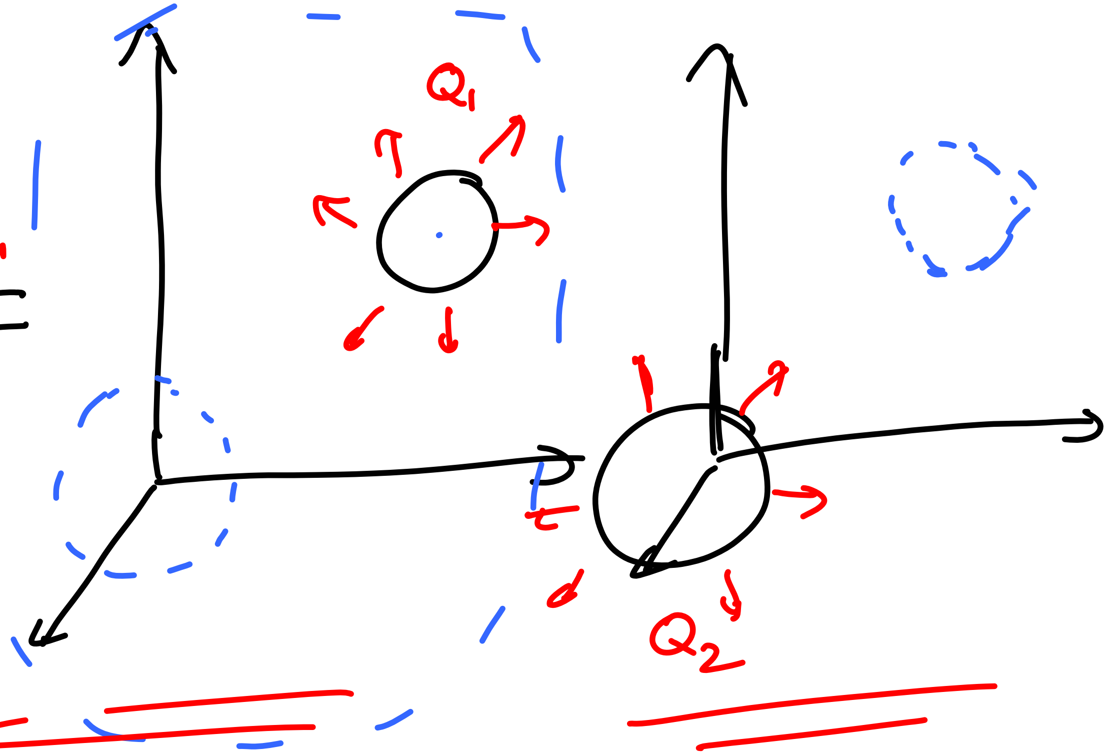

$$T_1 = \frac{Q_1}{4\pi k|\underline{x}-\underline{x}_1|}$$

$$T_2 = \frac{Q_2}{4\pi k|\underline{x}-\underline{x}_2|}$$

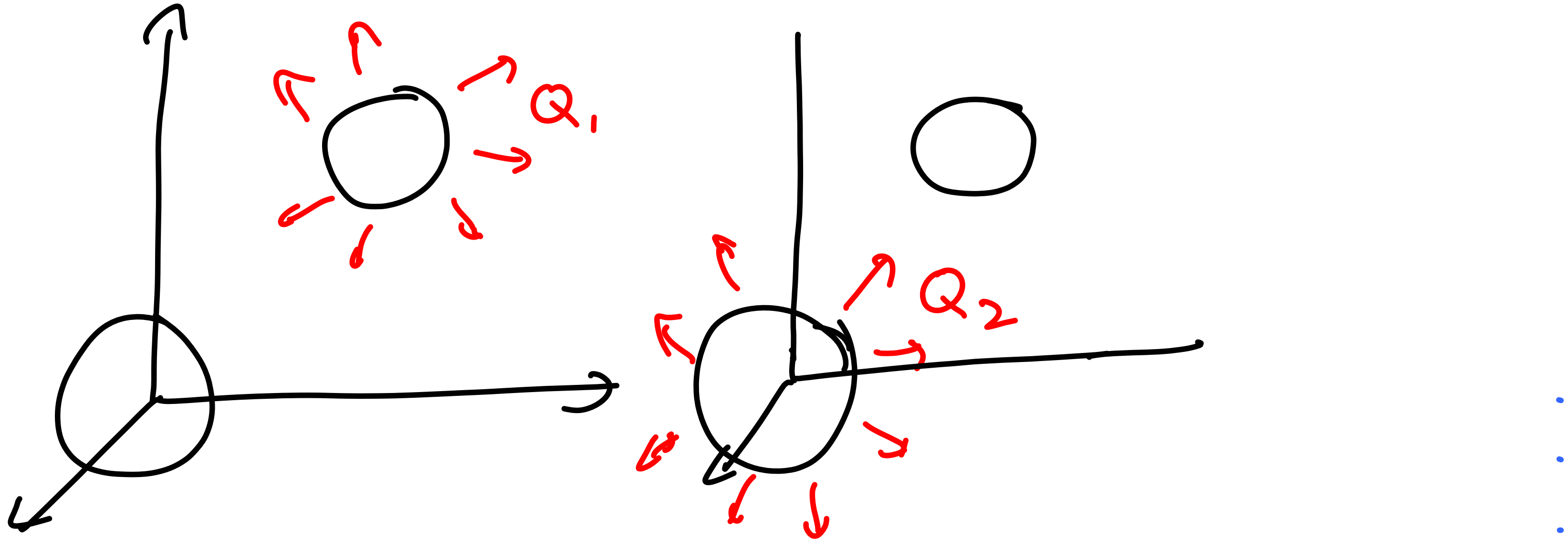
Linear superposition.

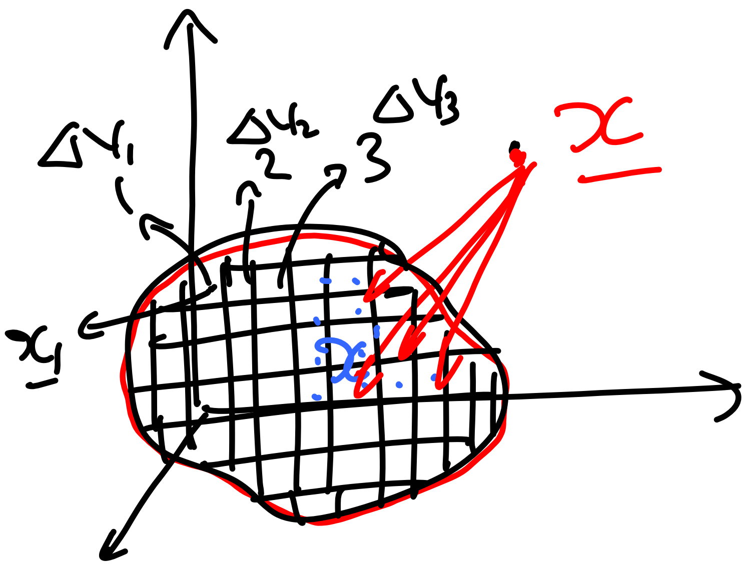



$\neq$




$=$






Heat generated = $q$ / unit volume / time

Heat generated / time in each volume

$= q(\underline{x}_1)\Delta V_1,\ q(\underline{x}_2)\Delta V_2, \ldots$

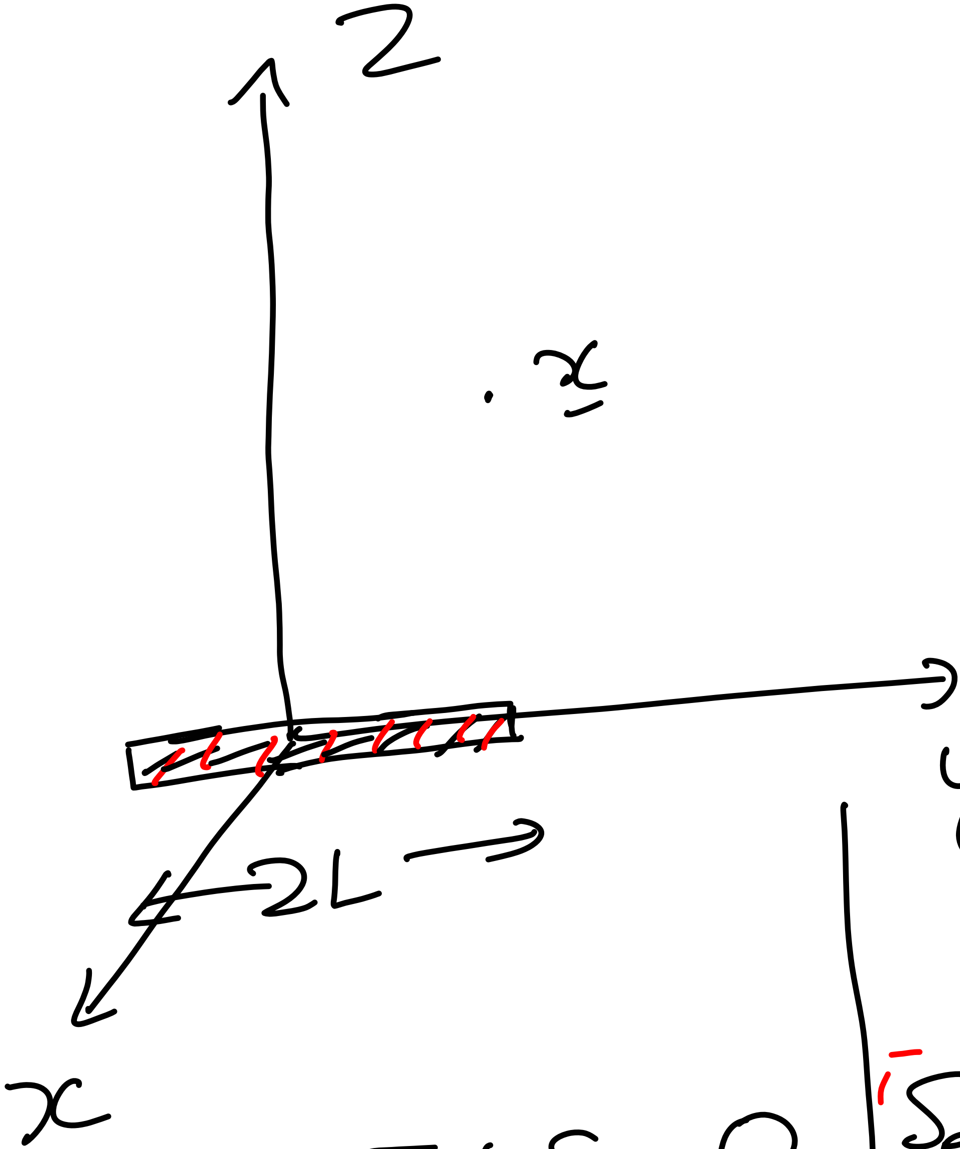
$T(\underline{x})$ due to source at $\underline{x}_1 = \dfrac{q(\underline{x}_1)\Delta V_1}{4\pi k|\underline{x}-\underline{x}_1|}$

$T(\underline{x})$ due to source at $\underline{x}_2 = \dfrac{q(\underline{x}_2)\Delta V_2}{4\pi k|\underline{x}-\underline{x}_2|}$

$$T(\underline{x}) = \sum_{i=1}^{N} \frac{q(\underline{x}_i)\,\Delta V_i}{|\underline{x} - \underline{x}_i|}$$

limit $\Delta V_i \to 0$

$$T(\underline{x}) = \int \frac{dV'\, q(\underline{x}')}{|\underline{x} - \underline{x}'|}$$




$$k\nabla^2 T + S_e = 0$$

$-L \leq y \leq L$

$Q$ / unit length / time

$$S_e(\underline{x}) = Q$$

$S_e(\underline{x}) \neq 0$ only for $x = 0$ & $z = 0$

$= 0$ otherwise

$$\int dx \int dz \, S_e(\underline{x}) = Q$$

$$S_e(\underline{x}) = Q\,\delta(x)\,\delta(z)$$

for $-L \leq y \leq L$

$$k\nabla^2 T + Q\,\delta(x)\,\delta(z) = 0$$

$$T(\underline{x}) = \frac{1}{4\pi k}\int dV' \frac{\delta(x')\,\delta(z')\,Q}{|\underline{x} - \underline{x}'|}$$

$$= \frac{1}{4\pi k}\int dx' \int dy' \int dz' \frac{Q\,\delta(x')\,\delta(z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$\int dx\,\delta(x)\,g(x) = g(0)$$

$$T(\underline{x}) = \frac{1}{4\pi k}\int dy' \int dz' \frac{Q\,\delta(z')}{\sqrt{x^2+(y-y')^2+z^2}}$$

$$= \frac{1}{4\pi k}\int dy' \frac{Q}{\sqrt{x^2+(y-y')^2+z^2}}$$

$$T(x) = \frac{Q}{4\pi k} \int_{-L}^{L} dy' \frac{1}{\sqrt{x^2 + (y-y')^2 + z^2}}$$

$$T(x) = \frac{Q}{4\pi k} \log\left[\frac{L + y + \sqrt{r^2 + (L+y)^2}}{-L + y + \sqrt{r^2 + (y-L)^2}}\right]$$

where $r^2 = (x^2 + z^2)$

Along the x-z plane, $y = 0$,

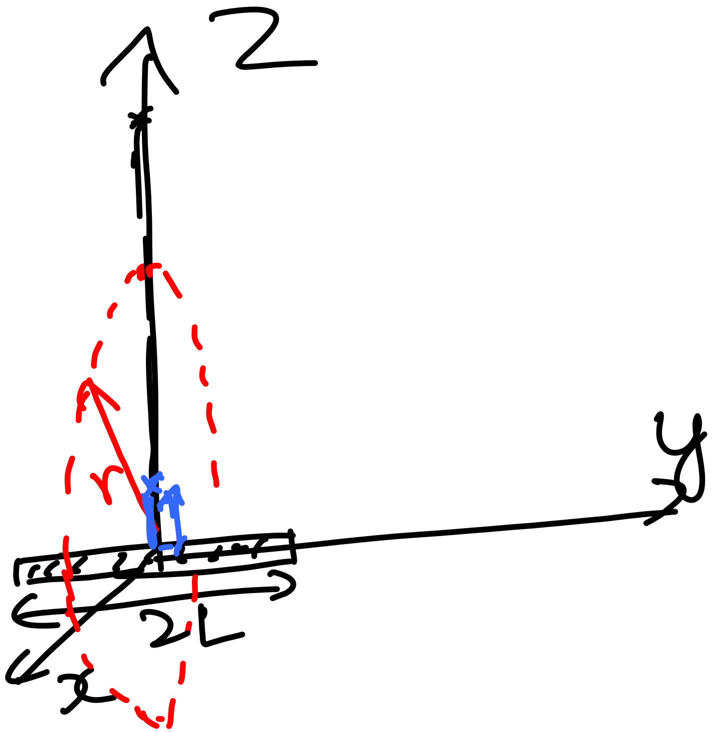
$$T = \frac{Q}{4\pi k} \log\left[\frac{L + \sqrt{r^2 + L^2}}{-L + \sqrt{r^2 + L^2}}\right]$$

① $r \gg L$

Expansion in small $(L/r)$

$$T = \frac{Q}{4\pi k} \log\left[\frac{(L/r) + \sqrt{1+(L/r)^2}}{(-L/r) + \sqrt{1+(L/r)^2}}\right]$$

$$= \frac{2QL}{4\pi k r}$$




② $r \ll L$

Expansion in small $(r/L)$

$$T = \frac{Q}{4\pi k} \log\left(\frac{1 + \sqrt{(r/L)^2+1}}{-1 + \sqrt{(r/L)^2+1}}\right)$$

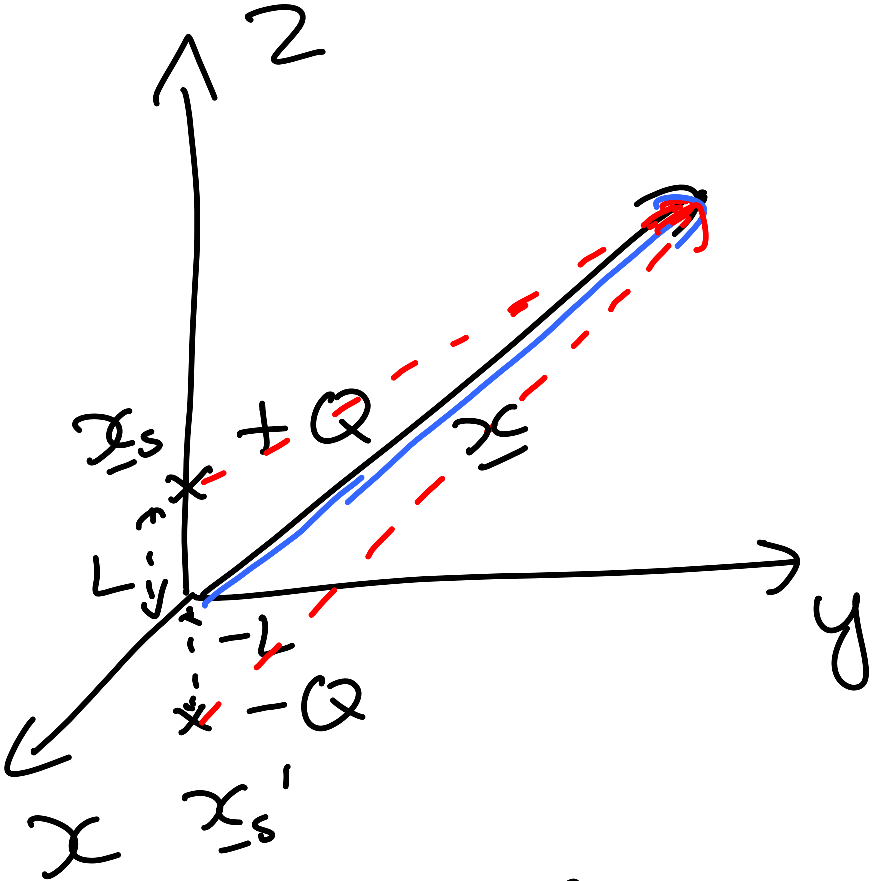
$$T(x) = \frac{Q}{4\pi k} \log\left(\frac{4L^2}{r^2}\right)$$

$$= \left(\frac{Q}{2\pi k}\right)\left(\log(2L) - \log(r)\right)$$

$$\nabla^2 T = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

$$T = C_1 \log r + C_2$$

$$= -\frac{Q}{2\pi k}\log r + C_2$$




Source $+Q$ at $(0,0,L)$

Sink $-Q$ at $(0,0,-L)$

$$T(\underline{x}) = \frac{Q}{4\pi k|\underline{x}-\underline{x}_s|} - \frac{Q}{4\pi k|\underline{x}-\underline{x}_s'|}$$

$$= \frac{Q}{4\pi k}\left[\frac{1}{\sqrt{x^2+y^2+(z-L)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+L)^2}}\right]$$

Expand in $(L/r)$:

$$T(\underline{x}) = \frac{Q}{4\pi k}\left[\frac{1}{\sqrt{x^2+y^2+z^2-2Lz+L^2}} - \frac{1}{\sqrt{x^2+y^2+z^2+2Lz+L^2}}\right]$$

$$= \frac{Q}{4\pi k}\left[\frac{1}{\sqrt{r^2-2Lz+L^2}} - \frac{1}{\sqrt{r^2+2Lz+L^2}}\right]$$

$$= \frac{Q}{4\pi k r}\left[\frac{1}{\left(1-2Lz/r^2+L^2/r^2\right)^{1/2}} - \frac{1}{\left(1+2Lz/r^2-L^2/r^2\right)^{1/2}}\right]$$

$$= \frac{Q}{4\pi k r}\left[\left(1+\frac{1}{2}\left(\frac{2Lz}{r^2}\right)+\cdots\right) - \left(1-\frac{1}{2}\left(\frac{2Lz}{r^2}\right)+\cdots\right)\right]$$

$$= \left(\frac{Q}{4\pi k r}\right)\left(\frac{2Lz}{r^2}\right]$$

$$= \frac{(2QL)}{4\pi k}\left(\frac{z}{r^3}\right)$$

$z = r\cos\theta$

$$= \frac{(2QL)}{4\pi k}\left(\frac{\cos\theta}{r^2}\right)$$

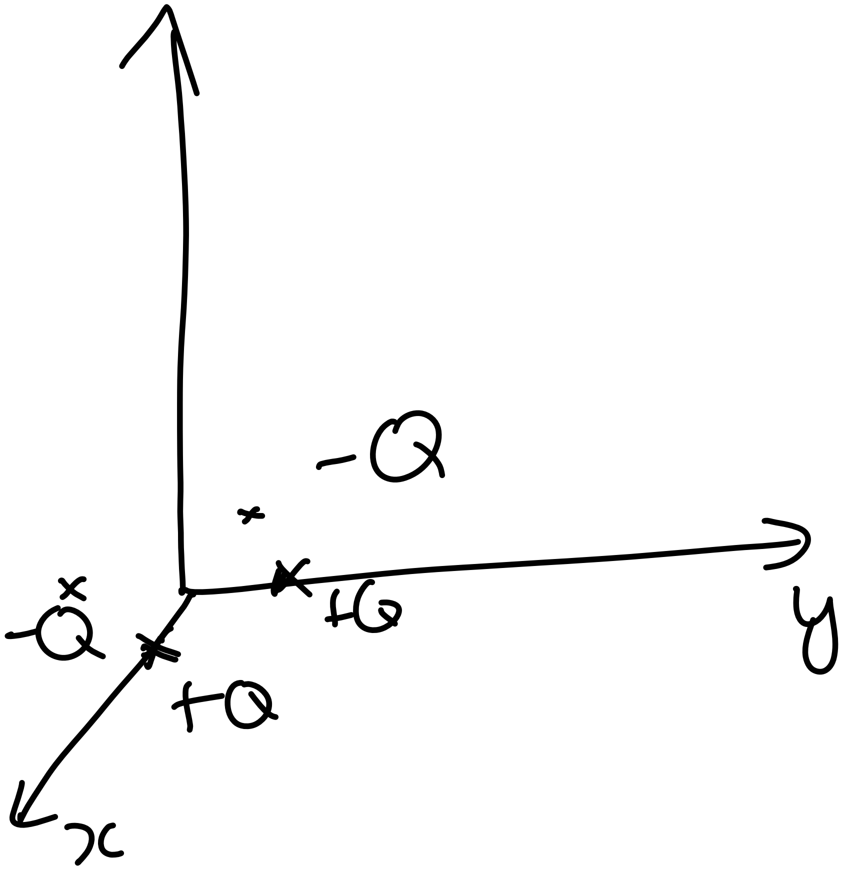
$$= \left(\frac{2QL}{4\pi k}\right)\frac{P_1^0(\cos\theta)}{r^2}$$

Spherical harmonic
$n=1$ & $m=0$

$$T = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(A_{nm}r^n + \frac{B_{nm}}{r^{n+1}}\right)P_n^m(\cos\theta)\left(\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)\right)$$

'Dipole'

$2QL$ = Dipole moment




$$T = \frac{2QL}{4\pi k} \frac{\sin\theta \cos\phi}{r^2}$$

$$= \frac{2QL}{4\pi k} \frac{P_1'(\cos\theta)\cos\phi}{r^2}$$

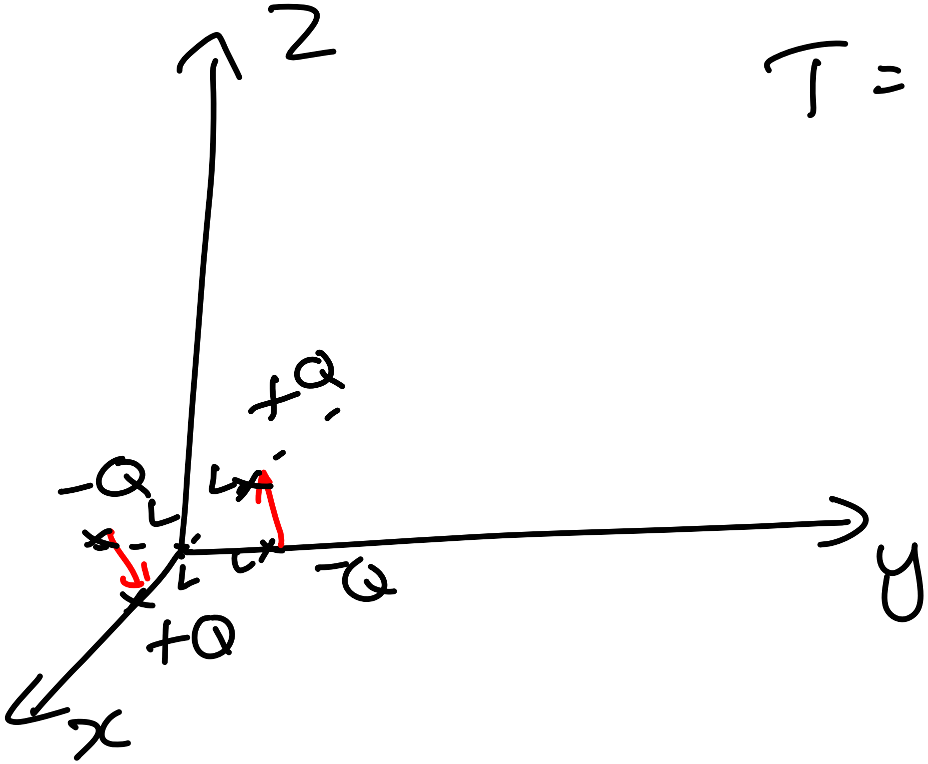
$$T = \frac{2QL}{4\pi k} \frac{\sin\theta \sin\phi}{r^2}$$

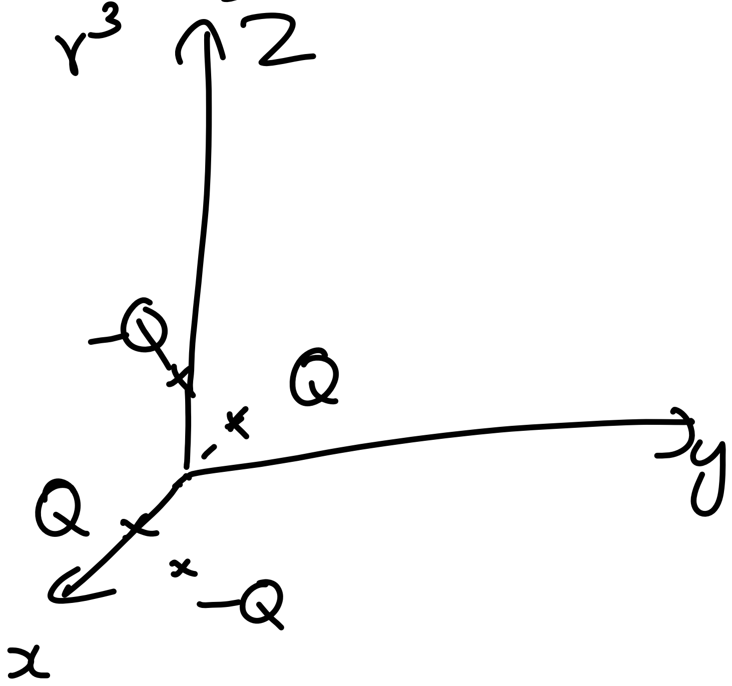
Quadrupole $n=2$

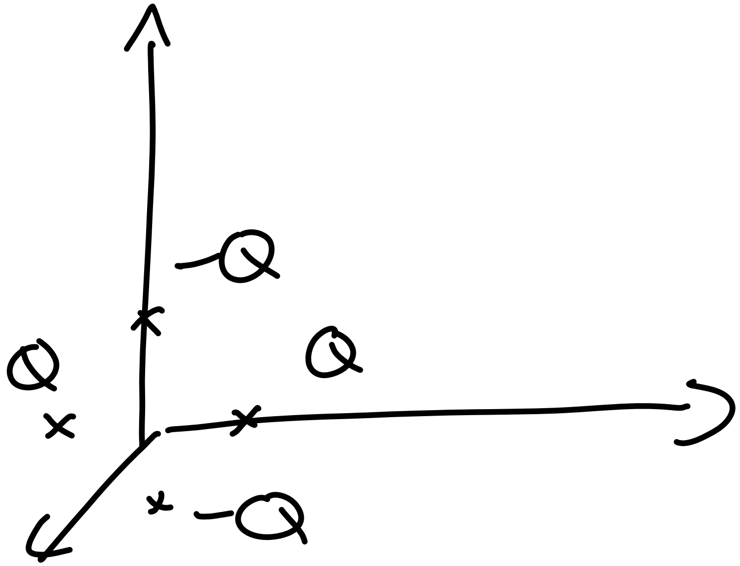
Two sources, two sinks of equal strength $Q$

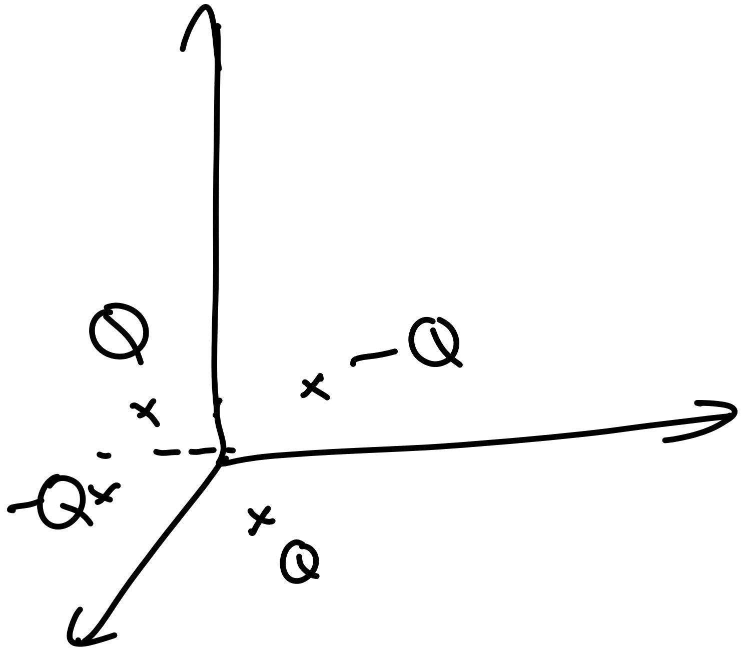
Arranged so that net source is zero & net dipole is zero.

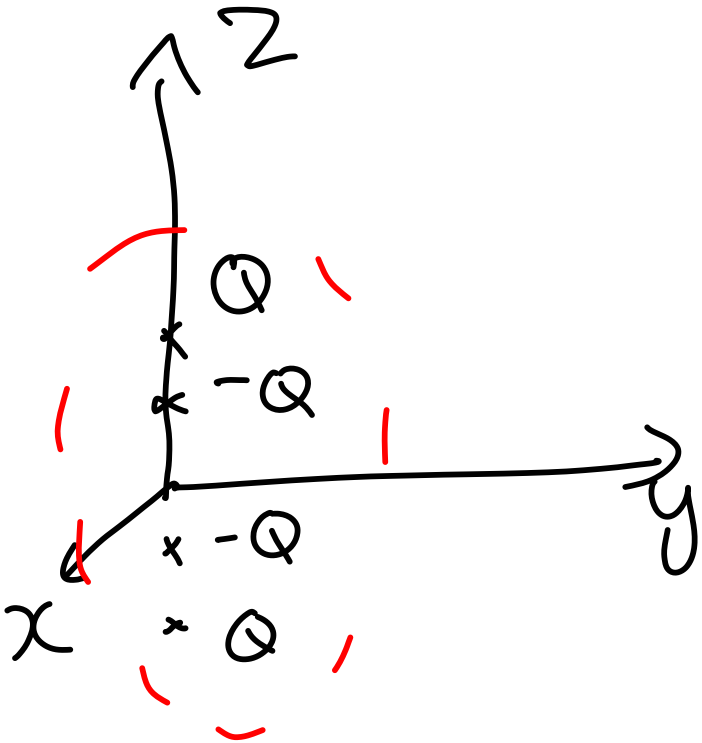
$$T = \frac{1}{r^3} P_2^m(\cos\theta) \begin{matrix}\cos \\ \sin\end{matrix} (m\phi)$$

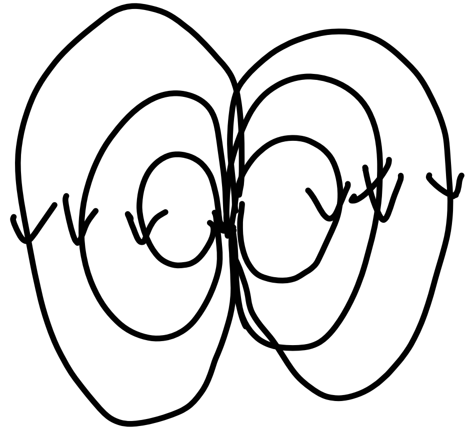


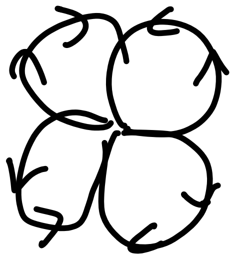




-Q
Q
Q
-Q


Q
-Q
-Q
Q


z
Q
-Q
y
-Q
x
Q



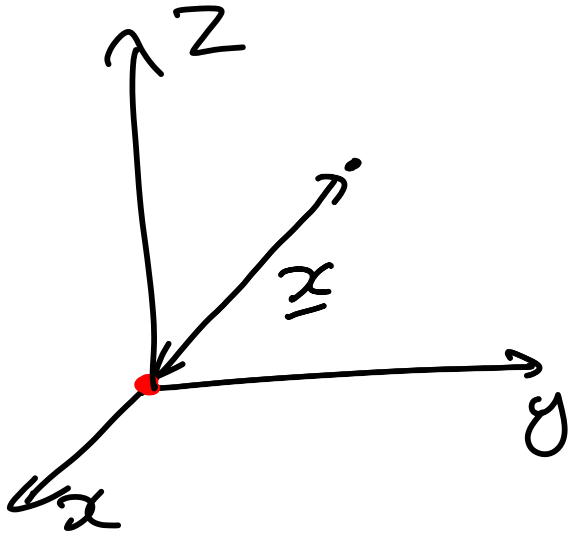


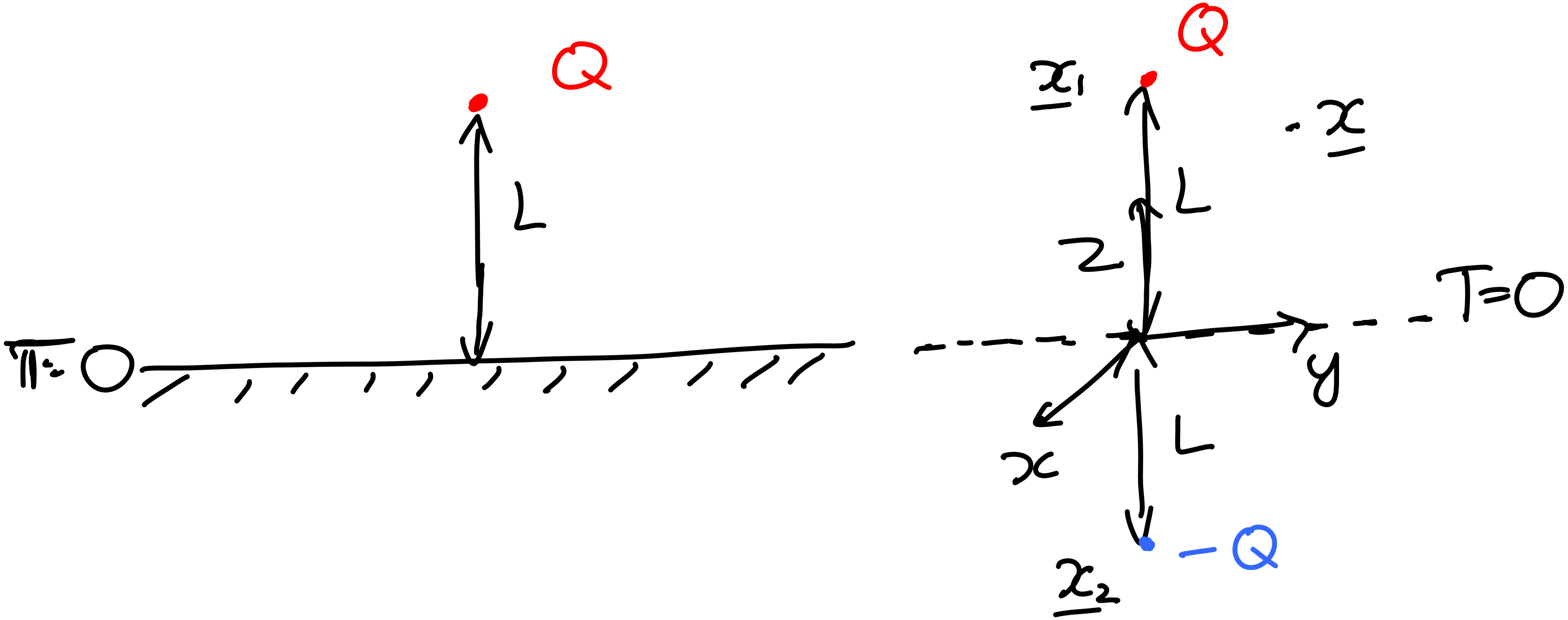
Green's function:

$$k\nabla^2 G(\underline{x}) = \delta(\underline{x})$$

$$G(\underline{x}) = \frac{1}{4\pi k|\underline{x}|}$$

$$= \frac{1}{4\pi k r}$$



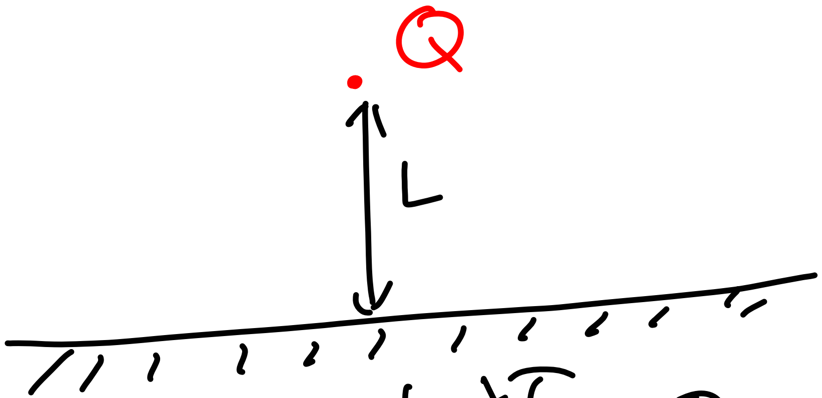



Source $Q$ at $(x,y,z) = (0,0,L)$

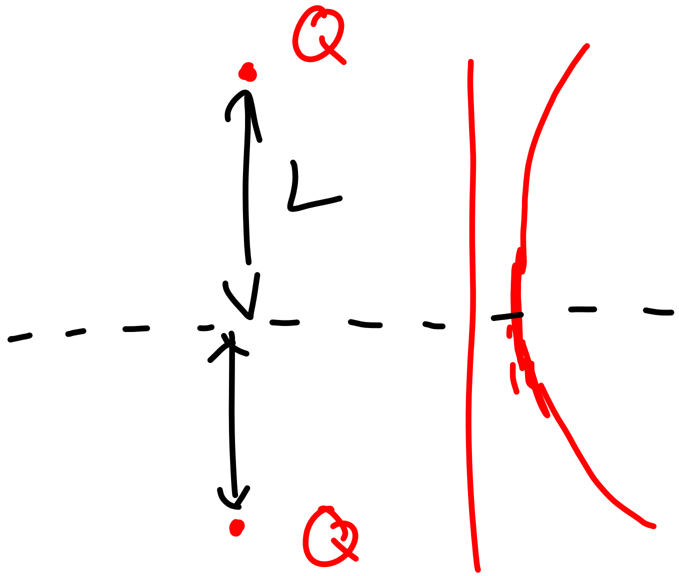
Sink $-Q$ at $(x,y,z) = (0,0,-L)$

$$T(\underline{x}) = \frac{Q}{4\pi k|\underline{x}-\underline{x}_1|} - \frac{Q}{4\pi k|\underline{x}-\underline{x}_2|}$$

$$= \frac{Q}{4\pi k\sqrt{x^2+y^2+(z-L)^2}} - \frac{Q}{4\pi k\sqrt{x^2+y^2+(z+L)^2}}$$




$$q_z = -k\frac{\partial T}{\partial z} = 0$$



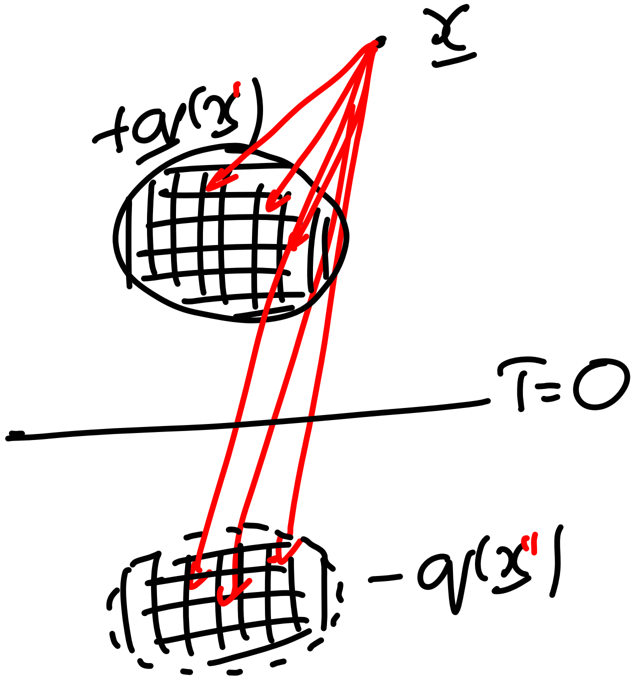
$$T = \frac{Q}{4\pi k|\underline{x}-\underline{x}_1|} + \frac{Q}{4\pi k|\underline{x}-\underline{x}_2|}$$

$$= \frac{Q}{4\pi k\left(x^2+y^2+(z-L)^2\right)^{1/2}} + \frac{Q}{4\pi k\left(x^2+y^2+(z+L)^2\right)^{1/2}}$$

Distributed source:

$$T(\underline{x}) = \frac{1}{4\pi k}\int dV' \frac{q(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

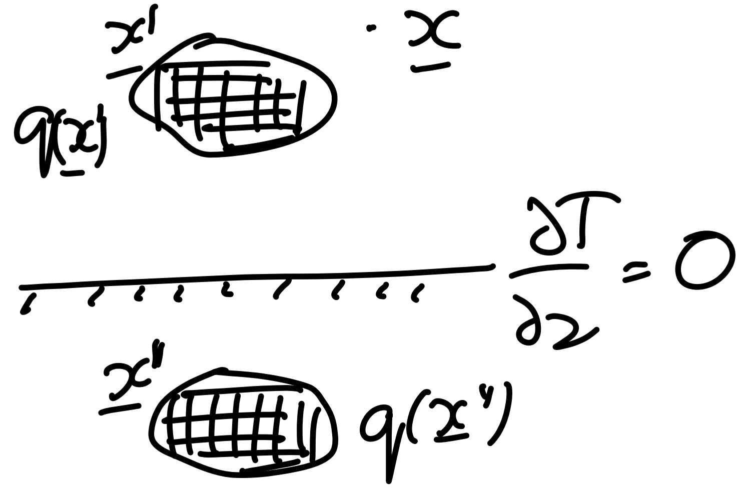
$$-\frac{1}{4\pi k}\int dV'' \frac{q(\underline{x}'')}{|\underline{x}-\underline{x}''|}$$

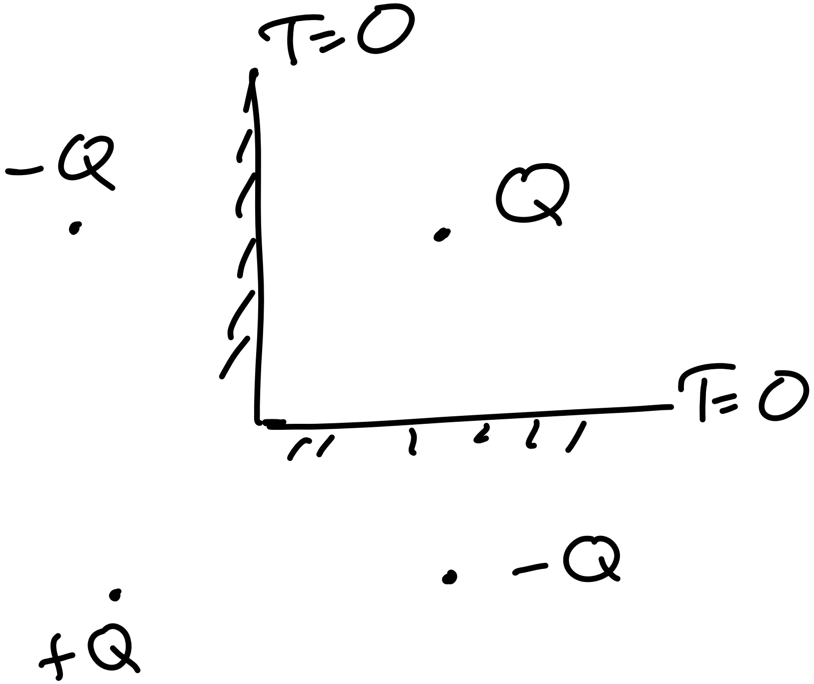


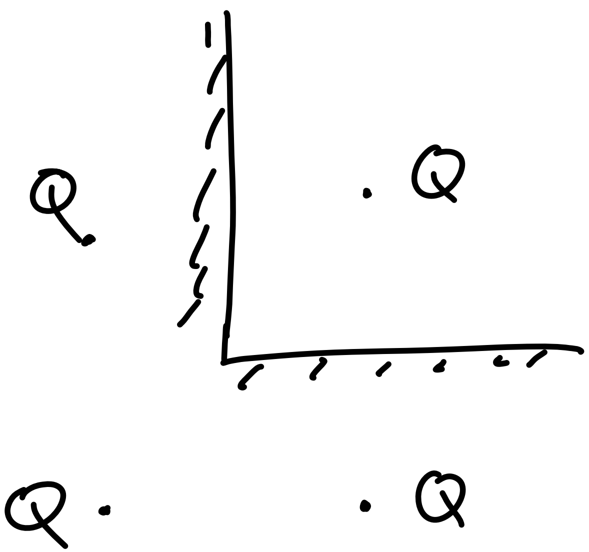

$$G(\underline{x}) = \frac{1}{4\pi k|\underline{x}-\underline{x}'|} - \frac{1}{4\pi k|\underline{x}-\underline{x}''|}$$

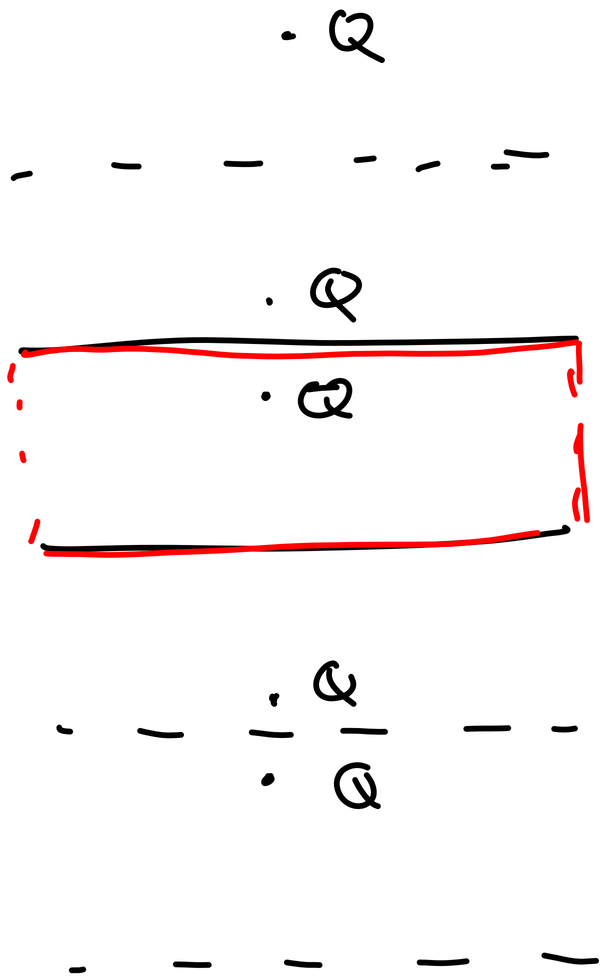
$$T(\underline{x}) = \frac{1}{4\pi k}\int dV' \, \frac{q(\underline{x}')}{|\underline{x} - \underline{x}'|}$$

$$+ \frac{1}{4\pi k}\int dV'' \, \frac{q(\underline{x}'')}{|\underline{x} - \underline{x}''|}$$



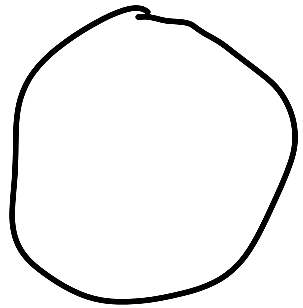




Q
Q
Q
Q
Q

$$k\nabla^2 T = -q(\underline{x})$$

Subject to boundary conditions:



$$k\nabla^2 G = -\delta(\underline{x})$$

$$\nabla' = \underline{e}_x \frac{\partial}{\partial x'} + \underline{e}_y \frac{\partial}{\partial y'} + \underline{e}_z \frac{\partial}{\partial z'}$$

$$G(\underline{x}) = \frac{1}{4\pi k |\underline{x}|}$$

$$\int dV' \, \nabla' \cdot \left( T(\underline{x}') \nabla' \left( G(\underline{x}-\underline{x}') \right) - G(\underline{x}-\underline{x}') \nabla' T(\underline{x}') \right)$$

$$= \int dV' \left( T(\underline{x}') \nabla'^2 G(\underline{x}-\underline{x}') - G(\underline{x}-\underline{x}') \nabla'^2 T(\underline{x}-\underline{x}') \right)$$

$$= \int dV' \, T(\underline{x}') \left( \frac{1}{k} \delta(\underline{x}-\underline{x}') \right) - G(\underline{x}-\underline{x}') \frac{1}{k} q(\underline{x}') )$$

$$= \frac{1}{k} T(\underline{x}) - \frac{1}{k} \int dV' \, G(\underline{x}-\underline{x}') q(\underline{x}')$$

$$\int dV' \, \nabla' \cdot \left( T(\underline{x}') \nabla' G(\underline{x} - \underline{x}') - G(\underline{x} - \underline{x}') \nabla' T(\underline{x}') \right)$$

$$= \int dS \, \underline{n}' \cdot \left( T(\underline{x}') \nabla' G(\underline{x} - \underline{x}') - G(\underline{x} - \underline{x}') \nabla' T(\underline{x}') \right)$$

$$T(\underline{x}) = \int dV \, G(\underline{x} - \underline{x}') \, q(\underline{x}')$$

$$+ \int dS \, \underline{n}' \cdot \left( T(\underline{x}') \nabla' G(\underline{x} - \underline{x}') - G(\underline{x} - \underline{x}') \nabla' T(\underline{x}') \right)$$

'Boundary integral technique'

$$\frac{\partial c}{\partial t} + \nabla \cdot (\underline{u}\, c) = D \nabla^2 c + S$$

$\underline{u}^* = \underline{u} / U$ ; $\underline{x}^* = (\underline{x}/L)$ $t^* = (t D / L^2)$

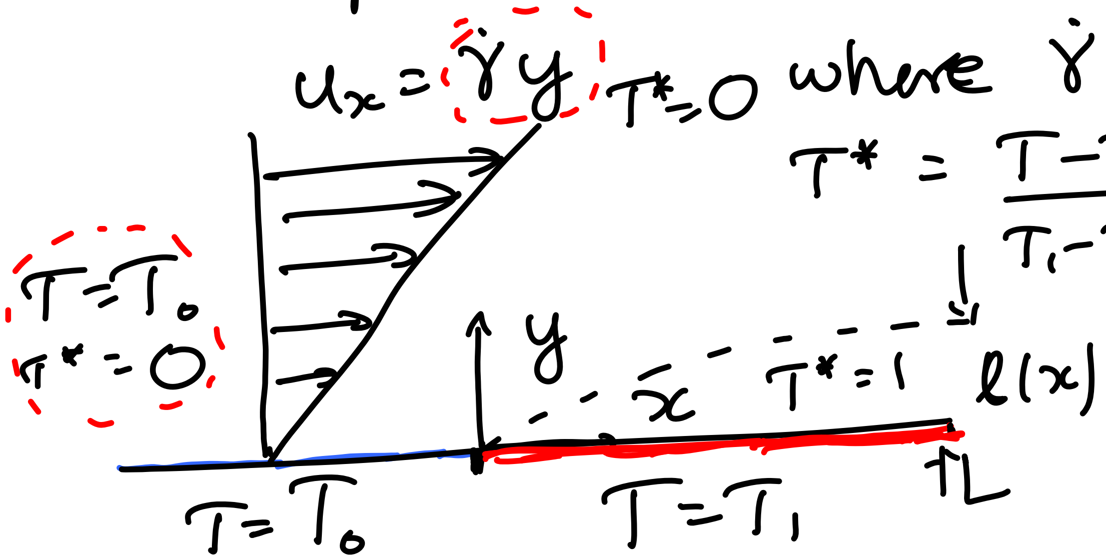
$$Pe \left( \frac{\partial c^*}{\partial t^*} + \nabla^* \cdot (\underline{u}^* c^*) \right) = \nabla^{*2} c^* + \frac{S}{(D/L^2)}$$

$$Pe = \left( \frac{UL}{D} \right)$$

So far, $Pe \ll 1 \Rightarrow D \nabla^2 c + S = 0$

$Pe \gg 1$ $\frac{\partial c}{\partial t} + \nabla \cdot (\underline{u}\, c) = 0$

# Flow past a flat plate:




where $\dot{\gamma}$ = strain rate

$$T^* = \frac{T - T_0}{T_1 - T_0}$$

$$\nabla \cdot (uT) = \alpha \nabla^2 T$$

Limit $Pe \gg 1$

$$Pe = \left( \frac{\dot{\gamma} L^2}{\alpha} \right)$$

$u_x$ is independent of $x$

$u_y = 0$

$$u_x \frac{\partial T}{\partial x} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$\dot{\gamma} y \frac{\partial T}{\partial x} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

Scale $x^* = (x/L)$, $y^* = (y/L)$

$$\left( \frac{\dot{\gamma} L^2}{\alpha} \right) y^* \frac{\partial T}{\partial x^*} = \left( \frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}} \right)$$

$$Pe\, y^* \frac{\partial T^*}{\partial x^*} = \frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}$$

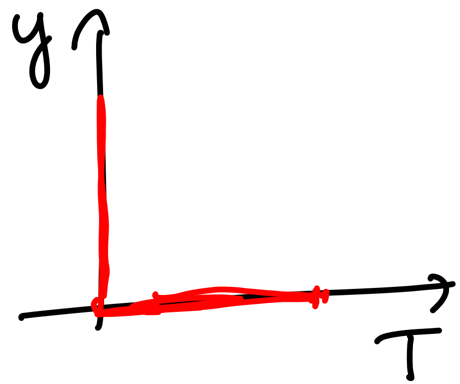
$Pe >> 1$

Boundary conditions:

$T^* = 1$ at $y^* = 0$ for $x^* > 0$

$T^* = 0$ as $y^* \to \infty$ for $x^* = 0$

$T^* = 0$ at $x^* = 0$ for $y^* > 0$



Naive approach:

Neglect diffusion

$$\frac{\partial T^*}{\partial x^*} = 0$$

Only solution $T^* = 0$ everywhere

$$y^* \frac{\partial T^*}{\partial x^*} = \frac{1}{Pe}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right)$$

$$x^* = (x/L) \quad y^* = (y/\ell) \quad T^* = \left(\frac{T - T_0}{T_1 - T_0}\right)$$

$$\dot{\gamma} y \frac{\partial T}{\partial x} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\dot{\gamma} \ell y^*}{L} \frac{\partial T^*}{\partial x^*} = \alpha\left(\frac{1}{L^2}\frac{\partial^2 T}{\partial x^{*2}} + \frac{1}{\ell^2}\frac{\partial^2 T}{\partial y^{*2}}\right)$$

$$y^* \frac{\partial T^*}{\partial x^*} = \frac{\alpha L}{l \dot{\gamma}} \left( \frac{1}{l^2} \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{1}{L^2} \frac{\partial^2 T^*}{\partial x^{*2}} \right)$$

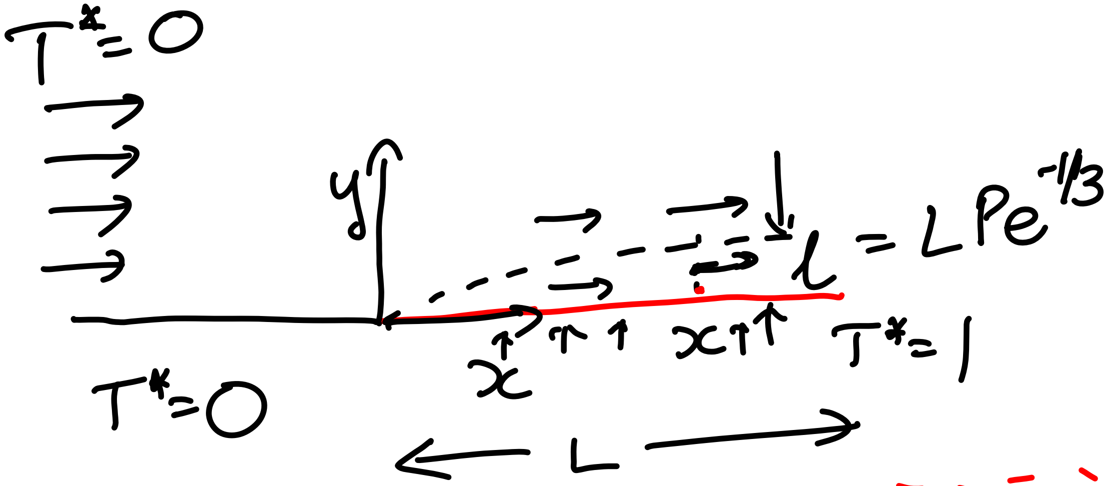
$$= \frac{\alpha L}{l^3 \dot{\gamma}} \left( \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{l^2}{L^2} \frac{\partial^2 T}{\partial x^{*2}} \right)$$

$$y^* \frac{\partial T^*}{\partial x^*} = \left( \frac{\alpha L}{l^3 \dot{\gamma}} \right) \left( \frac{\partial^2 T^*}{\partial y^{*2}} \right) \qquad Pe = \left( \frac{\dot{\gamma} L^2}{\alpha} \right)$$

$$\frac{l^3 \dot{\gamma}}{\alpha L} = 1 \implies \left( \frac{l}{L} \right)^3 = \left( \frac{\alpha}{\dot{\gamma} L^2} \right) = Pe^{-1}$$

$$\frac{l}{L} = Pe^{-1/3}$$

$$\dot{\gamma} y \frac{\partial T^*}{\partial x} = \alpha \frac{\partial^2 T^*}{\partial y^2}$$




$$\frac{l}{L} = Pe_L^{-1/3}$$

$$\frac{l(x)}{x} = Pe_x^{-1/3} = \left(\frac{\alpha}{\dot{\gamma} x^2}\right)^{1/3}$$

$$\eta = y / l(x) = y / (\alpha x / \dot{\gamma})^{1/3}$$

$$l(x) = \left(\frac{\alpha x}{\dot{\gamma}}\right)^{1/3}$$

$$\eta = y / (\alpha x / \dot{\gamma})^{1/3}$$

$$\dot{\gamma} y \frac{\partial T^*}{\partial x} = \alpha \frac{\partial^2 T^*}{\partial y^2}$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{(\alpha x/\dot{\gamma})^{1/3}}\frac{\partial T}{\partial \eta}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{(\alpha x/\dot{\gamma})^{2/3}}\frac{\partial T}{\partial \eta}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{-y}{3x(\alpha x/\dot{\gamma})^{1/3}}\frac{\partial T}{\partial \eta}$$

$$\dot{\gamma} y \left(\frac{-y}{3x(\alpha x/\dot{\gamma})^{1/3}}\right)\frac{\partial T}{\partial \eta} = \frac{\alpha}{(\alpha x/\dot{\gamma})^{2/3}}\frac{\partial^2 T}{\partial \eta^2}$$

$$y = \eta\left(\frac{\alpha x}{\dot{\gamma}}\right)^{1/3}$$

$$\frac{-\dot{\gamma}\,\eta^2\left(\frac{\alpha x}{\dot{\gamma}}\right)^{2/3}}{3x\,(\alpha x/\dot{\gamma})^{1/3}}\frac{\partial T}{\partial \eta} = \frac{\alpha}{(\alpha x/\dot{\gamma})^{2/3}}\frac{\partial^2 T}{\partial \eta^2}$$

$$-\eta^2\frac{\partial T}{\partial \eta} = \frac{\partial^2 T}{\partial \eta^2} \qquad \eta = \frac{y}{(\alpha x/\dot{\gamma})^{1/3}}$$

At $y=0$, $T^*=1 \Rightarrow \eta=0$

As $y\to\infty$, $T^*=0 \Rightarrow \eta\to\infty$

At $x=0$ for $y>0$, $T^*=0 \Rightarrow \eta\to\infty$

$$-\eta^2\frac{\partial T^*}{\partial \eta} = \frac{\partial^2 T^*}{\partial \eta^2}$$

$$\frac{\partial T^*}{\partial \eta} = C_1\exp\left(-\eta^3/3\right)$$

$$T^* = C_1 \int_0^{\eta} d\eta' \exp\left(-\eta'^3/3\right) + C_2$$

$T^* = 0$ as $\eta \to \infty$ & $T^* = 1$ at $\eta = 0$

$$T^* = \left[ 1 - \frac{\int_0^{\eta} d\eta' \, e^{\left(-\eta'^3/3\right)}}{\int_0^{\infty} d\eta' \, e^{-\eta'^3/3}} \right]$$

where $\eta = y/(\alpha x/\dot{\gamma})^{1/3}$

Heat flux

$$q_y = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} = \frac{-k}{(\alpha x/\dot{\gamma})^{1/3}} \left.\frac{\partial T}{\partial \eta}\right|_{\eta=0}$$

$$= \frac{-k}{(\alpha x/\dot\gamma)^{1/3}} \left[ - \frac{1}{\int_0^\infty d\eta' \, e^{-\eta'^3/3}} \right] (T_1 - T_0)$$

$$q_y = \frac{k(T_1 - T_0)}{(\alpha x/\dot\gamma)^{1/3}} \frac{3^{2/3}}{\Gamma(1/3)} \qquad \int_0^\infty d\eta' \, e^{-\eta'^3/3} = \frac{\Gamma(1/3)}{3^{2/3}}$$

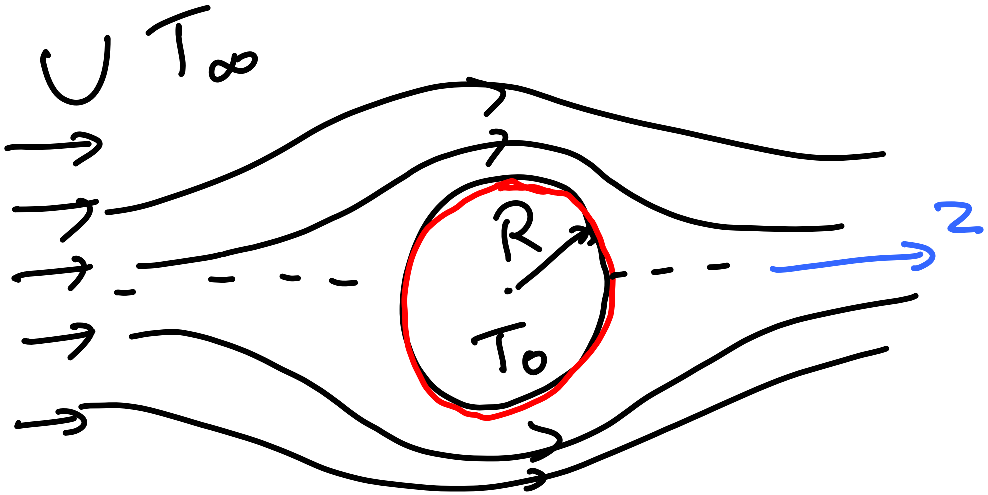
$$Q = \int_0^L dx \, q_y = \frac{k(T_1 - T_0)}{(\alpha/\dot\gamma)^{1/3}} \frac{3^{2/3}}{\Gamma(1/3)} \int_0^L \frac{dx}{x^{1/3}}$$

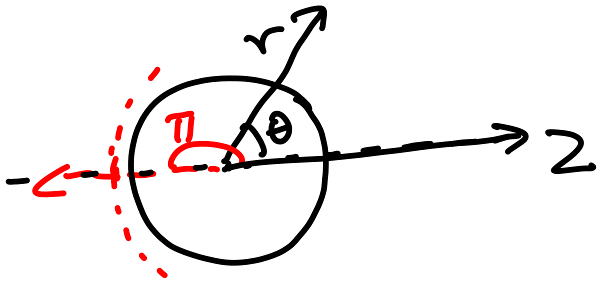
$$= \frac{k(T_1 - T_0)}{(\alpha/\dot\gamma)^{1/3}} \frac{3^{2/3}}{\Gamma(1/3)} \left[ \frac{3}{2} L^{2/3} \right]$$

$$= \frac{3^{5/3}}{2\Gamma(1/3)} \frac{k(T_1 - T_0)}{(\alpha/\dot\gamma L^2)^{1/3}}$$

$$Nu = \frac{2Q}{k(T_1 - T_0)} = \frac{3^{5/3}}{\Gamma(1/3)} Pe_L^{1/3} = \frac{3^{5/3}}{\Gamma(1/3)} Re^{1/3} Pr^{1/3}$$

# Heat transfer from a spherical particle:




$$u_r = U\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$u_\theta = -U\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$

$$T^* = \frac{T - T_\infty}{T_0 - T_\infty}$$

$$r^* = (r/R)$$

$$u_r^* = u_r/U$$

$$u_\theta^* = u_\theta/U$$

Boundary conditions:

At $r^* = 1$, $T^* = 1$

As $r^* \to \infty$, $T^* = 0$

$$u_r^* = \cos\theta\left(1 - \frac{3}{2r^*} + \frac{1}{2r^{*3}}\right)$$

$$u_\theta^* = -\sin\theta\left(1 - \frac{3}{4r^*} - \frac{1}{4r^{*3}}\right)$$

$$\nabla\cdot(\underline{u}T) = \alpha\nabla^2 T$$

$$Pe\left(u_r^*\frac{\partial T^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial T^*}{\partial\theta}\right) = \left(\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial T^*}{\partial r^*}\right)\right) + \frac{1}{r^{*2}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T^*}{\partial\theta}\right)$$

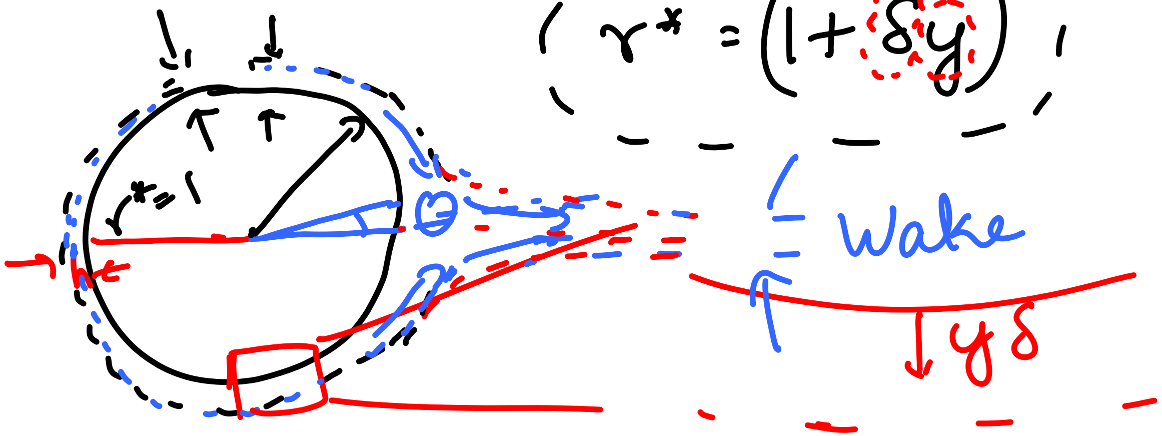
$$Pe = \left(\frac{UR}{\alpha}\right)$$

Limit $Pe \gg 1$

$$u_r^*\frac{\partial T^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial T^*}{\partial\theta} = 0$$

$$\underline{u}^*\cdot\nabla^* T^* = 0$$

$$r^* = (1+\delta y)$$




$$u_r^* = \cos\theta\left(1 - \frac{3}{2r^*} + \frac{1}{2r^{*3}}\right)$$

$$= \cos\theta\left(1 - \frac{3}{2(1+\delta y)} + \frac{1}{2(1+\delta y)^3}\right)$$

$$= \cos\theta\left(1 - \frac{3}{2}(1+\delta y)^{-1} + \frac{1}{2}(1+\delta y)^{-3}\right)$$

$$= \cos\theta\left(\cancel{1} - \cancel{\frac{3}{2}} + \cancel{\frac{3}{2}\delta y} - \frac{3}{2}(\delta y)^2 + \cancel{\frac{1}{2}} - \cancel{\frac{3}{2}\delta y} + 3(\delta y)^2\right)$$

$$\cong \cos\theta\ \frac{3}{2}\delta^2 y^2$$

$$u_\theta^* = -\sin\theta \left[ 1 - \frac{3}{4r^*} - \frac{1}{4r^{*3}} \right]$$

$$= -\sin\theta \left[ 1 - \frac{3}{4(1+\delta y)} - \frac{1}{4(1+\delta y)^3} \right]$$

$$= -\sin\theta \left[ \cancel{1} - \cancel{\frac{3}{4}} + \frac{3}{4}(\delta y) - \cancel{\frac{1}{4}} + \frac{3}{4}\delta y \right]$$

$$= \boxed{-\sin\theta \, \frac{3}{2}\delta y}$$

$$Pe\left[ u_r^* \frac{\partial T^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial T^*}{\partial \theta^*} \right] = \left[ \frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left( r^{*2} \frac{\partial T^*}{\partial r^*} \right) + \frac{1}{r^{*2}\sin\theta}\frac{\partial}{\partial \theta}\left( \sin\theta \frac{\partial T^*}{\partial \theta} \right) \right]$$

$$Pe\left[ \frac{3}{2}\delta^2 y^2 \cos\theta \frac{1}{\delta}\frac{\partial T}{\partial y} - \frac{3}{2}\frac{\delta y \sin\theta}{(1+\cancel{\delta y})}\frac{\partial T^*}{\partial \theta} \right]$$

$$= \left[ \frac{1}{(1+\cancel{\delta y})^2} \frac{1}{\delta} \frac{\partial}{\partial y} \left( (1+\cancel{\delta y})^2 \frac{1}{\delta} \frac{\partial T^*}{\partial y} \right) \right) + \frac{1}{(1+\cancel{\delta y})^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial T^*}{\partial \theta} \right)$$

$$Pe \, \frac{3}{2} \left[ \delta y^2 \cot\theta \frac{\partial T^*}{\partial y} - \delta y \sin\theta \frac{\partial T^*}{\partial \theta} \right] = \frac{1}{\delta^2} \frac{\partial^2 T^*}{\partial y^2}$$

$$+ \cancel{\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial T}{\partial \theta} \right)}$$

$$Pe \, \delta^3 \, \frac{3}{2} \left[ y^2 \cot\theta \frac{\partial T^*}{\partial y} - y \sin\theta \frac{\partial T^*}{\partial \theta} \right] = \frac{\partial^2 T^*}{\partial y^2}$$

$$\delta = Pe^{-1/3}$$

$$\boxed{\frac{3}{2} \left[ y^2 \cot\theta \frac{\partial T^*}{\partial y} - y \sin\theta \frac{\partial T^*}{\partial \theta} \right] = \frac{\partial^2 T^*}{\partial y^2}}$$

$$\eta = \frac{y}{h(\theta)}$$

$$\frac{\partial T^*}{\partial y} = \frac{\partial T^*}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{h(\theta)}\frac{\partial T^*}{\partial \eta}$$

$$\frac{\partial^2 T^*}{\partial y^2} = \frac{1}{h^2}\frac{\partial^2 T^*}{\partial \eta^2}$$

$$\frac{\partial T^*}{\partial \theta} = \frac{\partial T^*}{\partial \eta}\frac{\partial \eta}{\partial \theta} = \left(\frac{\partial T^*}{\partial \eta}\right)\left(-\frac{y}{h^2}\frac{dh}{d\theta}\right)$$

$$= \left(\frac{\partial T^*}{\partial \eta}\right)\left(-\frac{\eta}{h}\frac{dh}{d\theta}\right)$$

$$\frac{3}{2}\left[y^2\cos\theta \frac{1}{h(\theta)}\frac{dT^*}{d\eta} - y\sin\theta\left(\frac{-\eta}{h}\frac{dh}{d\theta}\right)\frac{dT^*}{d\eta}\right]$$

$$= \frac{1}{h^2}\frac{d^2T^*}{d\eta^2}$$

$$\eta^2\frac{dT^*}{d\eta}\boxed{\left[\frac{3}{2}\left(h^3\cos\theta + h^2\sin\theta\frac{dh}{d\theta}\right)\right]} = \frac{d^2T^*}{d\eta^2}$$

$$h^3\cot\theta + h^2\sin\theta\,\frac{dh}{d\theta} = -2$$

$$\frac{d^2T^*}{d\eta^2} - 3\eta^2\frac{dT^*}{d\eta} = 0$$

$$\frac{dT^*}{d\eta} = C_1 e^{-\eta^3}$$

$$T^* = C_1\int_0^\eta d\eta'\, e^{-\eta'^3} + C_2$$

$T^* = 0$ as $y \to \infty \Rightarrow \eta \to \infty$

$T^* = 1$ at $y = 0 \Rightarrow \eta = 0$

$$T^* = \left[1 - \frac{\int_0^\eta d\eta'\, e^{-\eta'^3}}{\int_0^\infty d\eta'\, e^{-\eta'^3}}\right]$$

$$\frac{\partial T^*}{d\eta} = -\frac{1}{\int_0^\infty d\eta'\, e^{-\eta'^3}}$$

$$h^3 \cos\theta + h^2 \sin\theta \frac{dh}{d\theta} = -2$$

$$\frac{\sin\theta}{3} \frac{d(h^3)}{d\theta} + h^3 \cos\theta = -2$$

$$x = \cos\theta \qquad dx = -\sin\theta \, d\theta$$

$$-\frac{\sin^2\theta}{3} \frac{d(h^3)}{dx} + h^3 x = -2$$

$$-\frac{(1-x^2)}{3} \frac{d(h^3)}{dx} + h^3 x = -2$$

$$h^3 = g(x) + p(x)$$

$$\frac{(1-x^2)}{3} \frac{dg}{dx} - xg = 0$$

$$\frac{dg}{dx} = \frac{3x\,g}{1-x^2} \Longrightarrow g = \frac{1}{(1-x^2)^{3/2}}$$

$$p(x) = g(x)\,q(x)$$

$$\frac{(1-x^2)}{3}\frac{d}{dx}\left(g(x)q(x)\right) - x\,g(x)q(x) = 2$$

$$\frac{(1-x^2)}{3}\,g(x)\,\frac{dq}{dx} = 2$$

$$\frac{dq}{dx} = \frac{6}{(1-x^2)\,g(x)}$$

$$q(x) = \int_{-1}^{x} dx' \frac{6}{(1-x'^2)\,g(x')} = \int_{-1}^{x} dx'\, 6(1-x'^2)^{1/2}$$

$$h^3 = \cancel{\frac{C}{(1-x^2)^{3/2}}} + \frac{6}{(1-x^2)^{3/2}}\int_{-1}^{x} dx'\,(1-x'^2)^{1/2}$$

$x = \cos\theta$

$$h^3 = \left[ \frac{6}{(1-x^2)^{3/2}} \int_{-1}^{x} dx' \, (1-x'^2)^{1/2} \right]$$

$$q_r \Big|_{r=1} = -k \frac{\partial T}{\partial r} \Big|_{r=1}$$

$$= -\frac{k(T_0 - T_\infty)}{R} \frac{\partial T^*}{\partial r^*} \Big|_{r^*=1}$$

$$\eta = \frac{y}{h(\theta)}$$

$$= -\frac{k(T_0 - T_\infty)}{R} \frac{1}{\delta} \frac{\partial T^*}{\partial y} \Big|_{y=0}$$

$$= \frac{-k(T_0 - T_\infty)}{R\,\delta\, h(\theta)} \frac{\partial T^*}{\partial \eta} \Big|_{y=0}$$

$$= -\frac{k(T_0 - T_\infty)}{R\,\delta\, h(\theta)} \left[ -\frac{1}{\int_0^\infty d\eta' e^{-\eta'^3}} \right]$$

$$= \frac{k(T_0 - T_\infty)}{R\delta\, h(\theta)} \left[ \frac{1}{\int_0^\infty d\eta' e^{-\eta'^3}} \right]$$

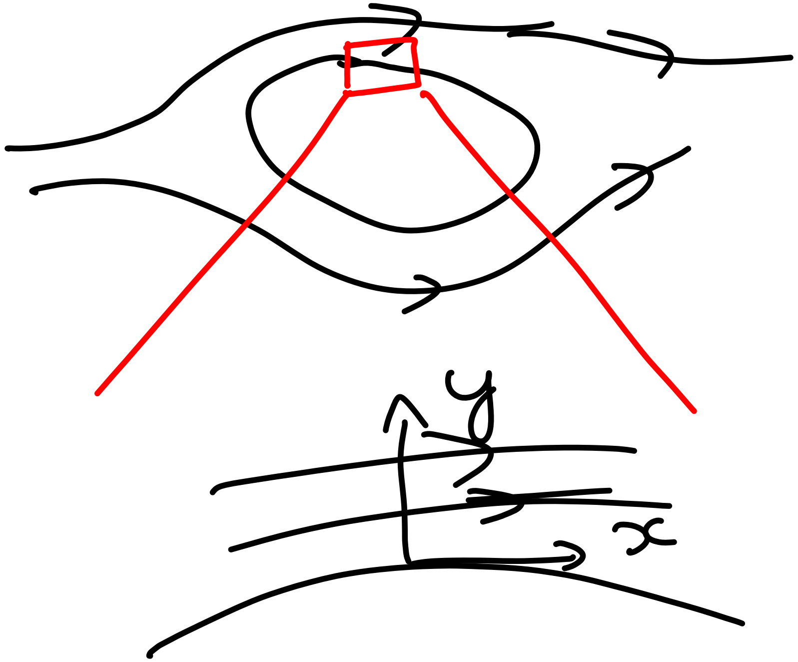
$$Q = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta \, d\theta \, d\phi \; q_r \Big|_{(\theta,\phi)}$$

$$= 2\pi R^2 \int_0^{\pi} \sin\theta \, d\theta \; q_r(\theta,\phi)$$

$$= (2\pi R^2) \frac{k(T_0 - T_\infty)}{R\delta \int_0^\infty d\eta' e^{-\eta'^3}} \int_0^{\pi} \sin\theta \, d\theta \left(\frac{1}{h(\theta)}\right)$$

$$= 1.2491 \left(2\pi R k (T_0 - T_\infty) Pe^{1/3}\right)$$

$$Nu = \frac{2Q}{(4\pi R^2) k (T_0 - T_\infty)/R} = 1.2491\, Pe^{1/3}$$




$u_x = 0$ at the surface

$= y A(x)$ near the surface

Slip at surface

$u_x = U$

$u_y \sim U \delta y$

For an incompressible flow, velocity components satisfy

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \qquad \nabla \cdot \underline{u} = 0$$

$$\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x} = -y\frac{dA}{dx}$$

$$u_y = -\frac{y^2}{2}\frac{dA}{dx}$$

$u_\theta \sim \delta y$

$u_x \sim (\delta y)^2$

$$u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

$$\frac{\partial T}{\partial x}(A\,y) - \frac{y^2}{2}\frac{dA}{dx}\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

$$y^* = (y/\delta)\,, \quad x^* = (x/L)$$

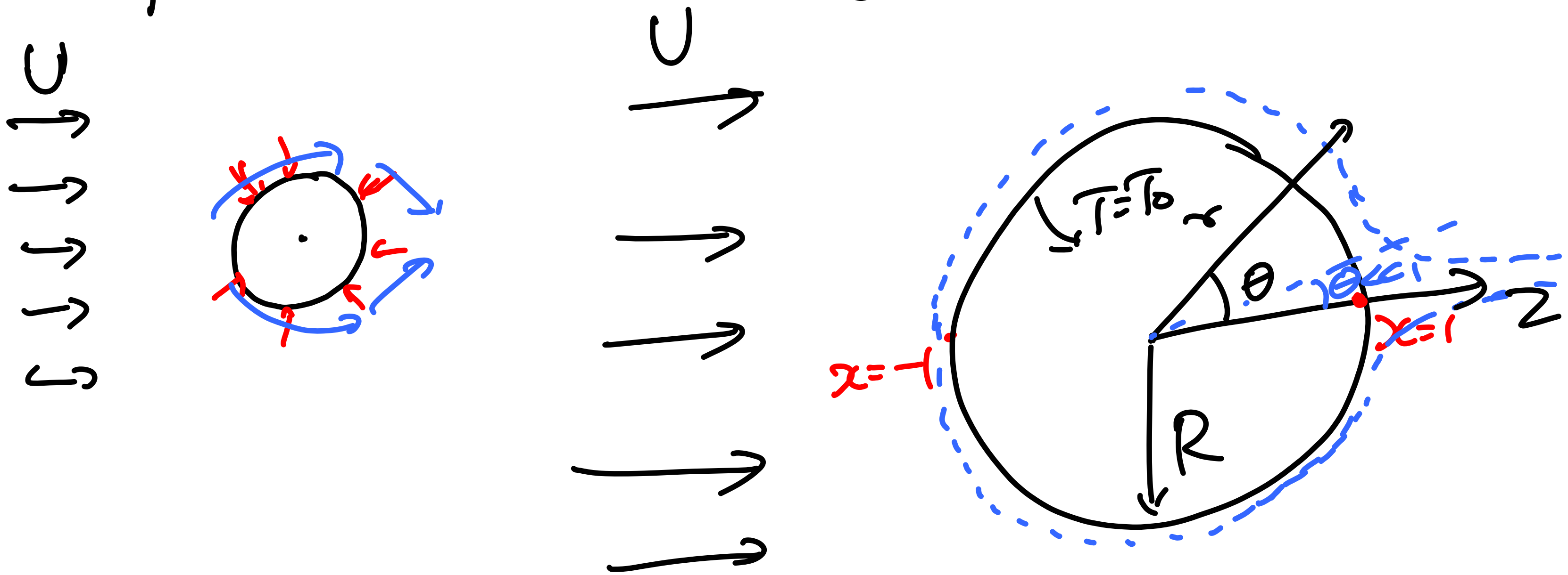
$$\frac{A\,y^*\delta}{L}\frac{\partial T}{\partial x^*} - \frac{y^{*2}\delta}{2\,L}\frac{dA}{dx^*}\frac{\partial T}{\partial y^*} = \frac{\alpha}{\delta^2}\frac{\partial^2 T}{\partial y^{*2}}$$

$$\left(\frac{\delta^3 A}{L\,\alpha}\right)\left(y^*\frac{\partial T}{\partial x^*} - \frac{y^{*2}}{2}\frac{1}{A}\frac{dA}{dx^*}\frac{\partial T}{\partial y^*}\right) = \frac{\partial^2 T}{\partial y^{*2}}$$

$$\frac{\delta}{L} \sim Pe^{-1/3} \sim \left(\frac{\alpha}{A L^2}\right)^{1/3} \qquad \frac{\delta}{L} \sim Pe^{-1/2}$$

$$Nu \sim Pe^{1/3} \qquad \eta = \left(\frac{y}{g(x)}\right)$$

Diffusion from a gas bubble: $T = T_\infty$ as $r \to \infty$




$$u_r = U\cos\theta\left(1 - \frac{R}{r}\right)$$

$$u_\theta = -U\sin\theta\left(1 - \frac{R}{2r}\right)$$

$$u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} = \alpha\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right)\right)$$

$$u_r^* = \frac{u_r}{U} \qquad T^* = \frac{T - T_\infty}{T_0 - T_\infty} \qquad r^* = \frac{r}{R}$$

$$Pe\left(u_r^* \frac{\partial T^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial T^*}{\partial \theta}\right) = \frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial T}{\partial r^*}\right) + \frac{1}{r^{*2}\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T^*}{\partial \theta}\right)$$

where $Pe = \left(\frac{UR}{\alpha}\right)$

$$r^* = 1 + \delta y$$

$$u_r^* = \left(1 - \frac{1}{r^*}\right)\cos\theta = \left(1 - \frac{1}{1+\delta y}\right)\cos\theta = \delta y \cos\theta$$

$$u_\theta^* = -\left(1 - \frac{1}{2r^*}\right)\sin\theta = -\left(1 - \frac{1}{2(1+\delta y)}\right)\sin\theta = -\frac{1}{2}\sin\theta$$

$$Pe\left(\delta y\cos\theta\,\frac{1}{\delta}\frac{\partial T^*}{\partial y}+\frac{\left(-\frac{1}{2}\sin\theta\right)}{(1+\cancel{\delta y})}\frac{\partial T^*}{\partial \theta}\right)$$

$$=\frac{1}{(1+\cancel{\delta y})^2}\frac{1}{\delta}\frac{\partial}{\partial y}\left((1+\cancel{\delta y})^2\frac{1}{\delta}\frac{\partial T^*}{\partial y}\right)+\frac{1}{(1+\cancel{\delta y})^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T^*}{\partial \theta}\right)$$

$$Pe\left[y\cos\theta\frac{\partial T^*}{\partial y}-\frac{1}{2}\sin\theta\frac{\partial T^*}{\partial \theta}\right]=\frac{1}{\delta^2}\frac{\partial^2 T^*}{\partial y^2}+\cancel{\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T^*}{\partial \theta}\right)}$$

$$(Pe\,\delta^2)\left[y\cos\theta\frac{\partial T^*}{\partial y}-\frac{1}{2}\sin\theta\frac{\partial T^*}{\partial \theta}\right]=\frac{\partial^2 T^*}{\partial y^2}$$

$$Pe\,\delta^2=1\implies\delta=Pe^{-1/2}$$

$$y\cos\theta \frac{\partial T^*}{\partial y} - \frac{1}{2}\sin\theta \frac{\partial T^*}{\partial \theta} = \frac{\partial^2 T}{\partial y^2}$$

$$\eta = \frac{y}{h(\theta)} \qquad \frac{\partial T^*}{\partial y} = \frac{1}{h}\frac{\partial T}{\partial \eta}$$

$$\frac{\partial^2 T^*}{\partial y^2} = \frac{1}{h^2}\frac{\partial^2 T^*}{\partial \eta^2}$$

$$\frac{\partial T^*}{\partial \theta} = -\frac{y}{h^2}\frac{dh}{d\theta}\frac{\partial T^*}{\partial \eta}$$

$$= -\frac{\eta}{h}\frac{dh}{d\theta}\frac{\partial T^*}{\partial \eta}$$

$$\frac{y\cos\theta}{h}\frac{\partial T^*}{\partial \eta} + \frac{1}{2}\sin\theta\,\frac{\eta}{h}\frac{dh}{d\theta}\frac{\partial T^*}{\partial \eta} = \frac{1}{h^2}\frac{\partial^2 T^*}{\partial \eta^2}$$

$$\eta\frac{\partial T^*}{\partial \eta}\left(h^2\cos\theta + \frac{1}{2}h\frac{dh}{d\theta}\sin\theta\right) = \frac{\partial^2 T^*}{\partial \eta^2}$$

$$h^2\cos\theta + \frac{1}{2}h\frac{dh}{d\theta}\sin\theta = -2$$

$$\frac{\partial^2 T^*}{\partial \eta^2} + 2\eta\frac{\partial T^*}{\partial \eta} = 0$$

Boundary conditions:

At $r^* = 1$, $y = 0$, $T^* = 1$

As $r^* \to \infty$ $(y \to \infty)$ $T^* = 0$

$$T^* = \left[1 - \frac{\int_0^\eta d\eta' \, e^{-\eta'^2}}{\int_0^\infty d\eta' \, e^{-\eta'^2}}\right] \Bigg|\Bigg|$$

$$h^2\cos\theta + \frac{1}{2}h\frac{dh}{d\theta}\sin\theta = -2$$

$$\cos\theta = x \Rightarrow dx = -\sin\theta \, d\theta$$

$$h^2 x - \frac{1}{4}(1-x^2)\frac{dh^2}{dx} = -2$$

$$\frac{(1-x^2)}{4}\frac{d(h^2)}{dx} - h^2 x = 2$$

$$h^2 = h_g^2 f$$

$$\frac{(1-x^2)}{4}\frac{d(h_g^2)}{dx} - h_g^2 x = 0$$

$$h_g^2 = \frac{c}{(1-x^2)^2} \qquad f = 8x$$

$$h^2 = \frac{c}{(1-x^2)^2} + \frac{8x}{(1-x^2)^2}$$

$$h = \frac{8(1+x)}{(1-x^2)} = h(\cos\theta)$$

$$q_r = -k\frac{\partial T}{\partial r} = -\frac{k(T_0-T_\infty)}{R}\frac{\partial T^*}{\partial r^*}$$

$$= -\frac{k(T_0-T_\infty)}{R\delta}\frac{\partial T}{\partial y} = \frac{-k(T_0-T_\infty)}{R\delta\, h(\theta)}\frac{\partial T}{\partial \eta}$$

Heat flux at the surface:

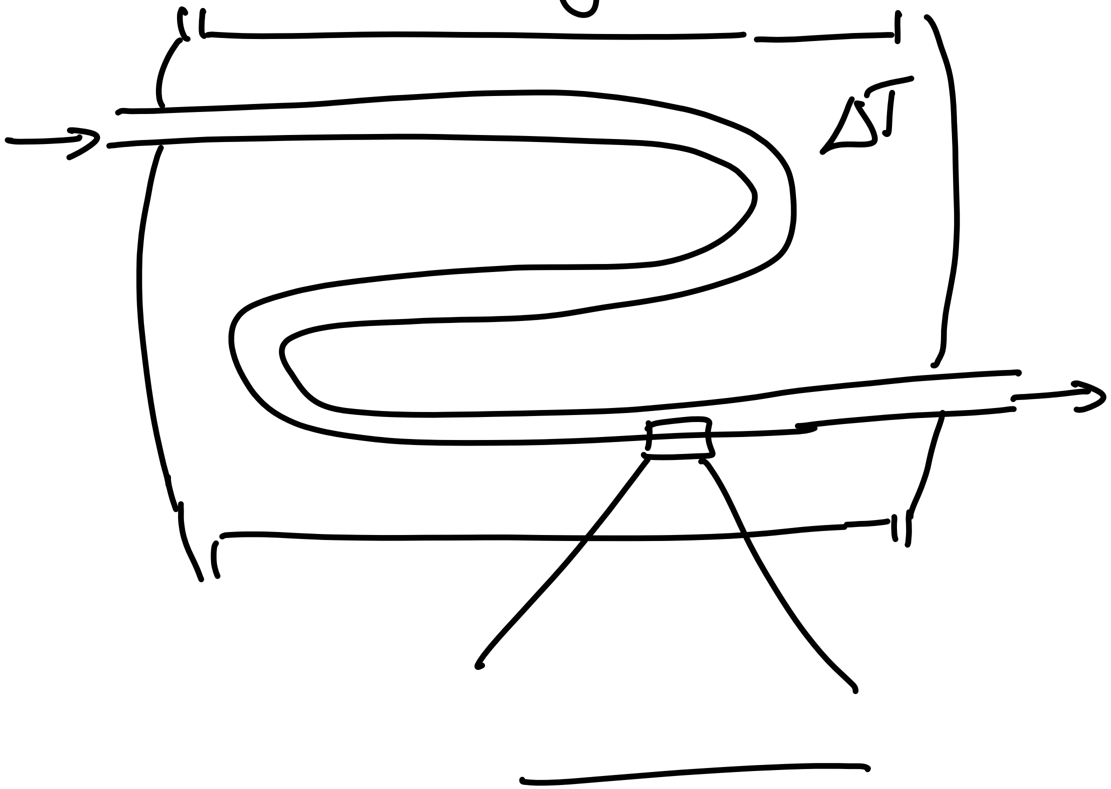
$$q_r\big|_{r=R} = \frac{-k(T_0-T_\infty)}{R\delta\, h(\theta)}\frac{\partial T^*}{\partial \eta}\bigg|_{\eta=0}$$

$$= \frac{-k(T_0-T_\infty)}{R\delta\, h(\theta)}\left(\frac{1}{\int_0^\infty d\eta'\, e^{-\eta'^2}}\right)$$

$$Q = 2\pi R^2\int_0^\pi \sin\theta\, d\theta\, q_r(\theta)$$

$$Nu = 0.9213\, Pe^{1/2}$$

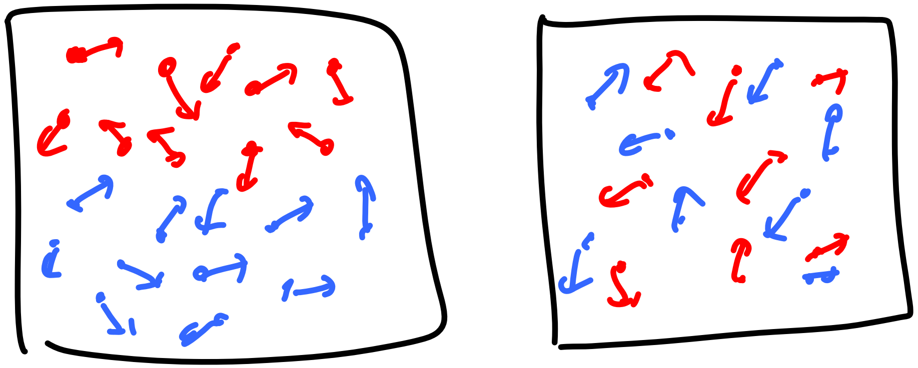
Summary:

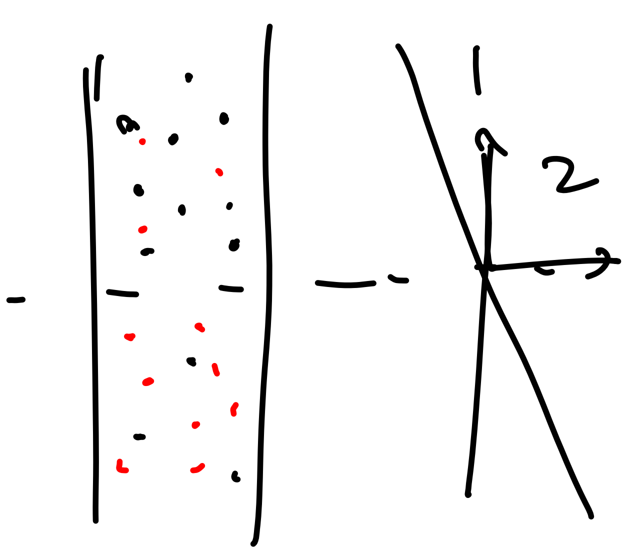


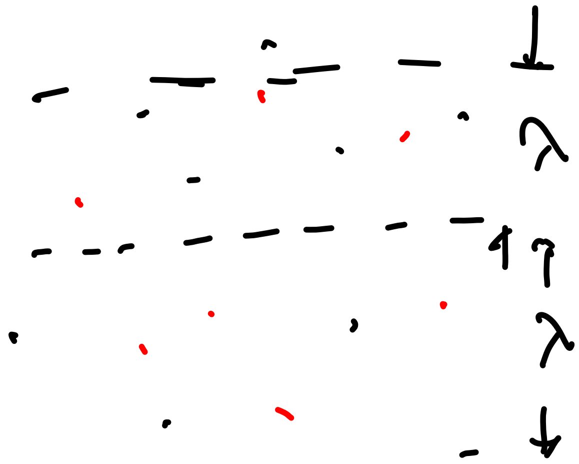

Convection

Diffusion

Diffusion:



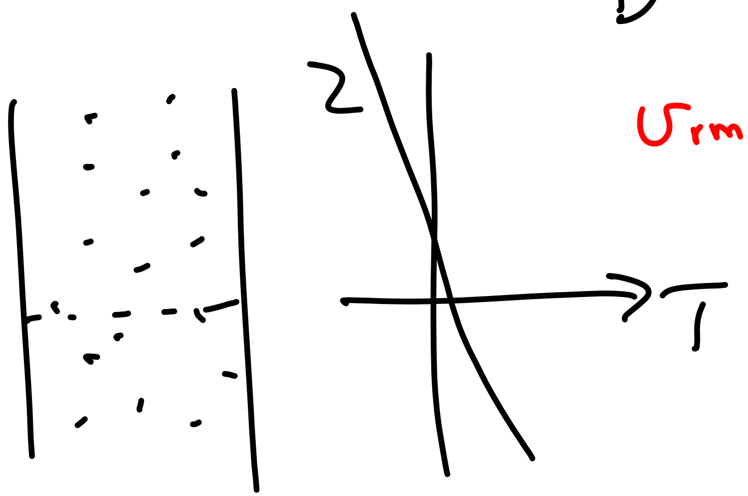



$$j_z = -D\frac{\partial c}{\partial z}$$

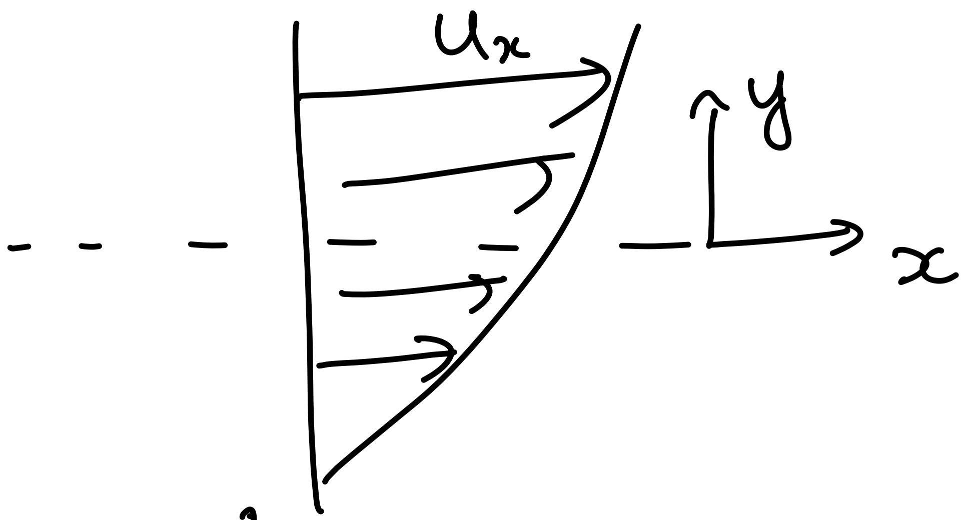
$$D = a\,\lambda\,v_{rms}$$

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

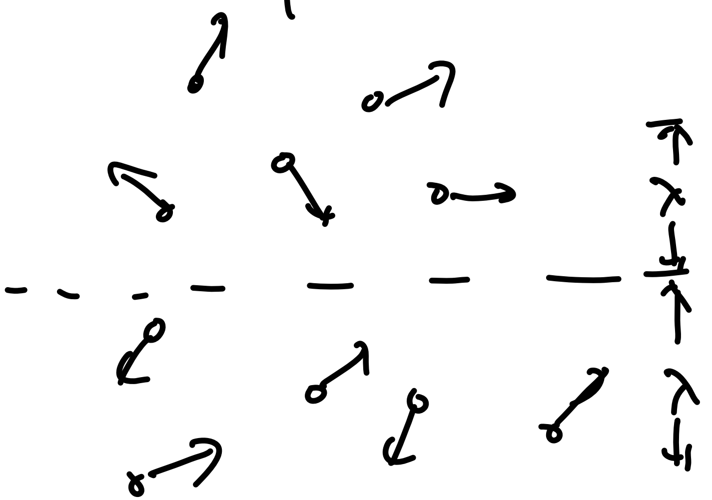



$$j_e = -k\frac{dT}{dz} = -\alpha\frac{de}{dz}$$

$\alpha$ = Thermal diffusivity = $(k/\rho C_p)$



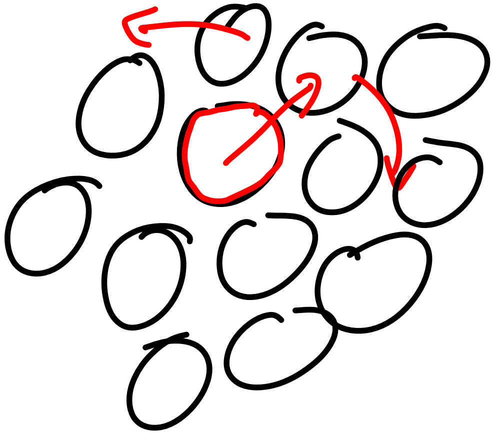

$u_x$ is a function of y



$\tau_{xy}$ = Force/Area at the surface in x direction at a surface with outward unit normal in y direction

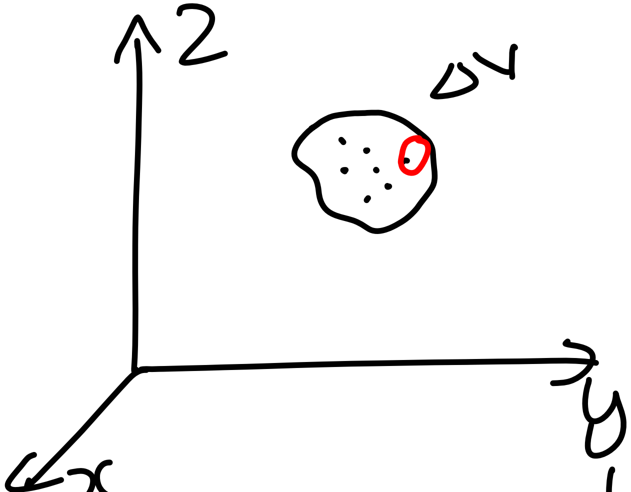
$$\tau_{xy} = \mu \frac{du_x}{dy} = \nu \frac{d}{dy}(\rho u_x)$$

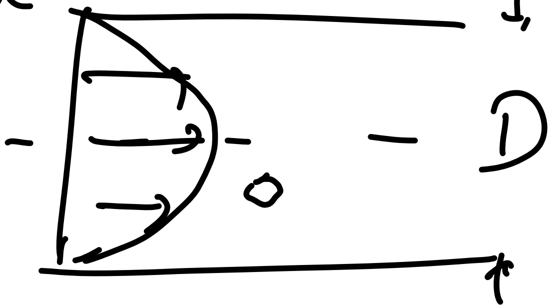
$\nu = (\mu/\rho)$



Fields:

$$\rho(x,y,z) = \lim_{\Delta V \to 0} \frac{\sum m_i}{\Delta V}$$



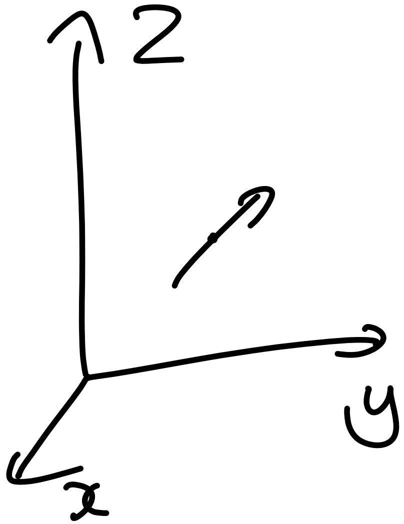




Continuum approximation:

$$(\rho \underline{u}) = \lim_{\Delta V \to 0} \frac{\sum m_i \underline{u}_i}{\Delta V}$$

$$\underline{u} = \frac{(\rho \underline{u})}{\rho}$$

$$e = \lim_{\Delta V \to 0} \frac{\sum e_i}{\Delta V} \qquad \delta C_p \Delta T = \Delta e$$




$$j_x = -D \frac{\partial c}{\partial x}$$

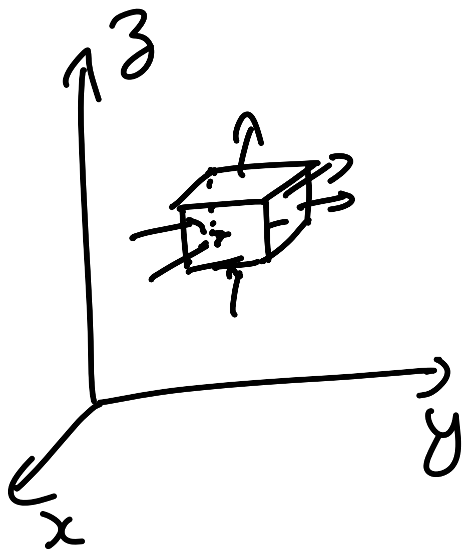
$$j_y = -D \frac{\partial c}{\partial y}$$

$$\underline{j} = j_x \underline{e_x} + j_y \underline{e_y} + j_z \underline{e_z}$$

$$= -D \left( \underline{e_x} \frac{\partial c}{\partial x} + \underline{e_y} \frac{\partial c}{\partial y} + \underline{e_z} \frac{\partial c}{\partial z} \right)$$

$$= -D \nabla c$$

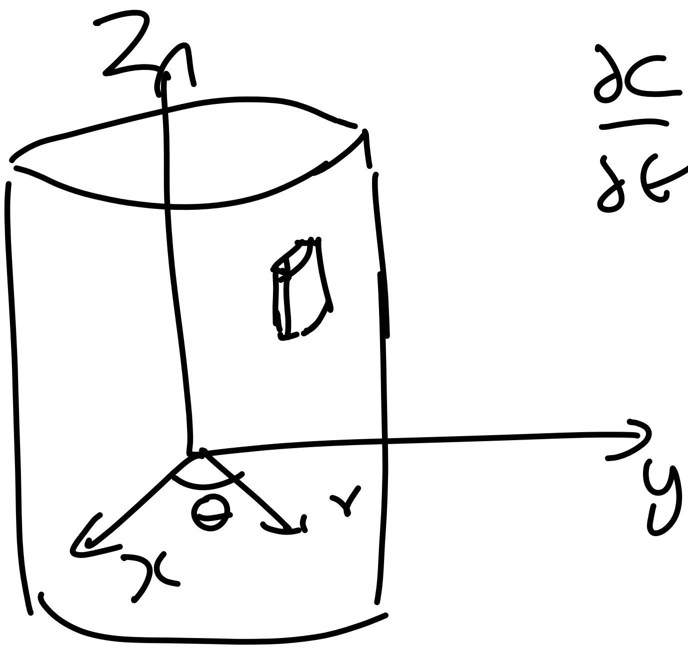
Shell balances:



$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(u_x c) + \frac{\partial}{\partial y}(u_y c) + \frac{\partial}{\partial z}(u_z c) =$$

$$D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) + S$$

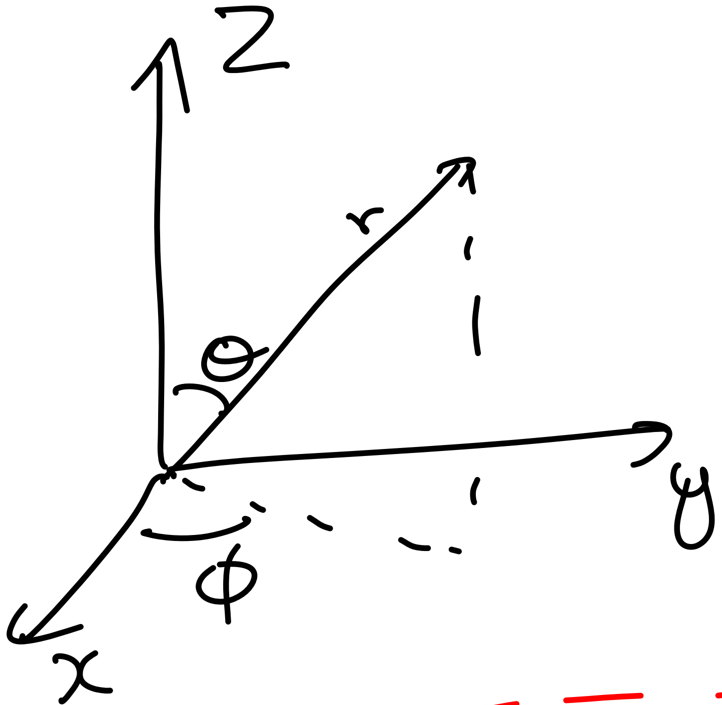
$$\frac{\partial c}{\partial t} + \nabla \cdot (\underline{u} c) = D\nabla^2 c + S$$




$$\frac{\partial c}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r c u_r) + \frac{1}{r}\frac{\partial (c u_\theta)}{\partial \theta} + \frac{\partial (c u_z)}{\partial z}$$

$$= D\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 c}{\partial \theta^2} + \frac{\partial^2 c}{\partial z^2}\right)$$

Spherical co-ordinate system:




$$\frac{\partial c}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 c u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, c u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial (c u_\phi)}{\partial\phi} =$$

$$D\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 c}{\partial\phi^2}\right)$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (\underline{u}\, c) = D\nabla^2 c + S$$

$$Pe\left(\frac{\partial c^*}{\partial t^*} + \nabla^* \cdot (\underline{u}^* c^*)\right) = \nabla^{*2} c + S^*$$

$$Pe = \left(\frac{UL}{D}\right)$$

$$Pe \ll 1 \quad D\nabla^2 c + S = 0$$

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} = 0$$

$$c = X(x)\, Y(y)\, Z(z)$$

$$X(x) = \sin(n\pi x^*)$$

$$Y(y) = \sin(m\pi y^*)$$

$$c = \sum_n \sum_m A_{nm} \sin(n\pi x^*) \sin(m\pi y^*) \left(e^{(m^2+n^2)^{1/2}\pi^2 z}\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 c}{\partial\phi^2}=0$$

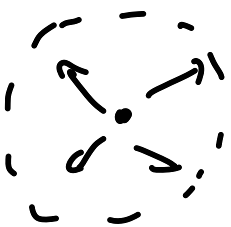
$$C=\sum_{nm}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)\boxed{Y_n^m(\theta,\phi)}$$

n, m are integers

$$Y_n^m(\theta,\phi)=P_n^m(\cos\theta)\begin{pmatrix}\cos\\ \sin\end{pmatrix}(m\phi)$$

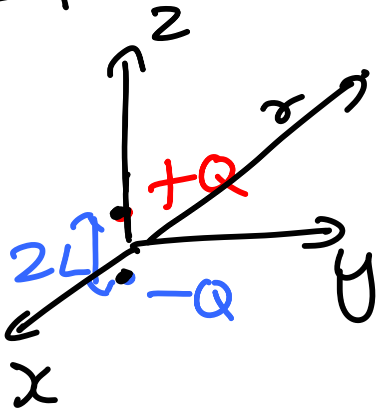
$$\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\;Y_n^m(\theta,\phi)\;Y_p^q(\theta,\phi)=\frac{2n}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{np}\delta_{mq}$$

$n=0$ & $m=0$

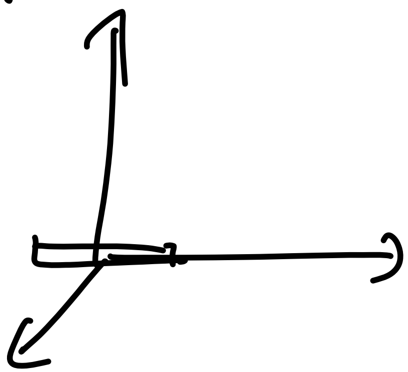


$$T=\frac{Q}{4\pi k r}\qquad k\nabla^2T+Q\delta(\underline{x})=0$$

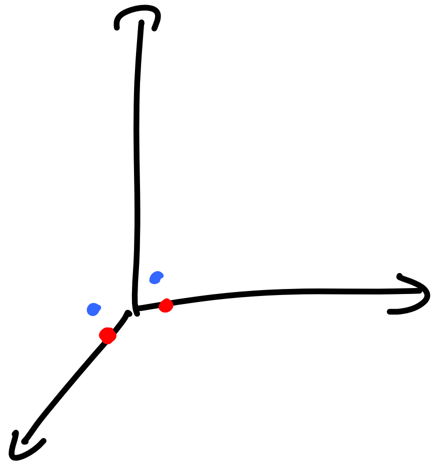
$n = 1 \quad m = 0$ Dipole




$$V = \frac{1}{r^2}\cos\theta = \frac{1}{r^2} P_1^0(\cos\theta)$$



$n = 1 \quad m = \pm 1$



High Peclet number limit:

$$Pe\left(\frac{\partial c^*}{\partial t^*} + \nabla^* \cdot (\underline{u}^* c^*)\right) = \nabla^{*2} c + S$$

$Pe \gg 1$

$l \sim Pe^{-1/3}$ for no-slip $Nu \propto Pe^{1/3}$

$l \sim Pe^{-1/2}$ for finite velocity at surface. $Nu \propto Pe^{1/2}$