Fundamentals of Transport Processes 1: Exercise solutions.

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1 Dimensional analysis:

1. The dimensionless number is a ratio of the rate of energy generated due to viscous dissipation, and the rate of energy conducted due to the temperature difference. From dimensional analysis, the rate of energy generated due to viscous dissipation per unit volume of the fluid, which has dimensions of $\mathcal{M}\mathcal{L}^{-2}\mathcal{T}^{-3}$, is proportional to $\mu(U/D)^2$, where μ, U and D are the viscosity, velocity and diameter of the tube. The rate of conduction of energy across the tube is $(k\Delta T/D^2)$, where k is the thermal conductivity and ΔT is the temperature difference. The ratio of these is the dimensionless Brinkman number $(\mu U^2/k\Delta T)$, which give the ratio of energy generated due to dissipation and energy conducted due to temperature variations.

2. In this problem, there are ten dimensional variables, and four dimensions, M, L, T, A. So there should be six dimensionless groups. Of these three are ratios, (R/L), (ϵ_w/ϵ_o) and (μ_w/μ_o). The other three dimensionless groups involve four physical mechanisms, inertia, viscosity, surface tension and electrical stress. On the basis of dimensional analysis, we can infer that the stress due to electrical effects,

$$\text{Stress} = \epsilon_w \frac{V}{L^4}\text{Function}(R/L, \epsilon_w/\epsilon_o) \tag{1}$$

If the droplet were small compared to the distance between plates, one would expect a constant potential gradient (V/L) around the droplet. In this case, the stress would be

$$\text{Stress} = \epsilon_w \left(\frac{V}{L^2}\right) \frac{1}{R^2} \text{Function}(\epsilon_w/\epsilon_o) \tag{2}$$

The dimensionless numbers include the Reynolds number (ratio of inertia and viscosity), the capillary number (ratio of surface tension and viscosity) and a number which gives the ratio of viscous stresses and electrical stresses, $(\epsilon V/R^3 \mu U)$.

3. The dimensions are,

$$\mathbf{E} \sim \mathcal{M}\mathcal{L}\mathcal{T}^{-3}\mathcal{A}^{-1} \tag{3}$$
$$\epsilon_0 \sim \mathcal{A}^2\mathcal{T}^4\mathcal{L}^{-3}\mathcal{M}^{-1} \tag{4}$$
$$\mathbf{B} \sim \mathcal{M}\mathcal{T}^{-2}\mathcal{A}^{-1} \tag{5}$$
$$\mu_0 \sim \mathcal{M}\mathcal{L}^2\mathcal{T}^{-2}\mathcal{A}^{-2} \tag{6}$$
$$\mathbf{J} \sim \mathcal{A}\mathcal{L}^{-2} \tag{7}$$

4. **Nozzle design** The nozzle diameter and the flow rate through the nozzle have to be adjusted in a manner that the desired drop size is obtained due to the flow through the nozzle. The drop size D_p, and the velocity of the drops U_p, will depend on the following parameters with their dimensions

   - Diameter of nozzle $D(\mathcal{L})$
   - Average flow speed through the nozzle $U(\mathcal{L}\mathcal{T}^{-1})$
   - The density of the liquid $\rho(\mathcal{M}\mathcal{L}^{-3})$
   - The viscosity of the liquid $\mu(\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1})$
   - The surface tension of the interface between the liquid and air $\gamma(\mathcal{M}\mathcal{T}^{-2})$
   - The surface tension between the liquid and the nozzle material $\gamma_s(\mathcal{M}\mathcal{T}^{-2})$

   In the above, there are six variables and three dimensions, so there should be three dimensionless numbers. The three are the Reynolds number, $(\rho UD/\mu)$, the capillary number (which gives the ratio of viscous and surface tension forces) $(\mu U/\gamma)$, and the ratio of surface tensions of the liquid - air and liquid - surface interfaces (γ_s/γ). So long as the material used in design remains unchanged, the ratio of surface tensions remains unchanged and so this can be neglected in the calculations.

It is first important to estimate the magnitudes of the Reynolds and capillary numbers. The Reynolds number scales as $(10^3 \times 0.5 \times 10^{-4}/0.1)$ for nozzle of diameter $100 \mu m$ and droplet speeds of 0.5 m/s. This indicates that inertial forces are not likely to be important, and the droplet size is likely to be determined by the balance between viscous and surface tension forces. The capillary number can be estimated as $0.1 \times 1/0.1$, which is 1 for a typical surface tension of 1 N/m, and a velocity of 1 m/s. Consequently, a dimensionless relationship for the droplet size and the droplet velocity at ejection can be determined as

$$\frac{D_p}{D} = \Phi\left(\frac{\mu U}{\gamma}\right)$$

$$\frac{U_p}{U} = \Psi\left(\frac{\mu U}{\gamma}\right)$$

The above scaling relations indicate that in the limit of low Reynolds number, the droplet diameter is a linear function of the nozzle diameter, and the droplet velocity is a linear function of the nozzle velocity. Consequently, it is sufficient to carry out experiments for a particular nozzle diameter and experimentally measure the range of droplet sizes obtained as a function of the capillary number. The nozzle and the flow speed can then be designed for the desired droplet size and droplet speed.

The next problem is to design the diameter of the spray drier, so that the droplet dries completely before it impacts the wall of the drier. For this, it is necessary to know the latent heat required for evaporating all the moisture in the droplet, and the rate of heat transfer which determines the rate at which heat is supplied to the droplet. The latent heat of water is about 1000 Btu/lb, which works out to about 2.2×10^6 J/kg. For a droplet of radius 100 μm, the amount of heat required to evaporate 80 % of the water is $0.8 \times 2.2 \times 10^6J/kg\times 1000kg/m^3 \times (\pi/6 \times (1 \times$

$10^{-4})^3)\text{m}^3 = 9.21 \times 10^{-4}$ J. The heat flux, which is the heat transfer rate per unit area that has to be achieved, is 3×10^4 J/m^2.

The heat flux at the droplet surface is related to the following parameters with the following dimensions

- Heat flux per unit area $(q/A)(\mathcal{H}\mathcal{L}^{-2}\mathcal{T}^{-1})$
- Droplet radius $D_p(\mathcal{L})$
- Droplet velocity $U_p(\mathcal{L}\mathcal{T}^{-1})$
- Thermal conductivity of air $k(\mathcal{H}\mathcal{L}^{-1}\mathcal{T}^{-1}\Theta^{-1})$
- Specific heat of air $c_p(H\mathcal{M}^{-1}\Theta^{-1})$
- Density of air $\rho_a(\mathcal{M}\mathcal{L}^{-3})$
- Viscosity of air $\mu_a(\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1})$
- Average temperature difference between the air and the droplet $\Delta T(\Theta)$

There are eight dimensional parameters, and five dimensions, so it is possible to obtain three dimensionless groups. These are the Reynolds number $(\rho_a U_p D_p/\mu_a)$, the Prandtl number $(c_p\mu_a/k)$, and the dimensionless heat flux $(q/A)(D_p/k\Delta T)$. Therefore, the heat average heat flux can be determined using the following relation

$$\frac{q}{A}\frac{D_p}{k\Delta T} = \Phi\left(\frac{\rho_a D_p U_p}{\mu_a}, \frac{c_p\mu_a}{k}\right)$$

The heat flux can be calculated by empirical relations of the form

$$\frac{q}{A}\frac{D_p}{k\Delta T} = 2.0 + 0.6\left(\frac{\rho_a D_p U_p}{\mu_a}\right)^{1/2}\left(\frac{c_p\mu_a}{k}\right)^{1/3}$$

which are available in standard text books.

In the above relationship, the Reynolds number can be estimated as 1kg / m$^3\times$1m/s$\times 10^{-4}$m/10^{-5}kg/m/s$\,=10$. This is not small, so inertia could be important for heat transfer through the air. The Prandtl number for gases is typically $O(1)$, so the dimensionless heat flux is between $2-10$ typically. Using the thermal conductivity for air (2×10^{-2}J/m/s/^{o}C), and an average temperature difference between the air and the droplet of 40^{o}C, the heat transfer rate is at least $2 \times 2 \times 10^{-2}$J/m/s/ ^{o}C $\times$ 40^{o}C/10^{-4}m $= 1.6 \times 10^4$J/m^2/s. Consequently, the time required for drying is about 2 seconds. At a speed of 0.5 m/s, it is necessary to have a spray drier of radius 1m.

It is not enough to ensure that the necessary latent heat is transferred to the particle, it is also necessary to determine whether all the water evaporated can diffuse out of the particle within the time required for the particle to reach the wall. This is a mass transfer problem, and the flux of particle from the surface j (mass per unit area per unit time) has to be predicted as a function of the radius of the particle, and the difference in moisture concentration between the surface of the particle and the ambient air. If we assume that the ambient air is dry, the water vapour concentration at the particle surface is the saturation concentration at that temperature. If the vapour pressure is 0.4 atm, then the concentration is 0.4 times the density of air, or about 0.4 kg m^{-3}. The mass diffusion coefficient for water in air is about 2×10^5 m^2/s. In order to ensure that all the moisture is transferred from the particle, it is necessary to evaporate about 1.33×10^{-2} kg/m^2 of the surface of the droplet. This is obtained as follows. The mass of water per droplet is 0.8×10^3kg/m^3 $\times$ $(\pi/6 \times (10^{-4})^3)$m^3, and this is divided by the area of the droplet $\pi \times (10^4)$m^2 to get the above mass transferred per unit volume.

The mass flux at the surface of the droplet depends on the following parameters, with their respective units.

- Mass flux per unit area $j(\mathcal{M}_w\mathcal{L}^{-2}\mathcal{T}^{-1})$
- Droplet radius $D_p(\mathcal{L})$
- Droplet velocity $U_p(\mathcal{L}\mathcal{T}^{-1})$
- Diffusivity of air $\mathcal{D}(\mathcal{L}\mathcal{T}^{-2})$
- Density of air $\rho_a(\mathcal{M}\mathcal{L}^{-3})$
- Viscosity of air $\mu_a(\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1})$
- Average concentration difference between the air and the droplet $\Delta c(\mathcal{M}_w\mathcal{L}^{-3})$

There are seven dimensional parameters, and four dimensions, so there are three dimensionless units. These are easily derived using methods similar to those used for heat transfer, and a general relationship of the following form is obtained

$$\frac{jD_p}{\mathcal{D}\Delta c} = \Phi\left(\frac{\rho_a D_p U_p}{\mu_a}, \frac{\mu_a}{\rho_a \mathcal{D}}\right)$$

This is the most we can do with dimensional analysis, but experiments have been done to determine the above relationship, and the result is exactly the same as that for heat transfer upto a Reynolds number of about 1000

$$\frac{jD_p}{\mathcal{D}\Delta c} = 2.0 + 0.6\left(\frac{\rho_a D_p U_p}{\mu_a}\right)^{1/2}\left(\frac{\mu_a}{\rho_a \mathcal{D}}\right)^{1/3}$$

For design purposes, we can take a lower limit of 2 for the dimensionless number on the left. Using this, the flux can be estimated as $2 \times (2 \times 10^{-5}\mathrm{m}^2/\mathrm{s} \times 0.4\mathrm{kg/m}^3/10^{-4}\mathrm{m}$, which is $8 \times 10^{-2}\mathrm{kg/m}^2/s$. Using this, the time required is estimated as 0.17 s.

5. First, it is useful to see why there is a limit on the velocity of water through the bed. This is because, if the force exerted by the water on the particles is greater than the difference between the weight and the buoyancy force of the particles, the particles will be swept away by the flow. The density of carbon ρ_c is 1.1 $\times 10^3$ kg / m^3, while that of water ρ_w is 1.0 $\times 10^3$ kg / m^3. The net force exerted by the particles (weight - buoyancy) is given by $(\rho_c - \rho_w)(4\pi/3)r^3$, where r is the radius of the particles. This works out to 5.236×10^{-9} N. The upward force exerted on the particles due to flow is $6\pi\mu r U$, which is $1.885 \times 10^{-6} U$ N, where U is measured in m/s. For the two to be equal, the velocity is 2.78 mm/s. Some safety factor must have been added to stipulate that the maximum velocity is 1 mm/s.

   The parameters which affect the design of the bed, and their dimensional dependences, can be listed as follows

   - Height of the bed $H(\mathcal{L})$
   - Diameter of the bed $D(\mathcal{L})$
   - Particle diameter $d(\mathcal{L})$
   - Inlet concentration $c_i(\mathcal{M}_o/\mathcal{L}^3)$

- Outlet concentration $c_o(\mathcal{M}_o/\mathcal{L}^3)$
- Adsorption rate constant $k_a(\mathcal{T}^{-1})$
- Diffusion constant for the organics $D_c(\mathcal{L}^2\mathcal{T}^{-1})$
- Porosity of the bed ϵ (dimensionless)
- Velocity $U(\mathcal{L}\mathcal{T}^{-1})$
- Density of fluid $\rho(\mathcal{M}\mathcal{L}^{-3})$
- Viscosity of fluid $\mu(\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1})$

Of these parameters, the density and viscosity are required for mechanical calculations, and not for mass transfer calculations. Since the diameter and height of the bed are determined from mass transfer considerations, these need not be included. However, they will be necessary for determining the pressure drop required, as we shall see later. Therefore, there are a total of nine dimensional variables, and these involve three dimensions $(\mathcal{M}_o, \mathcal{L}, \mathcal{T})$, and this leaves us with six dimensionless groups. One of these is the porosity, which is itself dimensionless. There are two length ratios, (D/d) and (H/d) relating the three length scales in the problem. There is one concentration ratio, which can be considered as the adsorption fraction $(c_i - c_o)/c_i$, of the organics. This leaves us with two other dimensionless parameters which need to be determined form the variables (k_a, D_c, U) and the other length parameters. This brings up the question about what is the appropriate length scale for scaling these parameters. The choice of the correct length scale requires physical arguments. The adsorption of the organics into the fluids occurs due to the diffusion of the organics from the fluid to the surface of the carbon particles over length scales comparable to the particle diameter, so it is appropriate to use the particle diameter as the appropriate length scale for dimensionless groups. With this, it is easy to see that one can get two dimensionless groups, $(k_a d^2/D_c)$, which gives the ratio of adsorption and diffusion, and (Ud/D_c), which gives the ratio of convection and diffusion. Therefore, the six dimensionless groups are

- (D/d)
- (H/d)
- $(c_i - c_o)/c_i$

- ϵ
- $(k_a d^2/D_c)$
- (Ud/D_c)

It is immediately apparent that there is a considerable separation of length scales in the problem. While the diameter and the height could be of the order of meters, the particle diameter is measured in millimeters. Consequently, the mass transfer process, which occur on the scale of the particle diameter locally, will be insensitive to the total diameter of the bed. In other words, every point along a horizontal cross section of the bed is equivalent to every other point. Therefore, if the velocity of the fluid through the bed is fixed, the diameter is determined from the flow rate. If the velocity through the bed is U, then the 'superficial' velocity (at the entrance) is ϵU, since the void fraction of the bed is ϵ. The flow rate is expressed in terms of the superficial velocity as $Q = \epsilon U(\pi D^2/4)$. Since the flow rate is given (180 l/min or 3×10^{-3} m^3/s), the diameter is related to the velocity U by $D = (12/Pi\epsilon U)^{1/2}$, where U is expressed in m/s. For a maximum velocity U = 1 mm/s, the bed diameter turns out to be 2.9 m. For lower velocities, the diameter will be correspondingly larger.

It is useful, at this point, to estimate the relative magnitudes of convection, diffusion and adsorption. The dimensionless parameter (Ud/D_c) is typically $O(10^3)$ in this case, since the diffusivities in most fluids is $O(10^{-9})$ m^2/s. Consequently, convective effects are large compared to diffusive effects even at velocities as small as 1 mm/s, and convection is important. However, as noted earlier, even when the ratio of convection and diffusion is large (high Peclet number), diffusion cannot be neglected since convection only transports material tangential to the particle surface, and not in the normal direction. The ratio of adsorption and diffusion, $k_a d^2/D_c$, is $O(1000)$, and so reaction is fast compared to diffusion. Consequently, diffusion is the rate limiting step and reaction at the particle surface is instantaneous.

Clearly, since convection is an important effect, the height required for effecting the necessary conversion will depend on the velocity U. To determine this without a knowledge of the microscopic processes occurring in the bed, it is necessary to carry out experiments with beds of various heights and speeds, and to find out the height required for 99 % conversion as a function of the speed U. Clearly, this is a difficult task. However, this task can be rendered much simpler using the following physical reasoning.

Since the height of the bed is large compared to the particle diameter, the microscopic processes that result in the adsorption of the solute are insensitive to the total height, but depend only on the local concentration and velocity of the fluid. Consequently, the variation in concentration with height of the bed can be related to the variation in concentration with time of a thin layer of carbon through which the effluent is circulated. The height required for conversion in the bed H is related to the time required for conversion in the recirculating bed τ by the relation $H = (U\tau)$. Consequently, it is necessary to do the experiment in the thin carbon layer for different velocities U, and determine the conversion time τ as a function of U. Using this, one can obtain H as a function of U.

The experiments in these circulating systems have been carried out, and they reveal that the time required for 99 % conversion typically vary as $U^{-1/2}$. For typical systems with a rate of diffusion 10^{-9} m^2/s, the relationship is $\tau = cU^{-1/2}$, where U is in m/s, and the coefficient c has a numerical value of $O(100)$ with units $m^{1/2}$ $s^{1/2}$. Consequently, the height required for the bed is $H = cU^{1/2}$, which for a velocity of 1mm/s is about 3.16 m.

2 Diffusion

1. The mean free path is given by the expression,

$$\lambda = \frac{1}{2\sqrt{\pi} n d^2}$$

where n is the number of molecules per unit volume and d is the molecular diameter. The number of molecules per unit volume is given by

$$n = \frac{p}{kT}$$

where the Boltzmann constant $k = 1.3807 \times 10^{-23} J/K$. This gives $n = 2.4143 \times 10^{25}$ molecules per cubic meter at $T = 300$K and $p = 10^5$

Pa. Using this, the mean free path of hydrogen is 1.097×10^{-7} m, and that of chlorine is 6.900×10^{-8} m. The ratio of mean free path and molecular diameter is 376 for hydrogen, and 134 for chlorine.

The mean molecular velocity is given by,

$$\bar{u} = \sqrt{\frac{8kT}{\pi m}}$$

where m is the molecular mass. The molecular mass for hydrogen is $2/(6.023 \times 10^{-26}) = 3.3206 \times 10^{-27}$kg, while that for chlorine is $71/(6.023 \times 10^{-26}) = 1.179 \times 10^{-25}$kg. Therefore, the mean molecular velocities for hydrogen and chlorine are 1782m/s and 299m/s.

In kinetic theory, the viscosity is given by,

$$\mu = \frac{1}{3} nm\bar{u}\lambda$$

The values for hydrogen and chlorine are 6.549×10^{-6}kg/m/s and $1.958 * 10^{-5}$kg/m/s.

2. The viscosity of pure nitrogen and oxygen, calculated as above, are 1.226×10^{-5} and 1.507×10^{-5} kg/m/s respectively.

3. The Knudsen number is the ratio of Mach and Reynolds numbers.

4. The mass flux at a surface is given by,

   $$j = au_{rms}c$$

   where a is a constant, u_{rms} is the root mean square velocity and c is the concentration. If the concentration is uniform and there is a gradient in the root mean square velocity, then the flux will be given by,

   $$j = a\lambda c \frac{\partial u_{rms}}{\partial x}$$

   This can be simplified to obtain,

   $$j = a\lambda c\sqrt{3k}2Tm \frac{\partial T}{\partial x}$$

5. The Stokes-Einstein formula states that

$$D = \frac{kT}{6\pi\mu R}$$

where R is the radius. If we assume that the viscosity of water is 10^{-3}kg/m/s, the radius of haemoglobin is 3.18471×10^{-9}m.

6. The diffusion coefficients of hydrogen, oxygen and benzene are 1.5077×10^{-9}, 1.28019×10^{-9} and 8.33948×10^{-10} m^2/s respectively.

3 Unidirectional flow in Cartesian co-ordinates:

1. - The mass and momentum equations for this velocity profile reduce to:

$$\frac{\partial u_x}{\partial x} = 0 \tag{8}$$

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2} \tag{9}$$

$$\frac{\partial p}{\partial y} = 0 \tag{10}$$

The boundary conditions are:

$$u_x = 0 \quad \text{at} y = 0 \tag{11}$$

$$u_x = V(t) \quad \text{at} y = H \tag{12}$$

In addition since the ends are closed, there should be no net flux of fluid across any cross section. For this, we require:

$$\int_0^H dy u_x = 0 \tag{13}$$

   - For a steady flow, the time derivation can be neglected in the momentum equation 9 and the solution is:

$$u_x = \frac{\partial p}{\partial x}\frac{y^2}{2} + C_1 y + C_2 \tag{14}$$

The boundary 11 requires that $C_2 = 0$ and the two other conditions 12 and 13 give the following equations:

$$\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{H^2}{2} + C_1 H = V \tag{15}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{H^3}{6} + \frac{C_1 H^2}{2} = 0 \tag{16}$$

The above equations can be solved to obtain:

$$\frac{\partial p}{\partial x} = \frac{6\mu V}{H^2} \tag{17}$$

$$u_x = \frac{3Vy^2}{H^2} - \frac{2Vy}{H} \tag{18}$$

- In the presence of an oscillating top plate $V\cos(\omega t)$, we would expect the velocity profile also to be oscillating because this is a linear problem. So we can choose the velocity u_x and pressure p as:

  $$u_x = \tilde{u}(y)\exp(i\omega t) \quad p = \tilde{p}(y)\exp(i\omega t) \tag{19}$$

  Inserting this into the momentum equation 9, we obtain:

  $$\rho\imath\omega\tilde{u} = -\frac{\partial\tilde{p}}{\partial x} + \mu\frac{\partial^2\tilde{u}}{\partial y^2} \tag{20}$$

  The solution for this equation is:

  $$\tilde{u} = \frac{\imath}{\rho\omega}\frac{\partial\tilde{p}}{\partial x} + C_1\exp(\sqrt{\imath\rho\omega/\mu}y) + C_2\exp(-\sqrt{\imath\rho\omega/\mu}y) \tag{21}$$

  The boundary conditions 11 and 12 give:

  $$C_1 + C_2 + \frac{\imath}{\rho\omega}\frac{\partial\tilde{p}}{\partial x} = 0 \tag{22}$$

  $$C_1\exp(\sqrt{\imath\rho\omega/\mu}H) + C_2\exp(-\sqrt{\imath\rho\omega/\mu}H) + \frac{\imath}{\rho\omega}\frac{\partial\tilde{p}}{\partial x} = V \tag{23}$$

  $$\frac{\mu}{\imath\rho\omega}\left[C_1\Big(\exp(\sqrt{\imath\rho\omega/\mu}H) - 1\Big) - C_2\Big(\exp(-\sqrt{\imath\rho\omega/\mu}H) - 1\Big)\right] + \frac{\imath H}{\rho\omega}\frac{\partial\tilde{p}}{\partial x} = 0 \tag{24}$$

2. The non-dimensional temperature field is defined as $T^* = (T - T_0)/(T_1 - T_0)$. In terms of this temperature field, the boundary conditions are,

$$T^* = 1 \text{ at } x = 0 \tag{25}$$
$$T^* = 0 \text{ at } x = L \tag{26}$$
$$T^* = 0 \text{ at } y = 0 \tag{27}$$
$$T^* = 0 \text{ at } y = H \tag{28}$$

The solutions of the heat conduction, which satisfy the homogeneous boundary conditions in the y direction, are,

$$T^* = \sum_{n=1}^{\infty} \sin(n\pi y/H)(A_n \exp(n\pi x/H) + B_n \exp(-n\pi x/H) \tag{29}$$

From the boundary condition $T^* = 0$ at $x = L$, we obtain one of the coefficients in the above equation,

$$T^* = \sum_{n=1}^{\infty} A_n \sin(n\pi y/H)(\exp(n\pi x/H) - \exp(n\pi(2L - x)/H)) \tag{30}$$

The coefficients A_n can be determined from the orthogonality relation at $x = 0$,

$$\sum_{n=1}^{\infty} A_n \int_0^H dy \sin(m\pi y/H) \sin(n\pi y/H)(1 - \exp(2n\pi L/H)) = \int_0^H dy \sin(m\pi y/H) \tag{31}$$

This equation reduces to,

$$\begin{aligned} A_m &= \frac{4}{m\pi}(1 - \exp(2n\pi L/H))^{-1} \text{ for odd } n \\ &= 0 \text{ for even } n \end{aligned} \tag{32}$$

Therefore, the solution for the temperature field is,

$$T^* = \sum_{n=1,3,\ldots}^{\infty} \left(\frac{4}{n\pi}\right) \frac{\sin(n\pi y/H)(\exp(n\pi x/H) - \exp(n\pi(2L - x)/H))}{(1 - \exp(2n\pi L/H))} \tag{33}$$

The heat flux from the surface can be calculated as the flux in the x direction at $x = 0$,

$$\begin{aligned} q_x &= k(T_1 - T_0)\left(\frac{dT^*}{dx}\right)_{x=0} \\ &= k(T_1 - T_0)\sum_{n=1,3,\ldots}^{\infty}\frac{4\sin(n\pi y)}{n\pi}\frac{n\pi}{H}\frac{(1+\exp(2n\pi L/H))}{1-\exp(2n\pi L/H)} \end{aligned} \tag{34}$$

The total heat flux (per length in the direction perpendicular to the plane of the rectangle) is,

$$\begin{aligned} Q &= \int_0^H dy q_x \\ &= k(T_1 - T_0)\sum_{n=1,3,\ldots}^{\infty}\frac{4}{n\pi}\frac{2H}{n\pi}\frac{n\pi}{H}\frac{(1+\exp(2n\pi L/H))}{1-\exp(2n\pi L/H)} \end{aligned} \tag{35}$$

3. It is convenient to use a scaled temperature $\Theta = (T - T_0)/(T_1 - T_0)$. The boundary conditions are,

$$\Theta = 0 \quad \text{at } y = 0 \tag{36}$$

$$\Theta = 0 \quad \text{at } y = L \tag{37}$$

$$\Theta = 1 \quad \text{at } x = 0 \tag{38}$$

$$\Theta = 0 \quad \text{at } x \to \infty \tag{39}$$

The conduction equation for the temperature field is,

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} = 0 \tag{40}$$

This is solved by separation of variables, $\Theta = X(x)Y(y)$. The solution in the y direction is easily obtained,

$$Y(y) = \sin\left(\frac{n\pi y}{L}\right) \tag{41}$$

where n is an integer. The solution in the x direction is, which goes to zero as x goes to infinity, is,

$$X(x) = \exp\left(\frac{n\pi x}{L}\right) \tag{42}$$

Therefore, the final solution is given by,

$$\Theta = \sum_n A_n \sin\left(\frac{n\pi y}{L}\right) \exp\left(-\frac{n\pi x}{L}\right) \tag{43}$$

The constants in the above equation can be obtained from the consideration that $\Theta = 1$ at $x = 0$. Multiplying Θ by $\sin(m\pi x/L)$ and integrating over x, we obtain

$$\int_0^L dy \sin\left(\frac{m\pi y}{L}\right) \sum_n \sin\left(\frac{n\pi y}{L}\right) = \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \tag{44}$$

Simplifying both sides, we get,

$$\begin{aligned} \frac{A_m L}{2} &= \frac{2L}{\pi m} \text{ for odd m} \\ &= 0 \text{ for even m} \end{aligned} \tag{45}$$

Therefore, the final solution is,

$$\Theta = \sum_{\text{odd}n} \frac{4}{\pi n} \sin\left(\frac{n\pi y}{L}\right) \exp\left(-\frac{n\pi x}{L}\right) \tag{46}$$

The heat flux at the surface $x = 0$ is given by,

$$\begin{aligned} q &= -K \left.\frac{dT}{dx}\right|_{x=0} \\ &= \frac{4K(T_1 - T_0)}{L} \sum_{\text{odd}n} \sin\left(\frac{n\pi y}{L}\right) \end{aligned} \tag{47}$$

The total heat transported is obtained by integrating the heat flux over the y co-ordinate,

$$\begin{aligned} Q &= \frac{4K(T_1 - T_0)}{L} \sum_{\text{odd}n} \int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \\ &= \frac{4K(T_1 - T_0)}{L} \sum_{\text{odd}n} \frac{2L}{n\pi} \end{aligned} \tag{48}$$

4. If x is the co-ordinate along the length of the channel, the equation of motion is,
$$-\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) = 0$$
The boundary conditions are $u_x = 0$ at $y = 0, W$ and $z = 0, H$. The solution is of the form,
$$u_x = \sum_{m,n} A_{mn} \sin(m\pi y/W) \sin(n\pi z/H)$$
The constants A_{mn} are determined by inserting the above into the equation of motion,
$$\sum_{m,n} A_{mn} \sin(m\pi y/W) \sin(n\pi z/H) \left((m^2\pi^2/W^2) + (n^2\pi^2/H^2) \right) = -\frac{\partial p}{\partial x}$$
To obtain the constants, we use the orthogonality conditions, where both sides or the above equation area multiplied by $\sin(m\pi y/W)\sin(n\pi z/H)$ and integrated over $0 < y < W$ and $0 < z < H$,
$$A_{mn} = \left(-\frac{\partial p}{\partial x} \right) \frac{4}{nm\pi^2} \left((m^2\pi^2/W^2) + (n^2\pi^2/H^2) \right)^{-1}$$
for odd n and m.

5. First, note that there is no variation in the z direction, because the flux on the two walls perpendicular to the z co-ordinate are zero. Therefore, the temperature is a function of x and y alone. The heat conduction equation at steady state is,
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{49}$$
First, we can make the temperature in the y direction homogeneous by defining a new temperature $\Theta = T - (T_C + (T_D - T_C)(y/L))$. Then the temperature on the two faces $y = 0$ and $y = L$ are $\Theta = 0$. We can now use separation of variables for the temperature Θ,
$$\Theta = X(x)Y(y) \tag{50}$$

The differential equation for X and Y now become,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \tag{51}$$

Since the left side is only a function of x and the right side is only a function of y, these are both equal to constants. We now need to decide whether the constant is positive or negative. Note that the homogeneous boundary conditions are in the y direction, so we should expect sine and cosine solution in that direction. Therefore, it is best to take the constant to be positive, α^2. In this case, the solution for Y is,

$$Y = A\sin(\alpha y) + B\cos(\alpha y) \tag{52}$$

The condition $Y = 0$ at $y = 0$ implies that $B = 0$. The second condition, $Y = 0$ and $x = L$, fixes the value of α to be $(n\pi/L)$, where n is an integer. Therefore, the solution for Y is,

$$Y = A\sin(n\pi y/L) \tag{53}$$

The equation for the x direction is, then,

$$\frac{d^2X}{dx^2} = (n\pi/L)^2 X \tag{54}$$

The solution for this is,

$$X = A_n \exp(n\pi x/L) + B_n \exp(-n\pi x/L) \tag{55}$$

Therefore, the final solution for the temperature field is,

$$\Theta = \sum_n (A_n \exp(n\pi x/L) + B_n \exp(-n\pi x/L))\sin(n\pi y/L) \tag{56}$$

The boundary conditions are,

$$\begin{aligned} \Theta &= T_A - (T_C + (T_D - T_C)(y/L)) \qquad \text{at} x = 0 \\ \Theta &= T_B - (T_C + (T_D - T_C)(y/L)) \qquad \text{at} x = L \end{aligned} \tag{57}$$

Written in terms of the solutions for Θ, we find,

$$\begin{aligned} \sum_n (A_n + B_n)\sin(n\pi y/L) &= T_A - (T_C + (T_D - T_C)(y/L)) \\ \sum_n (A_n \exp(n\pi) + B_n \exp(-n\pi))\sin(n\pi y/L) &= T_B - (T_C + (T_D - T_C)(y/L)) \end{aligned} \tag{58}$$

These equations are solved using the orthogonality relations,

$$\int_0^L dx \sin(n\pi x/L) \sin(m\pi x/L) = (L/2)\delta_{nm} \tag{59}$$

Therefore, we multiply the equations for the boundary conditions by $\sin(n\pi x/L)$ and integrate from 0 to L, to obtain,

$$\begin{aligned} (L/2)(A_m + B_m) &= (T_A - T_C)(1 + (-1)^{m-1})(L/m\pi) \\ &\quad +(T_D - T_C)((-1)^{m-1}L/n\pi) \\ (L/2)(A_m \exp(m\pi) + B_m \exp(-m\pi)) &= (T_B - T_C)(1 + (-1)^{m-1})(L/m\pi) \\ &\quad +(T_D - T_C)((-1)^{m-1}L/n\pi) \end{aligned} \tag{60}$$

These equations can be solved to obtain the solutions for A_m and B_m.

6. The rate of dissipation of energy due to fluid friction will be calculated later when we derive the mass, momentum and energy balance equations for a fluid. For a laminar shear flow where the velocity is in the x direction and the velocity variation is in the z direction, the rate of dissipation of energy (per unit volume per unit time) is given by,

$$\begin{aligned} S_e &= \tau_{xy}\frac{du_x}{dz} \\ &= \mu\left(\frac{du_x}{dz}\right)^2 \end{aligned} \tag{61}$$

where τ_{xy} is the shear stress, and (du_x/dy) is the strain rate. Using this velocity profile, we find that the dissipation rate per unit volume, S_e, is,

$$S_e = 16U^2\left(\frac{z}{H} - \left(\frac{z}{H}\right)^2\right)^2 \tag{62}$$

At steady state, the energy balance equation reduces to,

$$k\frac{d^2T}{dz^2} + \frac{16\mu U^2}{H^2}\left(1 - \frac{2z}{h}\right)^2 = 0 \tag{63}$$

The boundary conditions are,

$$T = T_0 \quad \text{at} \quad z = 0 \tag{64}$$
$$T = T_0 \quad \text{at} \quad z = H \tag{65}$$

It is natural to define a scaled z co-ordinate, $z^* = (z/H)$, and a scaled temperature, $T^* = ((T - T_0)/T_0)$. Defined this way, the scaled temperature is the ratio of the local temperature rise due to viscous heating and the wall temperature. With this non-dimensionalisation, the energy balance equation becomes,

$$\frac{d^2T^*}{dz^{*2}} + 16\text{Br}(1 - 2z^*)^2 = 0 \tag{66}$$

with boundary conditions,

$$T^* = 0 \quad \text{at} \quad z^* = 0 \tag{67}$$
$$T^* = 0 \quad \text{at} \quad z^* = 1 \tag{68}$$

where the Brinkman number is,

$$\text{Br} = \frac{\mu U^2}{kT_0} \tag{69}$$

Equation 66 can be easily solved, subject to boundary conditions 67 and 68, to obtain,

$$T^* = \text{Br}\left(\frac{8z^*(1 - z^*)(1 - 2z^* + 2z^{*2})}{3}\right) \tag{70}$$

The profile of the scaled temperature, divided by Br, is shown as a function of the scaled z co-ordinate in figure 6. The temperature profile is very flat at the center of the channel, because the strain rate (du_x/dz) decreases to zero at the center, and the rate of generation also decreases to zero. The rate of generation of heat is a maximum near the wall, where the strain rate is a maximum.

From equation 70 for the temperature profile, the fractional increase in the temperature within the channel is given by the Brinkman number. For Br $\ll$ 1, the temperature rise in the channel is small compared to the wall temperature, and so change in temperature due to viscous

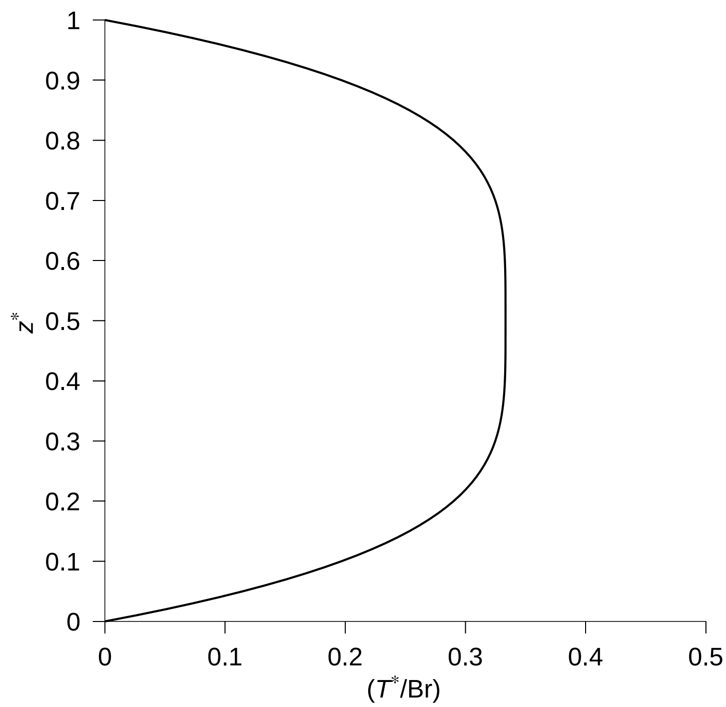

Figure 1: The ratio (T^*/Br) of the scaled temperature $T^* = ((T - T_0)/T_0)$, as a function of z^* a channel.

heating can be neglected. Viscous heating also results in a flux of energy across the wall of the channel, which is given by,

$$\begin{aligned} q_z &= -k\frac{dT}{dz} \\ &= -\frac{kT_0}{H}\frac{dT^*}{dz^*} \\ &= \frac{8kT_0(1-2z^*)^3\mathrm{Br}}{3H} \\ &= \frac{8\mu U^2(1-2z^*)^3}{3H} \end{aligned} \tag{71}$$

The heat flux is negative at the bottom surface at $z^* = 0$, because heat is transferred downwards from the fluid to the wall. At $z^* = 1$, the heat flux is positive because heat is transferred upwards to the wall. In both cases, the magnitude of the heat flux is given by $(8\mu U^2/3H)$.

4 Unidirectional flow in curvilinear co-ordinates:

1. The momentum eqution, obtained by shell balances, has the form,

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0$$

Since there is no pressure gradient in the z direction, the velocity profile is obtained by solving the above equation with $(\partial p/\partial z) = 0$. The solution of this equation is,

$$u_z = C_1 \log{(r)} + C_2$$

The boundary conditions are $u_z = U$ at the surface of the wire $r = R_w$, while $u_z = 0$ at the surface of the tube $r = R_t$.

$$u_z = \frac{U\log{(R_t/r)}}{\log{(R_t/R_w)}}$$

The total flow rate is determined by integrating the above equation from R_t to R_2,

$$Q = 2\pi\int_{R_w}^{R_t} r dr u_z = \pi(R_t^2 - R_w^2 - 2R_w^2\log{(R_t/R_w)})U/2\log{(R_t/R_w)}$$

2. The heat conduction equation will have the form

$$\frac{\partial T}{\partial t} = D_T \left(\frac{1}{r} \frac{d}{dr} r \frac{dT}{dr} \right) \tag{72}$$

One of the boundary conditions are that $T \to T_0$ in the limit $r \to \infty$. The other boundary condition is a flux condition. The total heat transmitted, per unit length, of the wire is Q. Therefore, the flux from a cylindrical surface of radius r is $Q/(2\pi r)$. Therefore, the requirement at the wire surface, in the limit $r \to 0$, is that

$$-K \frac{dT}{dr} = \frac{Q}{2\pi r} \tag{73}$$

where Q is a constant.

To solve for the temperature field, note that there is no length scale in the problem (the wire is infinitesimal in thickness, and the boundaries are at infinity). Therefore, a similarity solution can be used with the similarity variable $y = (r/\sqrt{Dt})$. The heat conduction equation, in terms of this variable, is

$$\frac{d^2 T}{dy^2} + \left(\frac{1}{y} + \frac{y}{2} \right) \frac{dT}{dy} = 0 \tag{74}$$

This equation can be solved to obtain the

$$\frac{dT}{dy} = \frac{C}{y} \exp\left(-y^2/4\right) \tag{75}$$

The temperature can be obtained by integrating the above equation with respect to y, and realising that $T = 0$ as $y \to \infty$.

$$T = \int_\infty^y dy' \frac{C}{y'} \exp\left(-y'^2/4\right) \tag{76}$$

The constant C can be determined from the flux condition, in the limit $r \to 0 (y \to 0)$,

$$\frac{dT}{dr} = -\frac{Q}{2\pi r K} \tag{77}$$

When expressed in terms of y, this is equivalent to

$$\frac{dT}{dy} = -\frac{Q}{2\pi K y} \tag{78}$$

Therefore, $C = -(Q/2\pi K)$.

3. For the case $v_r = 0$ and $v_z = 0$, the θ momentum equation is given by:

$$\frac{\partial v_\theta}{\partial t} = -\frac{\nu}{r^2}\frac{\partial (r v_\theta)}{\partial r} + \frac{\nu}{r}\frac{\partial^2 (r v_\theta)}{\partial r^2} \tag{79}$$

and the boundary conditions are:

$$\begin{array}{rr} \text{At } r = 0 & v_\theta = 0 \\ \text{For } r \to \infty & v_\theta = \dfrac{\Gamma}{2\pi r} \\ \text{At } t = 0 \text{ and } r > 0 & v_\theta = \dfrac{\Gamma}{2\pi r} \end{array} \tag{80}$$

Since v_θ is dependent on r, it is convenient to use a new variable $\gamma = v_\theta/(\Gamma/2\pi r)$. γ is called the 'circulation'. The equation for the variable γ becomes:

$$\frac{\partial \gamma}{\partial t} = -\frac{\nu}{r}\frac{\partial \gamma}{\partial r} + \nu\frac{\partial^2 \gamma}{\partial r^2} \tag{81}$$

with the boundary conditions:

$$\begin{array}{rr} \text{At } r = 0 & \gamma = 0 \\ \text{For } r \to \infty & \gamma = 1 \\ \text{At } t = 0 \text{ and } r > 0 & \gamma = 1 \end{array} \tag{82}$$

A similarity solution can now be used, since there are no length or time scales in the problem other than r, t and ν. The dimensionless variable η is defined as:

$$\eta = \frac{r}{\sqrt{\nu t}} \tag{83}$$

and $\gamma = \gamma(\eta)$. The differential equation for γ in terms of η is given by:

$$\frac{d^2\gamma}{d\eta^2} + \left(\frac{\eta}{2} - \frac{1}{\eta}\right)\frac{d\gamma}{d\eta} = 0 \tag{84}$$

with the boundary conditions:

$$\begin{array}{rr} \text{At } \eta = 0 & \gamma = 0 \\ \text{For } r \to \infty & \gamma = 1 \end{array} \tag{85}$$

This can be easily solved to give:

$$\gamma = [1 - \exp(-\eta^2/4)] \quad v_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(\frac{r^2}{4\nu t}\right)\right] \tag{86}$$

4. The equation for the temperature field is,

$$\rho C_v \frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{4\mu U^2 r^2}{R^4}$$

The scaled r co-ordinate and time are defined as $r^* = (r/R)$ and $t^* = (\alpha t/R^2)$, where $\alpha = (k/\rho C_v)$ is the thermal diffusivity, to provide,

$$\frac{\partial T}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T}{\partial r^*}\right) + \frac{4\mu U^2 r^{*2}}{k}$$

From the above equation, it is appropriate to define a scaled temperature as $T^* = (k(T - T_0)/\mu U^2)$, so that the scaled temperature is zero at the walls. With this, the equation for the scaled temperature field becomes,

$$\frac{\partial T^*}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + 4r^{*2}$$

At steady state, this equation can be easily solved to obtain,

$$T^* = 1 - \frac{r^{*4}}{4}$$

or

$$T - T_0 = \frac{\mu U^2}{4k}\left(1 - \frac{r^4}{R^4}\right)$$

If there is an unsteady forcing of the type $U(t) = U\cos(\omega t)$, then the heat generated per unit area per unit time can be written as,

$$Q = \frac{4\mu U^2 r^2 \cos(\omega t)^2}{R^4}$$

This is converted into two parts, a steady and an oscillatory part,

$$Q = \frac{2\mu U^2 r^2}{R^4}(1 + \cos(2\theta))$$

Therefore, we obtain solutions with two inhomogeneous terms, a steady term

$$Q_s = \frac{2\mu U^2 r^2}{R^4}$$

for which the temperature field is,

$$T - T_0 = \frac{\mu U^2}{8k}\left(1 - \frac{r^4}{R^4}\right)$$

The second part is obtained by solving the equation with the inhomogeneous term,

$$Q_t^* = \frac{2\mu U^2 r^2}{R^4}\exp(2\imath\omega t)$$

and taking the real part of the solution. For this, we define the temperature field as $(T - T_0) = \tilde{T}(r)\exp(2\imath\omega t)$. This is inserted into the conservation equation to obtain,

$$\rho C_v \imath\omega\tilde{T} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{T}}{\partial r}\right) + \frac{2\mu U^2 r^2}{R^4}$$

We use the same scalings $r^* = (r/R)$, $t^* = (t\alpha/R^2)$ and $\tilde{T}^* = (k\tilde{T}/\mu U^2)$ as before, to obtain,

$$\frac{\partial^2\tilde{T}^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial\tilde{T}^*}{\partial r^*} + 2r^{*2} - 2\imath\mathrm{Pe}_\omega\tilde{T} = 0$$

where $\mathrm{Pe}_\omega = (R^2\omega/\alpha)$ is the Peclet number based on the frequency of the base flow. This is easily solved to obtain the general solution and particular integral. The general solution is,

$$\tilde{T}_g^* = C_1 J_0(\sqrt{-2\imath\mathrm{Pe}_\omega}r^*) + C_2 Y_0(\sqrt{-2\imath\mathrm{Pe}_\omega}r^*) +$$

while the particular integral is,

$$\tilde{T}_p^* = -\frac{\imath r^{*2}}{\mathrm{Pe}_\omega} - \frac{2}{\mathrm{Pe}_\omega^2}$$

The constant C_2 in the general solution is zero since the general solution has to be finite at the origin. Therefore, the final solution which satisfies the boundary condition $\tilde{T}^* = 0$ at $r^* = 1$ is,

$$\tilde{T}^* = \left(-\frac{\imath}{\mathrm{Pe}_\omega} - \frac{2}{\mathrm{Pe}_\omega^2}\right)\left(1 - \frac{J_0(\sqrt{-2\imath\mathrm{Pe}_\omega}r^*)}{J_0(\sqrt{-2\imath\mathrm{Pe}_\omega})}\right) + \frac{\imath(1 - r^{*2})}{\mathrm{Pe}_\omega}$$

Therefore the final transient temperature field is,

$$T_t - T_0 = \frac{\mu U^2}{k}\mathrm{Real}\left(\exp(2\imath\omega t)\left(-\frac{\imath}{\mathrm{Pe}_\omega} - \frac{2}{\mathrm{Pe}_\omega^2}\right)\left(1 - \frac{J_0(\sqrt{-2\imath\mathrm{Pe}_\omega}r^*)}{J_0(\sqrt{-2\imath\mathrm{Pe}_\omega})}\right) + \frac{\imath(1 - r^{*2})}{\mathrm{Pe}_\omega}\right)$$

5. The appropriation coordinate system is the cylindrical coordinate system, in which the z axis is along the axis of the cylinder. Thus, the boundaries of the cylinder are $0 \leq r \leq R$ and $0 \leq z \leq H$, while the ϕ coordinate varies between 0 and 2π. The only non-zero component of the velocity is the u_ϕ component. The boundary conditions for u_ϕ are,

$$\begin{aligned} u_\phi &= \Omega \text{ at } r = R \\ u_\phi &= 0 \text{ at } z = 0 \\ u_\phi &= 0 \text{ at } z = H \end{aligned} \tag{87}$$

The mass conservation equation obtained using a shell balance, is,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\phi)}{\partial \phi} + \frac{\partial(\rho u_z)}{\partial z} = 0 \tag{88}$$

Since the density is a constant, and u_ϕ is the only non-zero component of the velocity, the mass conservation equation reduces to

$$\frac{\partial u_\phi}{\partial \phi} = 0 \tag{89}$$

Therefore, the velocity is independent of the ϕ coordinate.

The momentum conservation equation in the ϕ direction, using shell balances, will reduce to,

$$\frac{\partial u_\phi}{\partial t} = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(u_p hir)}{\partial r} + \frac{\partial^2 u_\phi}{\partial z^2}\right) \tag{90}$$

This equation can be simplified by taking one derivative with respect to ϕ. Since $(\partial u_\phi/\partial \phi) = 0$ from the mass conservation condition, we find that

$$\frac{\partial^2 p}{\partial \phi^2} = 0 \tag{91}$$

This equation has a solution of the form,

$$p = C_1(r,z)\phi + C_2(r,z) \tag{92}$$

However, we require that the pressure should have the same value at ϕ as it has at $(\phi + 2\pi)$, since these are the same points in space. Therefore, the pressure is only a function of (r, z), and is independent of ϕ. With this, the momentum balance equation reduces to,

$$\frac{\partial u_\phi}{\partial t} = \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r u_\phi)}{\partial r}\right) + \frac{\partial^2 u_\phi}{\partial z^2} \tag{93}$$

At steady state, the equation reduces to,

$$\nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r u_\phi)}{\partial r}\right) + \frac{\partial^2 u_\phi}{\partial z^2} = 0 \tag{94}$$

This solved using the method of separation of variables, where we write $u_\phi = R(r)Z(z)$. Inserting this into the above equation, and dividing throughout by RZ, we get

$$\left(\frac{1}{R}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r u_\phi)}{\partial r}\right) + \frac{1}{Z}\frac{\partial^2 u_\phi}{\partial z^2}\right) = 0 \tag{95}$$

Since the first term is only a function of r, and the second term is only a function of z, these two are individually equal to constants. First, we solve the equation for Z,

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\lambda^2 \tag{96}$$

This has solutions

$$Z = A \sin(\lambda z) + B \cos(\lambda z) \tag{97}$$

Since $Z = 0$ at $z = 0$, we require $B = 0$. Also, since $Z = 0$ at $z = H$, the solution for Z is

$$Z = A_n \sin(n\pi z/H) \tag{98}$$

Since $Z = 0$ at $z = 0$, we require $B = 0$. Also, since $Z = 0$ at $z = H$, the solution for Z is

$$Z = A_n \sin(n\pi z/H) \tag{99}$$

where n is an integer. The equation for the radial coordinate is given by,

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2} = \lambda_n^2 = \left(\frac{n\pi}{H}\right)^2 \tag{100}$$

This equation can be simplified to obtain,

$$\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda_n^2 r^2 R = 0 \tag{101}$$

The solution of this equation, which is finite at the origin, is,

$$R = J_0(\imath \lambda_n r) \tag{102}$$

Therefore, the final solution is of the form,

$$u_\phi = \sum_{n=0}^{\infty} A_n J_1\left(\frac{\imath n\pi r}{H}\right)\sin\left(\frac{n\pi z}{H}\right) \tag{103}$$

In this equation, we have not yet enforced the boundary condition at the surface of the sphere, $u_\phi = \Omega$ at $r = R$. This is enforced using the orthogonality condition,

$$\int dz \sin\left(m\pi z/H\right)\, u_\phi|_{r=R} = \int dz \sin\left(m\pi z/h\right)\Omega \tag{104}$$

The left side of the above equation can be easily solved to give

$$\begin{aligned} \frac{A_m J_1(\imath\lambda_m R)}{2} &= \frac{2}{m\pi} \text{ for odd m} \\ &= 0 \text{ for even m} \end{aligned} \tag{105}$$

This completes the evaluation of the constants in the above equation.

6. The equation for the velocity is,

$$-\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = 0$$

Since the pressure is not a function of r, the above equation can be solved to obtain,

$$v_z = \frac{\partial p}{\partial z}\frac{r^2}{2} + C_1 \log(r) + C_2$$

From the condition that the velocity is zero at R_s and U at R_p, the constants in the above equation are,

$$v_z = -\frac{\partial p}{\partial z}\left(-\frac{r^2}{4} + \frac{R_p^2 \log(r/R_s) - R_s^2 \log(r/R_p)}{r \log(R_p/R_s)} + \frac{U \log(r/R_s)}{\log(R_p/R_s)}\right.$$

The flow rate is determined from the condition,

$$Q = \int_{R_p}^{R_s} r dr v_z = 0$$

This can be used to determine the relationship between the velocity and pressure,

$$\frac{\partial p}{\partial z} = -\frac{4U(1 - R_r^2 + 2R_r^2 \log{(R_r)})}{(1 - R_r^2)(1 - R_r^2 + (1 + R_r^2)\log{(R_r)}}$$

7. The equation can be solved using separation of variables,

$$u_z(r, \theta) = R(r)T(\theta) \tag{106}$$

The equation in the θ direction is,

$$\frac{\partial^2 T}{\partial \theta^2} = -\frac{n^2\pi^2}{\Theta^2} \tag{107}$$

with the boundary conditions,

$$T = 0 \text{at} \theta = -\Theta, \Theta \tag{108}$$

The solution for this is,

$$T = \sin{(n\pi\theta/\Theta)} \tag{109}$$

where n is an integer to satisfy the boundary condition. The equation for R then becomes,

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + r^2 K R - \frac{n^2\pi^2}{\Theta^2} R = 0 \tag{110}$$

If we define $r^* = \sqrt{K} r$, then the above equation reduces to,

$$r^{*2} \frac{\partial^2 R}{\partial r^{*2}} + r^* \frac{\partial R}{\partial r^*} + \left(r^{*2} - \frac{n^2\pi^2}{\Theta^2}\right) R = 0 \tag{111}$$

The solution of this equation is,

$$R = AJ_m(r^*) + BY_m(r^*) \tag{112}$$

where $m = (n\pi/\Theta)$. The constant B is zero since Y_m diverges at $r^* = 0$. Therefore, the final solution is,

$$u_z = \sum_n A_n J_m(\sqrt{K/\mu} r) \sin(n\pi\theta/\Theta) \tag{113}$$

The final boundary condition is $u_z = 0$ at $r = R$.

$$0 = \sum_n A_n J_m(\sqrt{K/\mu} R) \sin(n\pi\theta/\Theta) \tag{114}$$

This can be enforced using the orthogonality conditions.

8. The differential equation is,

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \tag{115}$$

with boundary conditions,

$$T \rightarrow 0 \text{ for } r \rightarrow \infty \tag{116}$$

$$\left(-k\frac{\partial T}{\partial r}\right)(4\pi r^2) = Q \text{ for } r \rightarrow 0 \tag{117}$$

The similarity variable is $\eta = (r/\sqrt{\alpha t})$. Expressed in terms of this variable, the solution for the conservation equation becomes,

$$\frac{\partial^2 T}{\partial \eta^2} + \left(\frac{2}{\eta} + \frac{\eta}{2}\right)\frac{\partial T}{\partial \eta} = 0 \tag{118}$$

The solution of this is,

$$\frac{\partial T}{\partial \eta} = \frac{C}{\eta^2}\exp(-\eta^2/4) \tag{119}$$

where C is the constant of integration.

The boundary conditions are,

$$T \rightarrow 0 \text{ for } \eta \rightarrow \infty \tag{120}$$

$$\left(-k\frac{\partial T}{\partial r}\right)(4\pi r^2) = Q \text{ for } r \rightarrow 0 \tag{121}$$

When expressed in terms of the similarity variable, the boundary condition becomes,

$$\eta^2 \frac{\partial^2 T}{\partial \eta^2} = -\frac{Q}{4\pi k \sqrt{\alpha t}} \tag{122}$$

Clearly, a solution is not possible unless Q is proportional to $\sqrt{\alpha t}$.

9. Since the forcing far from the cylinder is of the form $T = T' r^2 \sin(2\theta)/2$, the temperature everywhere has to be of the same form.

$$T = \left(\frac{T' r^2}{2} + \frac{A}{r^2} \right) \sin(2\theta) + T_0$$

At the surface, the temperature is T_0, and so the boundary condition is,

$$A = -(T' R^4/2)$$

The temperature field is,

$$T = T' \left(\frac{r^2}{2} - \frac{R^4}{2r^2} \right) \sin(2\theta) + T_0$$

The flux at the surface perpendicular to the surface is,

$$q_r = -k \frac{\partial T}{\partial r} = 2RT' \sin(2\theta)$$

5 Mass and energy conservation equations:

1. (a) The coordinates in the two coordinate systems are related by

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi) \\ z &= z \end{aligned} \tag{123}$$

$$\begin{aligned} r &= (x^2 + y^2)^{1/2} \\ \tan(\phi) &= (y/x) \end{aligned} \tag{124}$$

The unit vectors in the cylindrical coordinate system are related to those in the Cartesian coordinate system by

$$\begin{aligned} \mathbf{e}_r &= \cos(\phi)\mathbf{e}_x + \sin(\phi)\mathbf{e}_y \\ \mathbf{e}_\phi &= -\sin(\phi)\mathbf{e}_x + \cos(\phi)\mathbf{e}_y \end{aligned} \tag{125}$$

(b) We consider a differential volume bounded by r and $r+\Delta r$ in the radial direction, by ϕ and $\phi+\Delta\phi$ in the azimuthal direction and z and $z+\Delta z$ in the axial direction. The widths of the differential volume are $(\Delta r, r\Delta\theta, \Delta z)$ in the three directions respectively. The change in the concentration within the volume in a time Δt is

$$(c(r,\phi,z,t+\Delta t) - c(r,\phi,z,t))\,(\Delta r)(r\Delta\phi)\Delta z) \tag{126}$$

The total input of solute into the volume through the surface at r is given by

$$((cv_r + j_r)(r\Delta\phi)(\Delta z))|_r \tag{127}$$

while the output of solute through the surface at $(r+\Delta r)$ is

$$((cv_r + j_r)(r\Delta\phi)(\Delta z))|_{r+\Delta r} \tag{128}$$

The net accumulation of solute due to the flow through these two surfaces is given by

$$\begin{aligned} ((cv_r + j_r)(r\Delta\phi)(\Delta z))|_r - ((cv_r + j_r)(r\Delta\phi)(\Delta z))|_{r+\Delta r} &= \\ (\Delta\phi)(\Delta z)\Delta r \left(-\frac{\partial}{\partial r}(r(cv_r + j_r))\right) \end{aligned} \tag{129}$$

Similar expressions can be obtained for the net accumulation of solute through the surfaces at ϕ and $\phi+\Delta\phi$,

$$\begin{aligned} ((cv_\phi + j_\phi)(\Delta r)(\Delta z))|_\phi - ((cv_\phi + j_\phi)(\Delta r)(\Delta z))|_{\phi+\Delta\phi} &= \\ (\Delta r)(\Delta z)\left(-\frac{\partial}{\partial\phi}((cv_\phi + j_\phi))\right) \end{aligned} \tag{130}$$

and through the surfaces at z and $z+\Delta z$,

$$\begin{aligned} ((cv_z + j_z)(\Delta r)(r\Delta\phi))|_z - ((cv_z + j_z)(\Delta r)(r\Delta\phi))|_{z+\Delta z} &= \\ (\Delta r)(r\Delta\phi)(\Delta z)\left(-\frac{\partial}{\partial z}(cv_z + j_z)\right) \end{aligned} \tag{131}$$

Equating the rate of accumulation of mass to the sum of the Input – Output, and dividing by the volume $(\Delta r)(r\Delta\phi)(\Delta z)$, the equation for the concentration field is

$$\frac{\partial c}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(r(cv_r + j_r)) - \frac{1}{r}\frac{\partial(cv_\phi + j_\phi)}{\partial\phi} - \frac{\partial(cv_z + j_z)}{\partial z} \tag{132}$$

This equation can be expressed in the form of the diffusion equation using the definition of the divergence operator ∇ in spherical coordinates,

$$\nabla.\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{133}$$

The components of the fluxes in the three directions are related to the variation of the concentration with position, which in the spherical coordinate system is given by

$$\begin{aligned} j_r &= -D\frac{\partial c}{\partial r} \\ j_\phi &= -D\frac{1}{r}\frac{\partial c}{\partial \phi} \\ j_z &= -D\frac{\partial c}{\partial z} \end{aligned} \tag{134}$$

When this is inserted into the diffusion equation, we obtain

$$\frac{\partial c}{\partial t} + \nabla.(c\mathbf{v}) = D\nabla^2 c \tag{135}$$

where the Laplacian is defined as

$$\nabla^2 = \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right) \tag{136}$$

(c) The solution for the Laplace equation can be obtained using separation of variables,

$$c = T(t)R(r)F(\phi)Z(z) \tag{137}$$

This is inserted into the conservation equation, which is then divided by $TRFZ$, to obtain

$$\frac{1}{T}\frac{\partial T}{\partial t} = D\left(\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R}{\partial r} + \frac{1}{F}\frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right) \tag{138}$$

Using separation of variables, we can infer that the left and right sides of the above equation are individually equal to constants. Setting

$$\frac{1}{T}\frac{\partial T}{\partial t} = -D\lambda^2 \tag{139}$$

so that T decays with time, the solution for T is

$$T = \exp(-D\lambda^2 t) \tag{140}$$

The solution for F is obtained by setting

$$\frac{1}{F}\frac{\partial^2 F}{\partial \phi^2} = -m^2 \tag{141}$$

The solution for this is $F = \exp(\imath m\phi)$, and m is required to be an integer so that $F(\phi + 2\pi) = F(\phi)$. The solution for Z is obtained by setting

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\alpha \tag{142}$$

The value of α is as yet undetermined, but we would require that α is positive if we are solving for the concentration field inside a finite domain in z, so that the solutions are $\cos$ and $\sin$ functions. If we are solving in an infinite domain, then α would have to be negative, so that we get exponentially decaying functions. So the choice of the sign of α depends on the domain.

The solution for R is determined from

$$\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial R}{\partial r} - \frac{m^2}{r^2} + (\alpha + \lambda^2) = 0 \tag{143}$$

This equation can be recast as

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} - r^2(\alpha - \lambda^2) - m^2 = 0 \tag{144}$$

The solution for this equation is the combination of modified Bessel functions,

$$R = C_1 K_m((\alpha - \lambda^2)^{1/2} r) + C_2 I_m((\alpha - \lambda^2)^{1/2} r) \tag{145}$$

For a system at steady state, where $\lambda = 0$, the solutions for R reduce to

$$R = C_1 K_m(\alpha r) + C_2 I_m(\alpha r) \tag{146}$$

6 Diffusive transport:

1. The Laplace equation in spherical co-ordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \tag{147}$$

   We use separation of variables $T = R(r)F(\theta)$ to write,

$$\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R}{\partial r} + \frac{1}{r^2}\frac{1}{F}\frac{\partial^2 F}{\partial \theta^2} = 0 \tag{148}$$

   The θ part of the equation is,

$$\frac{1}{F}\frac{\partial^2 F}{\partial \theta^2} = -m^2 \tag{149}$$

   where m is an integer, since we have to get the same temperature if we go around by an angle of (2π). Therefore, the solution is

$$F = \exp(\imath m\theta) \tag{150}$$

   The r part of the equation is,

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} - m^2 R = 0 \tag{151}$$

   This can be solved to obtain,

$$R = (r^m, r^{-m}) \tag{152}$$

   Therefore the final solutions are,

$$T = \sum_m (A_m r^m + B_m r^{-m}) \exp(\imath m\theta) \tag{153}$$

2. The temperature due to a point source in two dimensions can be determined by solving the conduction equation,

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} = 0 \tag{154}$$

   The solution of this is,

$$T = C_1 \log(r) + C_2 \tag{155}$$

The constant C_1 can be determined from the condition that the total flux coming out of the source is 1,

$$\int_0^{2\pi} r\, d\theta \left(-K\frac{\partial T}{\partial r}\right) = Q \tag{156}$$

This provides the solution,

$$C_1 = -\frac{Q}{2\pi} \tag{157}$$

Therefore, the solution is of the form,

$$T = -\frac{Q}{2\pi}\log(r) + C_2 \tag{158}$$

Note that it is not possible to apply boundary conditions at infinity for this problem.

3. The Laplace equation for a cylindrical coordinate system is

$$K\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} = \delta(\mathbf{x}) \tag{159}$$

The right side of the above equation is zero for $x \neq 0$, and so the solution for T is

$$T = C_1 \log(r) + C_2 \tag{160}$$

The constant C_1 is determined from the flux condition

$$K\int dS\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} = \int dSQ\delta(\mathbf{x}) \tag{161}$$

The right side of the above equation is just Q, while the left side is

$$2\pi K\int rdr\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} = 2\pi K C_1 \tag{162}$$

Therefore, the solution for the temperature field due to a point source is given by

$$T = \frac{Q\log(r)}{2\pi K} + T_\infty \tag{163}$$

4. For a source and a sink of strength $\pm Q$ placed at $x = \pm L$, the temperature field is given by

$$\begin{aligned} T &= \frac{Q}{4\pi K}[\log\left((x-L)^2+y^2\right) - \log\left((x+l)^2+y^2\right)] \\ &= \frac{Q}{4\pi K}[\log\left(x^2+y^2-2xL+L^2\right) - \log\left(x^2+y^2+2xL+L^2\right)] \\ &= \frac{Q}{4\pi K}\left[\log\left(x^2+y^2\right) + \log\left(1+\frac{-2xL+L^2}{x^2+y^2}\right) - \log\left(x^2+y^2\right) - \log\left(1+\frac{2xL+L^2}{x^2+y^2}\right)\right] \\ &= \frac{Q}{4\pi K}\frac{4xL}{x^2+y^2} \\ &= \frac{QL}{\pi K}\frac{x}{r^2} \\ &= \frac{QL}{\pi K}\frac{\cos(\theta)}{r} \end{aligned} \tag{164}$$

A similar calcultion can be carried out for a source and sink placed at $y = \pm L$ to provide

$$T = \frac{2QL}{\pi K}\frac{\sin(\theta)}{r} \tag{165}$$

5. For two sources placed at (L, L), $(-L, -L)$ and two sinks at $(L, -L)$ and $(-L, L)$, the temperature field is given by

$$\begin{aligned} T = \frac{Q}{4\pi K} &\left[\log\left((x-L)^2+(y-L)^2\right) + \log\left((x+L)^2+(y+L)^2\right)\right. \\ &\left. - \log\left((x-L)^2+(y+L)^2\right) - \log\left((x+L)^2+(y-L)^2\right)\right] \end{aligned} \tag{166}$$

The first logarithm in the above equation can be simplified as follows,

$$\log\left((x-L)^2+(y-L)^2\right) = \log(u+a) \tag{167}$$

where

$$\begin{aligned} u &= x^2+y^2 \\ a &= -2xL-2yL+2L^2 \end{aligned} \tag{168}$$

The log can be expanded in a Taylor series for $a \ll r$,

$$\log(u+a) = \log(u) + a\frac{d\log(u)}{du} + \frac{a^2}{2}\frac{d^2\log(u)}{du^2} \tag{169}$$

Using this, we get

$$\begin{aligned}\log(u+a) &= \log(u) + \frac{a}{u} - \frac{a^2}{2u^2} \\ &= \log(x^2+y^2) - \frac{2L(x+y)}{(x^2+y^2)} + \frac{2L^2}{(x^2+y^2)} - \frac{2L^2(x+y)^2}{(x^2+y^2)^2}\end{aligned} \tag{170}$$

A similar expansion can be carried out for the other three terms, which are, respectively,

$$\begin{aligned}&\log((x+L)^2+(y+L)^2) \\ &\quad = \log(x^2+y^2) + \frac{2L(x+y)}{(x^2+y^2)} + \frac{2L^2}{(x^2+y^2)} - \frac{2L^2(x+y)^2}{(x^2+y^2)^2} \\ &\log((x-L)^2+(y+L)^2) \\ &\quad = \log(x^2+y^2) - \frac{2L(x-y)}{(x^2+y^2)} + \frac{2L^2}{(x^2+y^2)} - \frac{2L^2(x-y)^2}{(x^2+y^2)^2} \\ &\log((x+L)^2+(y-L)^2) \\ &\quad = \log(x^2+y^2) + \frac{2L(x-y)}{(x^2+y^2)} + \frac{2L^2}{(x^2+y^2)} - \frac{2L^2(x-y)^2}{(x^2+y^2)^2}\end{aligned} \tag{171}$$

Upon substituting this into the equation for the temperature field, the first three contributions on the right side of the above equation will cancel out, while the fourth term alone will give a non-zero contribution when added up, to give,

$$\begin{aligned}T &= \frac{Q}{4\pi K}\frac{-16L^2xy}{(x^2+y^2)^2} \\ &= \frac{Q}{4\pi K}\frac{-16L^2\cos(\theta)\sin(\theta)}{r^2} \\ &= \frac{Q}{4\pi K}\frac{-8L^2\sin(2\theta)}{r^2}\end{aligned} \tag{172}$$

The temperature field for the combination of sources at $(L, 0)$ and $(-L, 0)$ and sink at $(0, L)$ and $(0, -L)$ can be carried out in a similar manner.

$$\begin{aligned} T &= \frac{Q}{4\pi K}\frac{-4L^2(x^2-y^2)}{(x^2+y^2)^2} \\ &= \frac{Q}{4\pi K}\frac{-4L^2(\cos(\theta)^2-\sin(\theta)^2)}{r^2} \\ &= \frac{Q}{4\pi K}\frac{-4L^2\cos(2\theta)}{r^2} \end{aligned} \tag{173}$$

The solution obtained by separation of variables for the steady temperature field was

$$T = \sum_m A_m r^{-m} \exp(\imath m\theta) \tag{174}$$

The first solution, which is a log, does not agree with this, since it is obtained by variation of parameters. However, the next two are in agreement.

6. (a) The temperature field has to have the same symmetry as the temperature field at infinity, which is $T'x = T'r\cos(\theta)$. Therefore, the temperature within and outside the cylinder have to be of the form

$$\begin{aligned} T_o &= T'r\cos(\theta) + \frac{A_o\cos(\theta)}{r} \\ T_i &= A_i r\cos(\theta) \end{aligned} \tag{175}$$

The boundary conditions at the surface of the cylinder at $r = R$ are the equality of temperature and flux,

$$\begin{aligned} T_i &= T_o \\ K_i\frac{dT_i}{dr} &= K_o\frac{dT_o}{dr} \end{aligned} \tag{176}$$

Using these, we get the constants

$$\begin{aligned} A_i &= \frac{2T'}{1+K_R} \\ A_o &= \frac{(1-K_R)R^2T'}{1+K_R} \end{aligned} \tag{177}$$

This gives the temperature field outside and inside the cylinder,

$$
\begin{aligned}
T_i &= \frac{2T'r\cos(\theta)}{1+K_R} \\
T_o &= \left(T'r + \frac{(1-K_R)R^2T'}{(1+K_R)r}\right)\cos(\theta)
\end{aligned} \tag{178}
$$

(b) The total flux can be separated into two parts, one due to the flux through the matrix and the other due to the flux through the cylinders,

$$
\begin{aligned}
\langle j_x^e \rangle &= -\frac{K_m}{V}\int_{\text{matrix}} dV \frac{\partial T}{\partial x} - \frac{K_p}{V}\int_{\text{cylinders}} dV \frac{\partial T}{\partial x} \\
&= -\frac{K_m}{V}\int_{\text{total}} dV \frac{\partial T}{\partial x} - \frac{(K_p - K_m)}{V}\int_{\text{cylinders}} dV \frac{\partial T}{\partial x} \\
&= -K_m - \frac{N(K_p-K_m)}{V}\int_{\text{1 cylinder}} \frac{\partial T}{\partial x} \\
&= -K_m - \frac{NV_p}{V}(K_p-K_m)\frac{2T'}{1+K_R} \\
&= -K_m\left(1 + \frac{2\phi(K_R-1)}{1+K_R}\right)
\end{aligned} \tag{179}
$$

7. Choose a co-ordinate system in which the origin is located on the wall at a distance L from the source. The position of the heat source is then $(0,0,L)$, and the temperature field due to the heat source without the wall would be,

$$
T = \frac{Q}{4\pi k(x^2+y^2+(z-L)^2)^{1/2}} \tag{180}
$$

In the presence of a conducting wall, it is appropriate to put a heat sink at a distance L below the wall, so that the heat flux lines are perpendicular to the wall. The temperature is then,

$$
T = \frac{Q}{4\pi k(x^2+y^2+(z-L)^2)^{1/2}} - \frac{Q}{4\pi k(x^2+y^2+(z+L)^2)^{1/2}} \tag{181}
$$

In the presence of an insulating wall, it is necessary to put a heat source at a distance L below the wall, so that the flux lines are parallel to the wall.

$$
T = \frac{Q}{4\pi k(x^2+y^2+(z-L)^2)^{1/2}} + \frac{Q}{4\pi k(x^2+y^2+(z+L)^2)^{1/2}} \tag{182}
$$

If a fraction f is perpenducular to the wall and $(1-f)$ is parallel to the wall, it is necessary to put a heat source of strength $(1-f)Q$ and a sink of strength fQ, so that the total source is $(1-2f)Q$. The temperature field is then

$$T = \frac{Q}{4\pi k(x^2+y^2+(z-L)^2)^{1/2}} + \frac{(1-2f)Q}{4\pi k(x^2+y^2+(z+L)^2)^{1/2}} \quad (183)$$

8. The temperature field at a position (x, y, z) is given by,

$$T = \int_0^{2\pi} d\theta \frac{Qa}{((x-a\cos(\theta))^2+(y-a\sin(\theta))^2+z^2)^{1/2}} \quad (184)$$

The temperature field along the z axis is,

$$T = \frac{Q2\pi a}{(a^2+z^2)^{1/2}} \quad (185)$$

For $z \gg a$, the temperature field decays as

$$T = \frac{Q2\pi a}{z} \quad (186)$$

For $z \ll g$, the temperature field is proportional to

$$T = 2\pi Q - \frac{\pi Q z^2}{a^2} \quad (187)$$

7 Convection-dominated transport:

1. The energy conservation equation is of the form,

$$k x^{1/2} \frac{\partial T}{\partial x} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

If we scale x and y by L, we obtain,

$$\frac{kL^{3/2}}{\alpha} \sqrt{x^*} \frac{\partial T}{\partial x^*} = \left(\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}} \right)$$

Therefore, the Peclet number is $(kL^{3/2}/\alpha)$. When this number is large, we scale the y co-ordinate by $y^* = (y/\delta)$. The differential equation now becomes,

$$\frac{k\delta^2}{\alpha L^{1/2}}\sqrt{x^*}\frac{\partial T}{\partial x^*} = \left(\frac{\delta^2}{L^2}\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial y^{*2}}\right)$$

For convection and diffusion to be of equal magnitude, we require that,

$$\frac{\delta^2}{L^2} = \frac{\alpha}{L^{3/2}k} = \mathrm{Pe}^{-1}$$

To find a similarity solution, neglect streamwise diffusion to obtain,

$$kx^{1/2}\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}$$

Substitute $\eta = (y/g(x))$, and simplify to obtain,

$$-\frac{kx^{1/2}yg'}{g^2}\frac{dT}{d\eta} = \frac{\alpha}{g^2}\frac{d^2T}{d\eta^2}$$

Multiplying throughout by (g^2/α) and simplifying, we obtain,

$$-\frac{kx^{1/2}gg'}{\alpha}\eta\frac{dT}{d\eta} = \frac{d^2T}{d\eta^2}$$

For a similarity solution, we require that

$$\frac{kx^{1/2}gg'}{\alpha} = 1$$

This can be solved, with the condition $g = 0$ at $x = 0$, to obtain,

$$g = \sqrt{\frac{4\alpha x^{1/2}}{k}}$$

The solution for the differential equation in terms of the similarity variable is,

$$T = \left(1 - \frac{\int_0^{y/g}\exp\left(-\eta^2/2\right)}{\int_0^{\infty}\exp\left(-\eta^2/2\right)}\right)$$

2. The convection-diffusion equation is,

$$Ay^{((n+1)/n)}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

where

$$A = \left(-\frac{1}{c}\frac{dp}{dx}\right)^{1/n}\frac{n}{n+1}$$

Substituting the similarity variable $\eta = (y/g(x))$, the above equation becomes,

$$-(A/\alpha)\eta^{((2n+1)/n)}g^{((2n+1)/n)}\frac{dg}{dx}\frac{dT}{d\eta} = \frac{d^2T}{d\eta^2}$$

This has a similarity solution only for,

$$\frac{Ag^{((2n+1)/n)}}{alpha}\frac{dg}{dx} = 1$$

or

$$\frac{A}{\alpha}\frac{n}{3n+1}\frac{dg^{((3n+1)/n)}}{dx} = 1$$

This gives the solution for the similarity variable,

$$g = \left(\frac{3n+1}{n}\frac{\alpha x}{A}\right)^{(n/(3n+1))}$$

The equation for the temperature field is,

$$\frac{d^2T}{d\eta^2} + \eta^{((2n+1)/n)}\frac{dT}{d\eta} = 0$$

This can be solved to provide,

$$T = 1 - \frac{\int_0^\eta dx \exp\left(-((3n+1)/n)x^{((3n+1)/n)}\right)}{\int_0^\infty dx \exp\left(-((3n+1)/n)x^{((3n+1)/n)}\right)}$$

3. The scaled equations are of the form,

$$Pe(-(1-r^{*2})\cos(\theta)\frac{\partial T}{\partial r^*}+(1+r^{*2})\sin(\theta)\frac{\partial T}{\partial \theta} = \left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2 T}{\partial \theta^2}\right) \quad (188)$$

where $r^* = (r/R)$, and the Peclet number is UR/α.

In the thin boundary layer at the surface of the cylinder, we can define the scaled co-ordinate y as $\delta y = (1 - r^*)$, where δ is the boundary layer thickness. In the leading order approximation in δ, the conservation equation becomes,

$$Pe(2y\cos(\theta)\frac{\partial T}{\partial y} + 2\sin(\theta)\frac{\partial T}{\partial \theta}) = \frac{1}{\delta^2}\frac{\partial^2 T}{\partial y^2} \tag{189}$$

Clearly, for a balance between the right and left sides of the equation, we require $\delta = Pe^{-1/2}$. With this, the equation becomes,

$$2(y\cos(\theta)\frac{\partial T}{\partial y} + \sin(\theta)\frac{\partial T}{\partial \theta} = \frac{\partial^2 T}{\partial y^2} \tag{190}$$

Alternatively, if we assume $x = \cos(\theta)$, we obtain,

$$2(yx\frac{\partial T}{\partial y} - (1 - x^2)\frac{\partial T}{\partial x}) = \frac{\partial^2 T}{\partial y^2} \tag{191}$$

In order to solve the equation, we use a similarity transform of the type,

$$z = \frac{y}{g(x)} \tag{192}$$

where z is a similarity variable. The equation for the temperature field now becomes,

$$2\left(\frac{yx}{g} + (1 - x^2)\frac{yg'}{g^2}\right)\frac{\partial T}{\partial z} = \frac{1}{g^2}\frac{\partial^2 T}{\partial z^2} \tag{193}$$

where the prime denotes a derivative with respect to x. Multiplying throughout by g^2, and transforming from y to z, we get,

$$2z\frac{\partial T}{\partial z}(g^2 x + (1 - x^2)gg') = \frac{\partial^2 T}{\partial z^2} \tag{194}$$

Clearly, for a similarity solution to be valid, the function g has to satisfy the equation,

$$g^2 x + (1 - x^2)gg' = 1 \tag{195}$$

This equation can be solved by expressing $g^2 = u$, in which case the equation for u becomes,

$$\frac{1-x^2}{2}u' + xu = 1 \tag{196}$$

This equation can be solved by integrating factors, where the integrating factor for u is the solution of the homogeneous equation, $(1-x^2)$. Using this, the final solution for u is,

$$u = g^2 = x + \frac{1-x^2}{2}\log\left(\frac{1+x}{1-x}\right) \tag{197}$$

With this transform, the equation for the temperature field becomes,

$$2z\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial z^2} \tag{198}$$

The solution for the temperature field that satisfies all the boundary conditions is,

$$T = 1 - \frac{\int_0^z dz' \exp\left(-z'^2\right)}{\int_0^\infty dz' \exp\left(-z'^2\right)} \tag{199}$$

4. Near the surface, we can assume that $u_x = U$, the slip velocity, and $u_y = -\frac{dU}{dX}Y$ to satisy mass conservation. With this, the convection-diffusion equation becomes,

$$U\frac{\partial T}{\partial x} - \frac{dU}{dX}Y\frac{\partial T}{\partial Y} = \alpha\frac{\partial^2 T}{\partial Y^2}$$

Substituting $\eta = Y/g(X)$, where $g(X)$ is the boundary layer thickness, the equation becomes,

$$-\eta\frac{dT}{d\eta}\left(\frac{Ug}{\alpha}\frac{dg}{dX} + \frac{dU}{dX}\frac{g^2}{\alpha}\right) = \frac{d^2T}{d\eta^2}$$

A similarity solution exists if $g(x)$ satisfies the equation,

$$\left(\frac{Ug}{\alpha}\frac{dg}{dX} + \frac{dU}{dX}\frac{g^2}{\alpha}\right) = 1$$

The above equation can be solved using an integrating factor, $g(X) = f(X)/U(X)$, to obtain an equation for $f(X)$,

$$\frac{d(f^2)}{dX} = 2\alpha U$$

This is integrated to obtain,

$$f = \sqrt{\alpha \int_0^X dX\, U(X)}$$

This gives,

$$g(X) = \frac{\sqrt{\alpha \int_0^X dX\, U(X)}}{U(X)}$$