

# Fundamentals of Transport Processes 1

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## 1 Course outline and reading material:

	Subject	Lectures	Reading
1.	Dimensional analysis.	1-4	
2.	Diffusion.	5-7	BSL 1,8,15, ELC 5
3.	Unidirectional transport Cartesian co-ordinates	8-15	BSL 2.1, 2.2, 4.1, 4.4, 9.6, 11.1, 11.2, MNO 2.
4.	Unidirectional transport. Curvilinear co-ordinates.	16-24	BSL 2.3, 2.4, 2.6, 3.4, 3.5, 9.6, 10.2, 10.5, 10.6, MNO 3,4.
5.	Mass and energy conservation.	25-28	BSL 3,10,18
6.	Transport due to diffusion.	29-35	MNO 2, 3, 4.
7.	Transport at high Peclet number	36-39	LGL 9 G-L
8.	Summary	40	

1. Bird, Stewart and Lightfoot (BSL), Transport Phenomena, Wiley International, 1960.
2. L. G. Leal (LGL), Advanced Transport Phenomena, Cambridge University Press, 2007.
3. E. L. Cussler (ELC), Diffusion: Mass transfer in fluid systems, Cambridge University Press, 1984.
4. M. N. Ozisik (MNO), Boundary Value Problems of Heat Conduction, Dover 2002.

## 2 Exercises:

### 2.1 Dimensional analysis:

1. The dimensionless groups for the heat flux in a heat exchanger were determined assuming that there is no inter-conversion between mechanical and thermal energy. If mechanical energy can be converted to thermal energy, there would be one additional dimensionless group which would be relevant for the heat flux. What is this dimensionless group, and what is its significance?
2. The Maxwell equations in electrodynamics are,

$$\nabla \cdot \mathbf{E} = (\rho_s / \epsilon_0) \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

where  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic field vectors,  $\rho_s$  is the charge density,  $\epsilon_0$  and  $\mu_0$  are the permittivity and permeability of free space, and  $\mathbf{J}$  is the current density. The dimensions of electric field  $\mathbf{E}$  is (Volt/meter), and that of charge density  $\rho_s$  is (Coulombs/  $\text{m}^3$ ), or (Amps  $\times$  s /  $\text{m}^3$ ), where  $m$  is meters and  $s$  is seconds. Using the above equations, **determine the dimensions of  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\epsilon_0$ ,  $\mu_0$  and  $\mathbf{J}$  in terms the fundamental dimensions of mass, length, time and amperes.** Note that  $(\partial/\partial t)$  is the partial derivative with respect to time,  $\nabla \cdot$  and  $\nabla \times$  are the divergence and curl operators with dimensions of inverse length, and

$$\text{Power} = \text{Volt} \times \text{Ampere} \quad (5)$$

3. Use dimensional analysis to obtain the dimensionless groups in the problem of a droplet of oil placed in a water bath placed in an electric field between two plane electrodes separated by a distance  $L$ , and moving with velocity  $U$ , as shown in figure 1. Assume the density of oil and water are equal. The important dimensional parameters are,

- (a) the radius of the droplet  $R$ ,
- (b) the distance between the electrodes  $L$  (assume they are of infinite extent in the  $x - y$  plane),
- (c) the velocity of the droplet,
- (d) the viscosity of water,
- (e) the viscosity of oil,
- (f) the density of water/oil,
- (g) the surface tension between water and oil,
- (h) the voltage difference between the two electrodes with dimension Volts,
- (i) the dielectric constant of water  $\epsilon_w$ , with dimension  $(A^2s^4kg^{-1}m^{-3})$ , where  $A$  is amperes, and
- (j) the dielectric constant of oil  $\epsilon_o$ , with dimension  $(A^2s^4kg^{-1}m^{-3})$ .

Note that the dimension of potential, Volt, is related to the dimension of current, amperes, by the relation

$$\text{Power} = \text{Volt} \times \text{Ampere} \tag{6}$$

How many dimensionless groups are there in the above problem? Of these, three are easily obtained as the ratio of lengths ( $L/R$ ), the ratio of viscosities and the ratio of dielectric constants of water and oil. The other dimensionless groups are expressed as the ratios of stresses caused by different physical mechanisms. Identify the dimensionless groups, and the physical mechanisms.

How would the problem simplify if the distance between plates is large compared to the droplet radius?

- (a) How many dimensions are there in the above problem, and what are they?
- (b) On the basis of the number of dimensional quantities and the number of dimensions, how many dimensionless groups that can be obtained?
- (c) Of these, three dimensionless groups are easily obtained, the ratio of lengths ( $L/R$ ), the ratio of viscosities of water and oil, and the

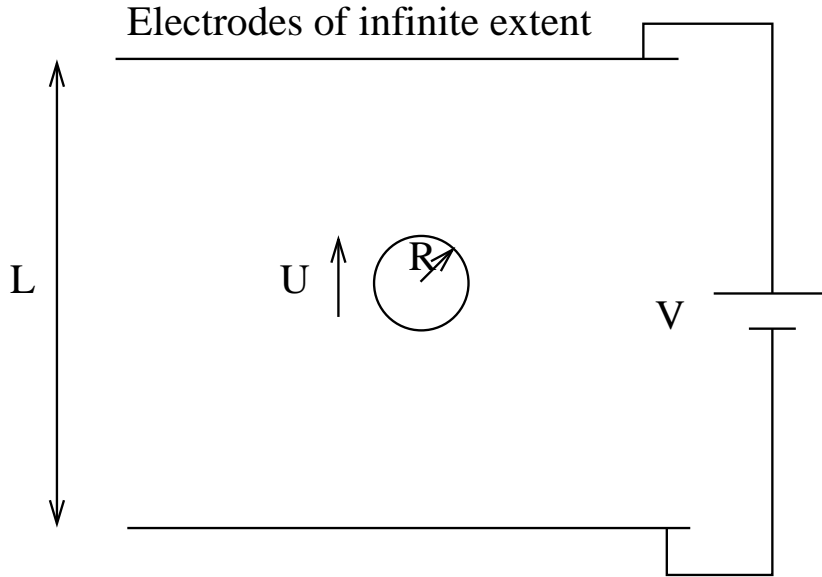


Figure 1: Motion of oil droplet in a water bath.

ratio of dielectric constants of water and oil. The other dimensionless groups can be expressed as ratios of stresses caused by different physical mechanisms. What are these physical mechanisms.

- (d) Of these, there should be a stress due to the electric potential, which involves the potential, dielectric constants, the droplet diameter and distance between plates. Can you obtain an expression for this on the basis of dimensional analysis? What stress would you expect when the distance between plates is large compared to the droplet diameter?
  - (e) Write down a complete set of dimensionless groups relevant in this problem, apart from those listed in part (c).
4. It is desired to set up a spray drier for drying a solution of viscosity  $0.1 \text{ kg/m/s}$  and density  $1000 \text{ kg/m}^3$ , and containing 80 % by weight of water, into particles of diameter about  $100 \mu\text{m}$ . In order to achieve this, it is necessary to design the nozzle used for ejecting the spray, the diameter of the spray drier and the hot air to be circulated through the drier. Use dimensional analysis to determine the design considerations.

Due to the temperature sensitivity of the product, it is not possible to have an average difference in temperature between the droplet and the air of more than  $40^{\circ}\text{C}$ .

This example is an instance where fluid flow is coupled with heat and mass transfer. In the spray drier, the flow through a very small nozzle breaks up the liquid into small drops, and the water in these drops is dried by the heat transferred from the air as the drops move through the air. The drops in nozzle spray driers are usually coarse, and sizes upto  $100\ \mu\text{ m}$  can be achieved. For finer drops, it is necessary to use a spray disk drier, where the spraying is done by a disk of about 1 ft in diameter rotating at speeds as large as 1000 rev/s. In the present example, we consider a nozzle type spray drier, since a relatively coarse size is required. The droplets are usually ejected with high velocities, as large as 0.1 - 1 m/s, and so the time required for drying the droplets is usually very small, of the order of seconds.

The spraying process can be separated into two distinct steps,

- (a) The ejection of the drop from the nozzle. The size of the drops, and the velocity of the drop at the nozzle, are determined primarily by the nozzle geometry, fluid properties and the flow rate of fluid through the nozzle.
- (b) The subsequent drying of the drop as it passes through the air. Here, it is necessary to ensure that the spray drier has a sufficient radius that the drop is completely dried before it hits the wall of the drier. Important for determining this are the velocity with which the drop leaves the nozzle, the heat transfer rate for transporting heat from the drop to the air, the latent heat of evaporation which determines the heat required to dry the drop.

For the second process above, determine the radius and height of the drier form consideration that the droplet should have dried before it reaches the side or bottom walls as follows.

- (a) What are the dimensional groups on which the heat flux and mass flux depends? Organise this into dimensionless parameters and calculate the numerical values. Assume droplet diameter of 100 microns ejected with velocity of 0.5 m/s.
- (b) Which procedure is rate limiting, the heat or mass transfer?

(c) What is the diameter and height required for complete drying?

**Data:** Air thermal conductivity is  $2^{-2} J/m/s/^{\circ}C$ , latent heat of water  $2.2 \times 10^6 J/kg$ .

5. The waste water generated from a chemical process has to be passed through activated carbon in order to remove organic matter by adsorption. The carbon particles are, on average, 1mm in diameter. The process of adsorption is described by a first order process, with a rate constant  $1.6 s^{-1}$ . It is desired to treat 160 l/min of the waste water, and the maximum speed of the water through the activated carbon bed is 1 mm/s, and the porosity (void fraction) of the bed is 0.45. First, discuss why there is a limit on the maximum velocity in the bed. Then, design the height and the cross sectional area of the bed.

The dimensional material parameters that the rate of adsorption can depend on are the adsorption rate constant  $k_a$ , which has units of inverse time for a first order process, and the diffusion constant for the organics. The porosity can also be a factor, in addition to the fluid properties, density and viscosity.

## 2.2 Diffusion

1. Compute the mean free path and the mean molecular velocity of hydrogen molecules (molecular diameter 2.915??) and chlorine molecules (molecular diameter 4.115??) at 300 K temperature and  $10^5$  Pa pressure. What is the ratio of the mean free path and the molecular diameter? Compute the viscosity from kinetic theory.
2. Compute the viscosity of air (mixture of oxygen and nitrogen) at 300 K and  $10^5$  Pa. First, compute the viscosity of pure oxygen (molecular diameter 3.433??) and pure nitrogen (molecular diameter 3.681??). Then, compute the viscosity of the mixture using the semi-empirical formula of Wilke for mixtures,

$$\mu_{mix} = \sum_{i=1}^n \frac{x_i \mu_i}{\sum_{j=1}^n x_j \Phi_{ij}}$$

in which

$$\Phi_{ij} = \frac{1}{\sqrt{8}} \left(1 + \frac{M_i}{M_j}\right)^{-1/2} \left[1 + \frac{\mu_i^{1/2} M_i^{-1/4}}{M_j}\right]^2$$

Compare the result with the measured value of  $1.813 \times 10^{-5}$  kg/m/s.

3. In the kinetic theory of gases, there are two dimensionless numbers that relate the macroscopic flow properties to the molecular properties. The Mach number is defined as  $(U/c)$ , where  $U$  is the flow velocity and  $c$  is the speed of sound. The Knudsen number is defined as  $(\lambda/L)$ , where  $\lambda$  is the mean free path and  $L$  is the macroscopic length scale. If the speed of sound in a gas is approximately equal to the molecular velocity, how is the ratio of convection and diffusion in a gas (the Peclet number for concentration and diffusion or the Reynolds number for momentum diffusion) related to the Mach number and the Knudsen number?
4. Estimate the mass flux in a gas with uniform density and a gradient in temperature.

What is the mass flux when there is both a gradient in density and temperature? What is the relation between the density and temperature gradients when the mass flux is zero?

5. The haemoglobin molecule has a diffusivity of  $0.069 \times 10^{-9}$  m<sup>2</sup>/s in water. Using the Stokes-Einstein relation, estimate the diameter of this molecule. Assume water has a viscosity of  $10^{-3}$  kg/m/s.
6. Use the Stokes-Einstein relation to determine the diffusion coefficient of hydrogen (molecular diameter 2.915??), oxygen (molecular diameter 3.433??) and benzene (molecular diameter 5.270??) in water at 300K. Compare with the measured values of  $4.5 \times 10^{-9}$ ,  $2.1 \times 10^{-9}$  and  $1.02 \times 10^{-9}$  m<sup>2</sup>/s respectively. For which molecule would you expect the best and worst agreement with measured values?

### 2.3 Unidirectional flow in Cartesian co-ordinates:

1. Consider a long and narrow channel two - dimensional of length  $L$  and height  $H$ , where  $H \ll L$ , as shown in figure 2. The ends of the

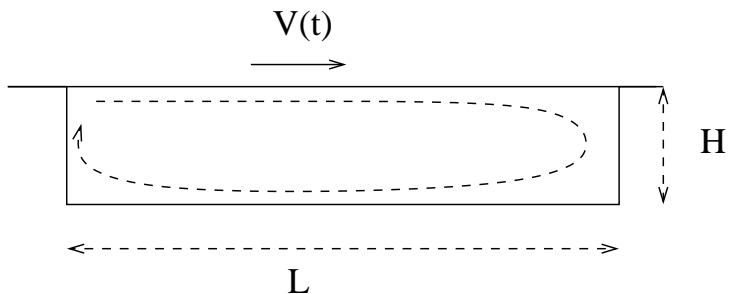


Figure 2: Flow in a thin slot due to a moving wall.

channel are closed so that no fluid can enter or leave the channel. The bottom and side walls of the channel are stationary, while the top wall moves with a velocity  $V(t)$ . Since the length of the channel is large compared to the height, the flow near the center can be considered as one dimensional. Near the ends, there will be some circulation due to the presence of the side walls, but this can be neglected far from the sides. For the flow far from the walls of the channel,

- (a) Write the equations for the unidirectional flow. What are the boundary conditions? What restriction is placed on the velocity profile due to the fact that the ends are closed and fluid cannot enter or leave the channel?
  - (b) If the wall is given a steady velocity  $V$  which is independent of time, solve the equations (neglecting the time derivative term). Calculate the gradient of the pressure.
  - (c) If the wall is given an oscillating velocity  $V \cos(\omega t)$ , obtain an ordinary differential equation to obtain the velocity profile. Get an analytical solution for this which involves the constants of integration. Use the boundary conditions to determine all unknown constants.
2. In shell-and-tube heat exchangers, the tube side often has fins in order to increase the conduction rate, as shown in figure 3. The fin can be modeled, in two dimensions, as a rectangular block of length  $L$ , height  $H$  and with thermal conductivity  $k$ . One surface (outer wall of the tube of the heat exchanger) is at the temperature  $T_t$ , which is the temperature of the tube side fluid. The other three surfaces are



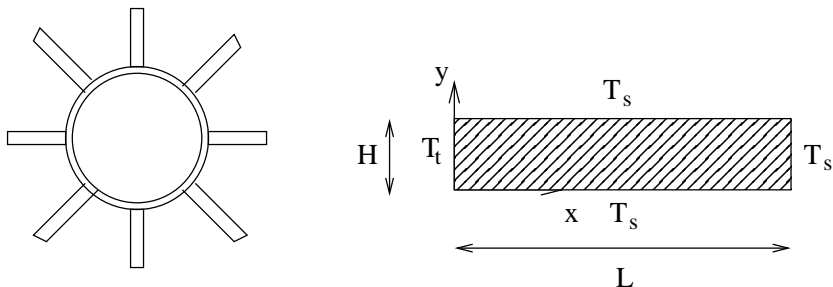


Figure 3: Fin of a shell-and-tube heat exchanger.

at the temperature  $T_s$ , which is the temperature of the shell side fluid. Determine the heat flux from the fin as follows.

- (a) Write down the conduction equation,  $\nabla^2 T = 0$ , in two dimensions, and specify the boundary conditions.
  - (b) Define a non-dimensional temperature in such a way that both boundary conditions are homogeneous along one of the co-ordinates.
  - (c) Use separation of variables to obtain separate the dependence of  $T$  on the  $x$  and  $y$  co-ordinates.
  - (d) Write down the final solution for the temperature field which satisfies homogeneous boundary conditions.
  - (e) Determine the coefficients using orthogonality relations along the inhomogeneous direction. From this, calculate the heat flux as a function of the temperature difference.
3. Consider a solid block extending from  $y = 0$  to  $y = L$  in the  $y$  coordinate, and from  $x = 0$  to  $x \rightarrow \infty$  in the  $x$  co-ordinate, as shown in the figure 4. The top and bottom faces at  $y = 0$  and  $y = L$  are at temperature  $T_0$ , face at  $x = 0$  is at temperature  $T_1$ . Obtain the temperature profile of the block at steady state as follows.
- (a) Which coordinate system would you choose for analysing the problem? Write down the conservation equation for the temperature field at steady state for this system.
  - (b) Transform the temperature to a new coordinate  $\Theta$  in such a way that  $\Theta$  satisfies the same governing equations, but there is a homogeneous boundary condition in all directions except one.

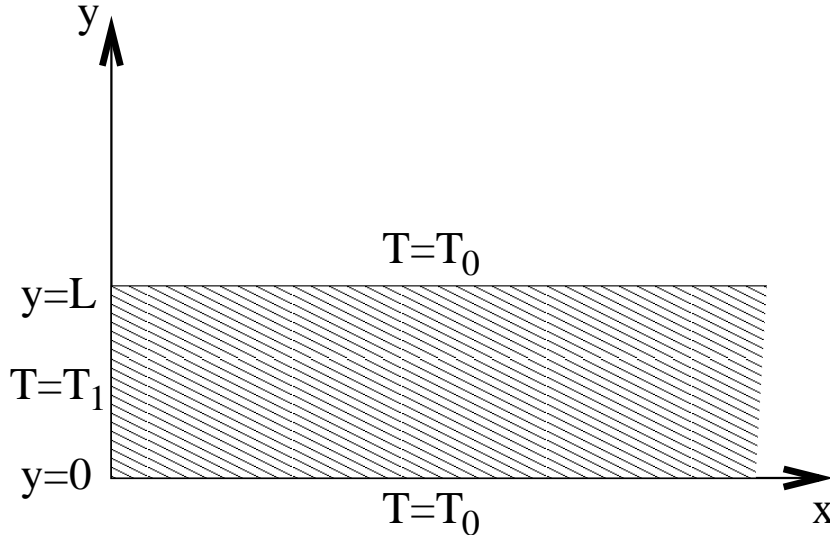


Figure 4: Conduction into a semi-infinite slab.

- (c) Use separation of variables to solve for the temperature field. Determine the constants in the solution for the temperature.
  - (d) What is the total heat transfer at the surface at temperature  $T_1$ ?
4. A rectangular channel of width  $W$  and height  $H$  is used for transporting fluid of density  $\rho$  and viscosity  $\mu$ . If a steady pressure difference  $\Delta p$  is applied across the length  $L$  of the channel, determine the flow rate.
    - (a) First, obtain the momentum balance for the streamwise velocity for a differential volume in the channel.
    - (b) Solve the equation using separation of variables to obtain the velocity. Note that the fluid velocity is zero on all the walls of the channel.
  5. A cubic solid of side  $a$  is initially held at a temperature  $T_0$ . At times  $t \geq 0$ , its lateral faces are held at temperatures  $T_A$ ,  $T_B$ ,  $T_C$  and  $T_D$  as illustrated in figure 5. The top and bottom faces are insulated so that no heat is transferred through them. The cube has heat conductivity  $C$ , density  $\rho$  and thermal conductivity  $K$ .
    - (a) Solve for the steady state temperature in the cube.

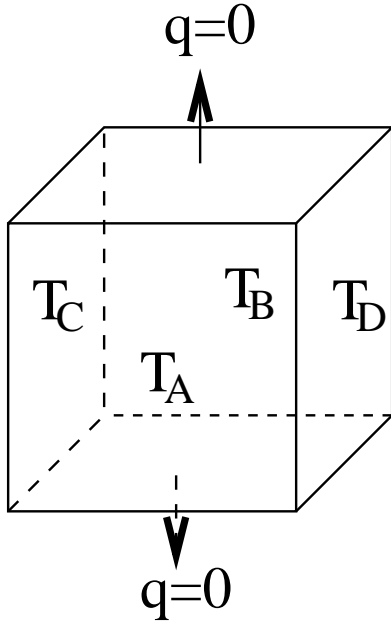


Figure 5: Conduction from a cube.

- (b) Show how the transient problem may be set up in a form to which separation of variables can be applied.
6. There is a dissipation of energy during the shear flow of a viscous liquid due to fluid friction, and this energy increases the temperature of the fluid. We consider the specific example of a pressure-driven flow in a channel between two infinite flat plates located at  $z = 0$  and  $z = H$ . There is no variation in the  $y$  direction perpendicular to the plane of the flow. The temperature at both the bounding surfaces is  $T_0$ , but there is an increase in the temperature within the channel due to the heat generated by viscous dissipation, and the source of energy per unit volume is  $\tau_{xz}(du_x/dz)$ . At steady state, the velocity profile in the channel is given by,

$$\begin{aligned}
 u_x &= -\frac{1}{2\mu} \frac{dp}{dx} z(H-z) \\
 &= 4U \left( \frac{z}{H} - \left( \frac{z}{H} \right)^2 \right)
 \end{aligned}
 \tag{7}$$

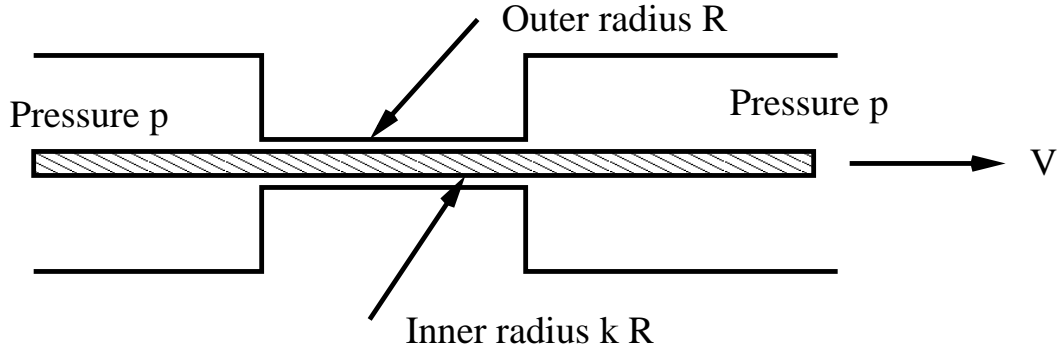


Figure 6: Wire coating.

where  $(dp/dx)$  is the pressure gradient, and  $U$  is the maximum velocity at the center of the channel. Determine the temperature profile in the channel, and the heat flux through the wall.

## 2.4 Unidirectional flow in curvilinear co-ordinates:

1. In wire coating, a cylinder in a thin annular region, as shown in figure 6. The cylinder is pulled with a constant velocity  $V$ . The pressure is equal on both sides of the cylinder. Determine the fluid velocity, and the flow rate.
2. A resistance heating apparatus for a fluid consists of a thin wire immersed in a fluid. In order to design the apparatus, it is necessary to determine the temperature in the fluid as a function of the heat flux from the wire. For the purposes of the calculation, the wire can be considered of infinite length so that the heat conduction problem is effectively a two dimensional problem. In addition, the thickness of the wire is considered small compared to any other length scales in the problem, so that the wire is a line source of heat. The wire and the fluid are initially at a temperature  $T_0$ . At time  $t = 0$ , the current is switched on so that the wire acts as a source of heat, and the heat transmitted per unit length of the wire is  $Q$ . The heat conduction in the fluid is determined by the unsteady state heat conduction equation

$$\partial_t T = K \nabla^2 T \quad (8)$$

and the heat flux (heat conducted per unit area) is

$$K\nabla T \quad (9)$$

where  $K$  is the thermal conductivity of the fluid.

- (a) Choose an appropriate coordinate system, and write down the unsteady heat conduction equation.
  - (b) What are the boundary conditions? Give special attention to the heat flux condition at the wire, and note that the wire is considered to be of infinitesimal radius.
  - (c) Solve the heat conduction equation using the simplest method, and determine the temperature field in the fluid.
  - (d) Use the boundary conditions to determine the constants in the expression for the temperature field.
3. An ideal vortex is a flow with circular streamlines where the particle motion is incompressible and irrotational. The velocity profile obeys the equation in cylindrical coordinates:

$$v_\theta = \frac{\Gamma}{2\pi r} \quad (10)$$

with  $v_r = v_z = 0$ . At the origin, the above equation indicates that the velocity becomes infinite. But this is prohibited because viscous forces become important and the flow is rotational in a small region near the core.

Consider an ideal vortex in which the velocity is given by the above equation for  $t < 0$ , and the core velocity is constrained to be zero at  $t = 0$ . Find the velocity profile for  $t > 0$ . Assume that  $v_\theta$  is the only non-zero velocity component. The momentum equation for  $v_\theta$  is,

$$\frac{\partial v_\theta}{\partial t} = \nu \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} \right)$$

4. Consider the fully developed flow in a circular tube with velocity profile

$$u_x = U(t) \left( 1 - \frac{r^2}{R^2} \right)$$

as shown in figure 7, where the maximum velocity  $U$  could be a function of  $t$ , but is independent of the stream-wise co-ordinate  $x$ . There is viscous dissipation which generates heat within the fluid, and the heat generated per unit volume of the fluid per unit time is given by,

$$Q = \mu \left( \frac{\partial u_x}{\partial r} \right)^2$$

Due to this, there is a temperature variation across the tube, and the temperature field is governed by the convection-diffusion equation,

$$\rho C_v \left( \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = k \nabla^2 T + Q$$

The temperature at the wall of the tube is maintained at  $T = T_0$ , and **we assume that the temperature could be a function of time, but the temperature field is ‘fully developed’ so that the temperature is independent of the flow ( $x$ ) direction.**

- (a) Choose a suitable co-ordinate system, and write down the convection-diffusion equation for the time-dependent but ‘fully developed’ temperature field.
  - (b) Scale the co-ordinates and time. What would you use to scale the temperature?
  - (c) Obtain the solution for the temperature at steady state, where both the maximum velocity  $U$  and temperature are independent of time.
  - (d) If the maximum velocity has a sinusoidal variation,  $U(t) = U \cos(\omega t)$ , what is the value of the heat source  $Q$ ? How would you express the inhomogeneous term in the time-dependent convection-diffusion equation for the temperature in order to obtain a solution?
  - (e) Determine the solution for the temperature field.
5. A rotating cylinder geometry consists of a cylinder of radius  $R$  and height  $H$ , filled with fluid, with two end caps. The cylinder rotates with an angular velocity  $\Omega$ , while the end caps are stationary. Determine the fluid velocity field using separation of variables as follows.

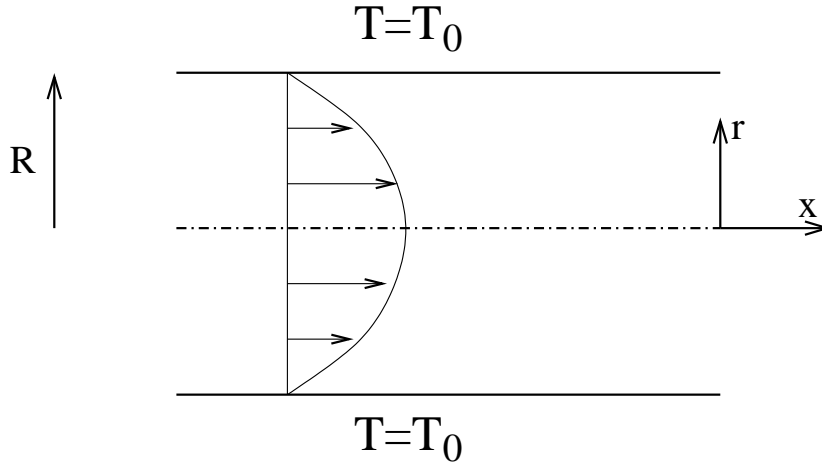


Figure 7: Viscous heating due to the flow in a tube.

- Choose a coordinate system for the problem. Clearly, the only non-zero component of the velocity is  $u_\phi$ . Determine the boundary conditions for this component of the velocity.
- Write down the mass balance condition for an incompressible fluid. For a uni-directional flow in which the density is a constant, what does this reduce to?
- The steady state momentum conservation equation for  $u_\phi$  is,

$$-\frac{\partial p}{\partial \phi} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (ru_\phi)}{\partial r} \right) + \frac{\partial^2 u_\phi}{\partial z^2} \right) = 0$$

Can you eliminate pressure in the above equation by taking a derivative with respect to  $\phi$  and using mass balance.

- Solve the conservation equation at steady state using the method of separation of variables. Frame the orthogonality conditions which would be required to solve the problem.

**Data:**

- Bessel equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

Solution:

$$y = A_1 J_n(x) + A_2 Y_n(x)$$

where  $J_n(x)$  is bounded for  $x \rightarrow 0$ , and  $Y_n(x)$  is bounded for  $x \rightarrow \infty$ . tem Modified Bessel equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2)y = 0$$

Solution:

$$y = A_1 I_n(x) + A_2 K_n(x)$$

where  $I_n(x)$  is bounded for  $x \rightarrow 0$ , and  $K_n(x)$  is bounded for  $x \rightarrow \infty$ .

6. A piston damper assembly, shown in figure 8, consists of a cylindrical piston rod moving through a stationary cylindrical sleeve which is filled with fluid. The sleeve is closed at both ends so that the fluid cannot move in or out of the sleeve. The radius of the piston rod is  $R_p$ , while the outer radius of the sleeve is  $R_s$ . The length of the sleeve  $L$  is large compared to the radius  $R_s$ . The piston rod moves with a velocity  $U$  with respect to the sleeve. Consider the region away from the ends of the sleeve, where the flow is expected to be in only one direction (parallel to the walls).
- (a) Choose a co-ordinate system for analysing the configuration. Mark the boundaries of the fluid in this co-ordinate system.
  - (b) Which is the non-zero component of the velocity in this co-ordinate system, and which spatial co-ordinate does the velocity depend on? Write the boundary conditions in the center region away from the ends of the sleeve, where the flow is expected to be unidirectional.
  - (c) What condition can be obtained about the velocity in the center of the channel (away from the ends) from mass conservation?
  - (d) Use a shell balance to derive the momentum conservation equation for the unidirectional flow in the center of the sleeve away from the ends.
  - (e) Solve the momentum conservation equation to determine the velocity field.



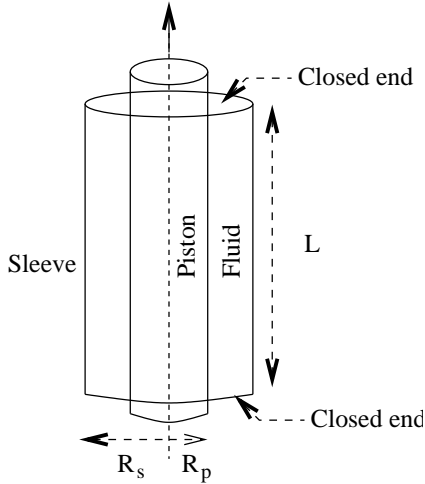


Figure 8: Piston-damper assembly.

- (f) Determine all the constants in the expression for the velocity from the conditions obtained above.
7. Consider an annular channel described in a cylindrical  $(r, \theta, z)$  coordinate system. The cross section of the channel is shown in figure 9, and the  $z$  direction is perpendicular to the plane of the cross section. The channel is bounded by solid walls at  $r = 0$  and  $r = R$ , and at  $\theta = 0$  and  $\theta = \Theta$ . The wall at  $r = R$  is moving in the  $z$  direction with a velocity  $U$ , while those at  $\theta = 0$  and  $\theta = \Theta$  are stationary. The flow is a unidirectional, fully developed, and steady flow with velocity only in the  $z$  direction. The equation for the velocity field in the  $z$  direction is,

$$\mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} \right) = 0$$

where  $\mu$  is the viscosity.

- (a) What are the boundary conditions required for solving the above equation?
- (b) Use separation of variables, by writing  $u_z = F(r)T(\theta)$ , and obtain equations for  $F$  and  $T$ . [2]
- (c) Find the solution for  $T(\theta)$  that satisfies the boundary conditions.

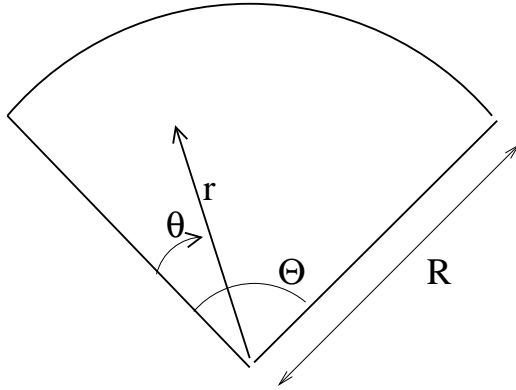


Figure 9: Annular sleeve.

- (d) Find the solution for  $F(r)$ , and enforce boundary conditions to find the final solution.
8. Consider a point source of heat located in an infinite medium of thermal conductivity  $k$  maintained at temperature  $T_0$  far from the source. For time  $t < 0$ , the source does not generate any heat, and the temperature everywhere is  $T_0$ . At time  $t > 0$ , the source generates  $Q$  units of energy per unit time.
- What co-ordinate system will you use for solving the problem? Obtain the unsteady energy conservation equation using a shell balance.
  - What are the boundary conditions?
  - Use similarity transform obtain a solution for the differential equation. What is the boundary condition in terms of the similarity variable?
  - Can a similarity solution be obtained if  $Q$  is a constant? What should be the dependence of  $Q$  on time in order to obtain a similarity solution?
9. Solve the two-dimensional heat diffusion equation,  $\nabla^2 T = 0$ , for the temperature field around a cylinder, as shown in figure 10, when the temperature at a large distance from the cylinder is  $T = T'xy$ , and the temperature at the cylinder surface is a constant,  $T_0$ . Note that

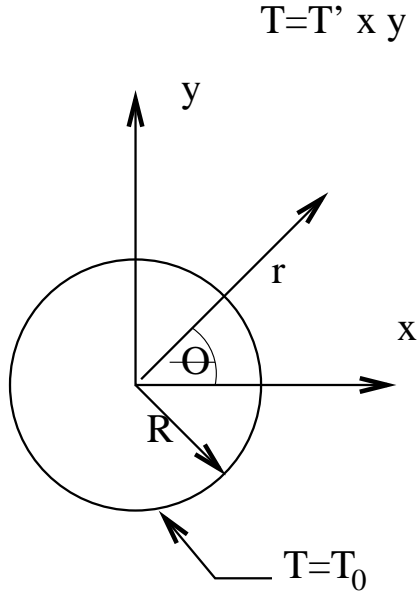


Figure 10: Cylinder in a temperature field.

the solutions of the diffusion equation in a two-dimensional cylindrical co-ordinate are,

$$T = \sum_{m=0}^{\infty} \left( \left( \frac{A_m}{r^m} + B_m r^m \right) \cos(m\phi) + \left( \frac{C_m}{r^m} + D_m r^m \right) \sin(m\phi) \right) \quad (11)$$

and the basis functions,  $\cos(m\phi)$  and  $\sin(m\phi)$  are orthogonal basis functions, that is,

$$\begin{aligned} \int_0^{2\pi} d\phi \cos(m\phi) \cos(n\phi) &= \int_0^{2\pi} d\phi \sin(m\phi) \sin(n\phi) = \pi \delta_{mn} \\ \int_0^{2\pi} d\phi \cos(m\phi) \sin(n\phi) &= 0 \end{aligned} \quad (12)$$

- (a) Determine the temperature field.
- (b) What is the heat flux, and the total heat transfer from the cylinder?

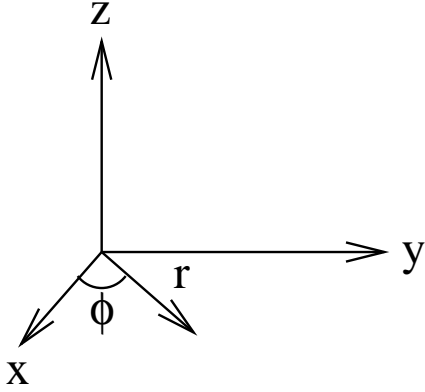


Figure 11: Cylindrical coordinate system.

## 2.5 Mass and energy conservation equations:

1. The cylindrical coordinate system consists of the coordinates  $(r, \phi, z)$ , where  $r$  is the distance from the  $z$  axis, and  $\phi$  is the angle made by the position projection of the position vector on the  $x - y$  plane with the  $x$  axis, as shown in figure 1. For this coordinate system,
  - (a) Determine the coordinates  $(x, y, z)$  in terms of  $(r, \phi, z)$ , and the coordinates  $(r, \phi, z)$  in terms of  $(x, y, z)$ . How are the unit vectors  $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$  related to  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ .
  - (b) Write down the conservation equation for the concentration field for the appropriate differential volume in cylindrical coordinates. What is the divergence operator  $\nabla \cdot$  in this coordinate system?
  - (c) Express the flux in terms of the gradient of concentration in the cylindrical coordinate system. What is the Laplacian operator  $\nabla^2$  in this coordinate system?
  - (d) Solve the differential equation for the concentration in cylindrical coordinates using the separation of variables, in a manner similar to that for spherical coordinate system in class.

## 2.6 Transport due to diffusion:

1. Derive the harmonic expansion for a two dimensional cylindrical coordinate system with coordinates  $(r, \theta)$ . Use separation of variables to

solve the equation  $K\nabla^2 T = 0$  in cylindrical co-ordinates.

2. For a point source, solve the heat equation  $K\nabla^2 T = Q\delta(\mathbf{x})$  in cylindrical coordinates, to obtain the temperature distribution due to a point source.
3. What is the temperature field when two sources are located as shown in figure 5(a) and 5(b), and  $L \ll r$ ? Compare with the second terms in the cylindrical harmonic expansion.
4. What is the temperature field when four sources are located as shown in figures 5(c), and (d)? Compare with the third terms in the cylindrical harmonic expansion.
5. Determine the second and third terms in the harmonic expansion by successively taking gradients of the temperature field due to the point source.
6. Determine the effective thermal conductivity for a dilute array of infinitely long circular cylinders along the plane perpendicular to the axis of the cylinders, when the area fraction of the cylinders is  $\phi$ . Use the following steps.
  - (a) Consider an infinitely long cylinder with conductivity  $K_p$  in a matrix of conductivity  $K_m$ , and determine the temperature field around the cylinder when a uniform gradient  $T'$  is imposed in the  $x$  direction perpendicular to the axis of the cylinder.
  - (b) Write the heat flux as the sum of the flux over the matrix and the sum over the cylinders. When the array is dilute, write the integral as the sum over one cylinder, and determine the thermal conductivity.
  - (c) What is the effective conductivity along the axis of the cylinders?
7. A point source of heat of strength  $Q$  (in units of heat energy per unit time) is placed at a distance  $L$  from a wall.
  - (a) If the wall is perfectly conducting, so that the flux lines at the wall are perpendicular to the wall as shown in figure 13, determine the temperature as a function of position.

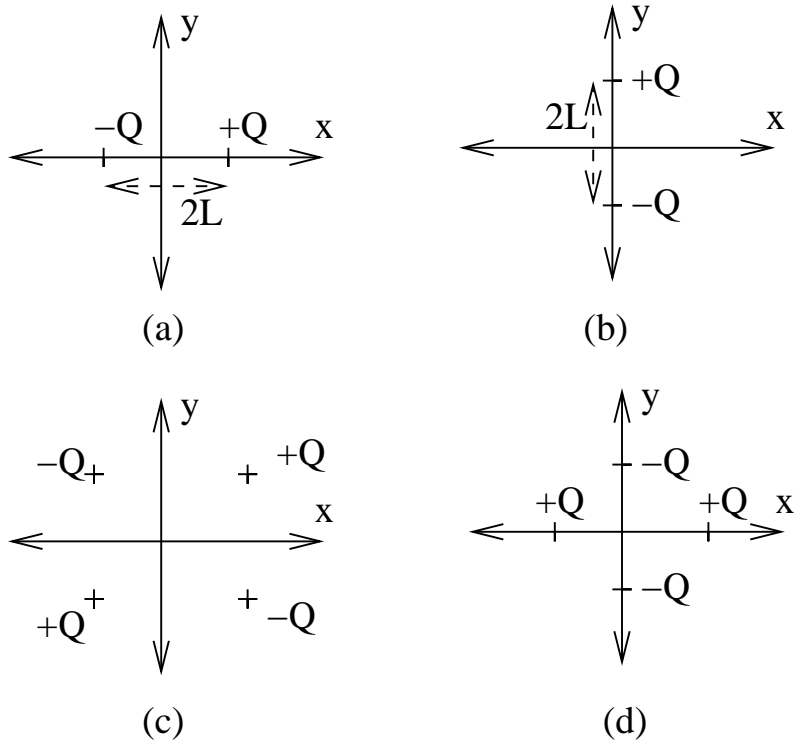


Figure 12: Different arrangements of sources corresponding to higher harmonics in two dimensions.

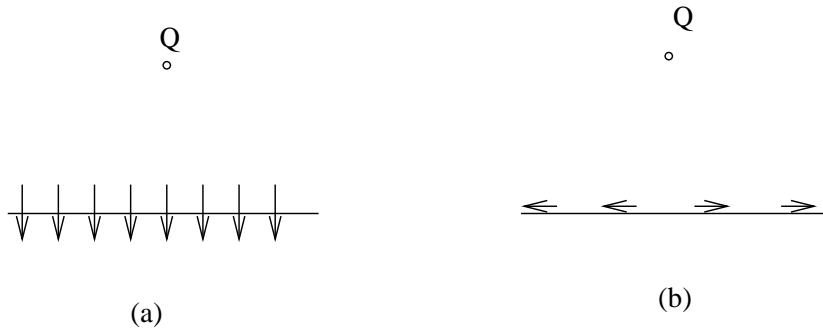


Figure 13:

- (b) If the wall is perfectly insulating, so that the flux lines at the wall are parallel to the wall as shown in figure 13, determine the temperature as a function of position.
  - (c) If the wall is not perfectly conducting, but only a fraction  $f$  of the heat on the wall penetrates it, while a fraction  $(1 - f)$  does not penetrate the wall, determine the temperature field as a function of position.
8. A heater coil in the form of a ring of radius  $a$  in the  $x-y$  plane generates heat  $Q$  per unit length of the coil per unit time, as shown in figure 14
- (a) If the heater is placed in an unbounded medium of thermal conductivity  $K$ , write an equation for the temperature as a function of position in the medium.
  - (b) Plot the temperature as a function of position along the symmetry axis of the heater ( $z$  axis in the figure 14). Simplify the expressions for the temperature for  $z \ll a$  and  $z \gg a$ . What does the expression for  $z \gg a$  correspond to?
9. Solve the two-dimensional heat diffusion equation,  $\nabla^2 T = 0$ , for the temperature field around a cylinder, as shown in figure 1, when the temperature at a large distance from the cylinder is  $T = T'xy$ , and the temperature at the cylinder surface is a constant,  $T_0$ . Note that the solutions of the diffusion equation in a two-dimensional cylindrical

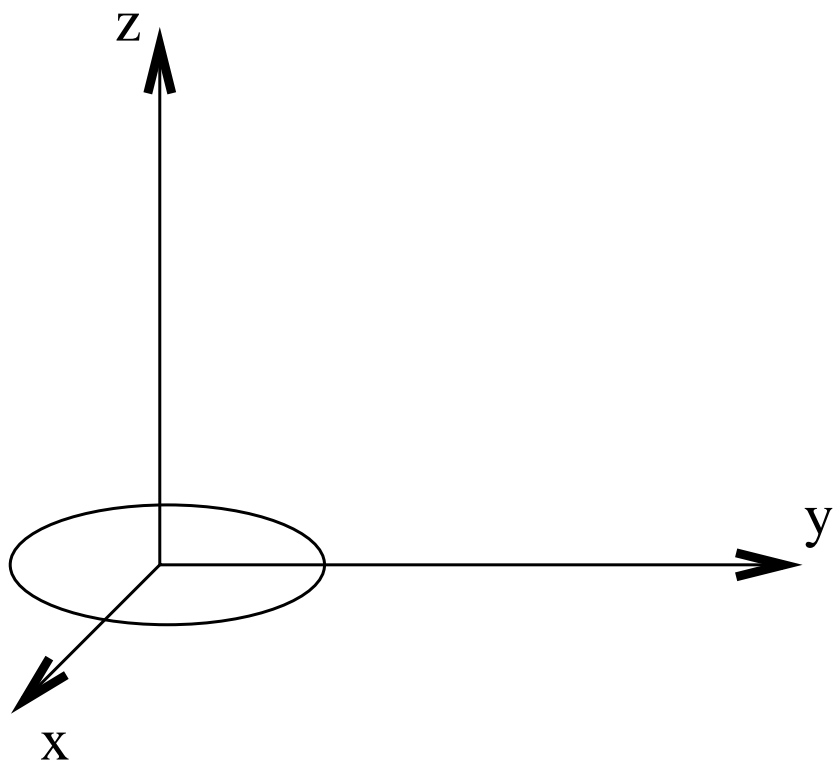


Figure 14:



co-ordinate are,

$$T = \sum_{m=0}^{\infty} \left( \left( \frac{A_m}{r^m} + B_m r^m \right) \cos(m\phi) + \left( \frac{C_m}{r^m} + D_m r^m \right) \sin(m\phi) \right) \quad (13)$$

and the basis functions,  $\cos(m\phi)$  and  $\sin(m\phi)$  are orthogonal basis functions, that is,

$$\begin{aligned} \int_0^{2\pi} d\phi \cos(m\phi) \cos(n\phi) &= \int_0^{2\pi} d\phi \sin(m\phi) \sin(n\phi) = \pi \delta_{mn} \\ \int_0^{2\pi} d\phi \cos(m\phi) \sin(n\phi) &= 0 \end{aligned} \quad (14)$$

- (a) Determine the temperature field.
- (b) What is the heat flux, and the total heat transfer from the cylinder?

## 2.7 Transport at high Peclet number:

1. Consider the heat transfer in the flow past a corner, shown in figure 15. The temperature in the fluid is  $T_0$  for  $x \leq 0$ , and there is a heated section with temperature  $T_1$  for  $x \geq 0$ . The fluid far from the surface is at temperature  $T_0$ . Adjacent to the heated section, the velocity profile in the fluid close to the surface is given by,  $u_x = kx^{1/2}$ ,  $u_y = 0$ .
  - Write down the steady state convection-diffusion equation for this case.
  - If we scale distance along the heated surface ( $x$ ) and perpendicular to heated surface ( $y$ ) by the characteristic length  $L$ , what is the Peclet number?
  - When the Peclet number is large, the effect of heating is confined to a thin boundary layer of thickness  $\delta$  near the surface. Rescale the equations, and determine the dependence of  $\delta$  on Peclet number.
  - Solve the simplified convection-diffusion equation, which includes streamwise convection and cross-stream diffusion, using a similarity transform to obtain a similarity solution for the temperature.

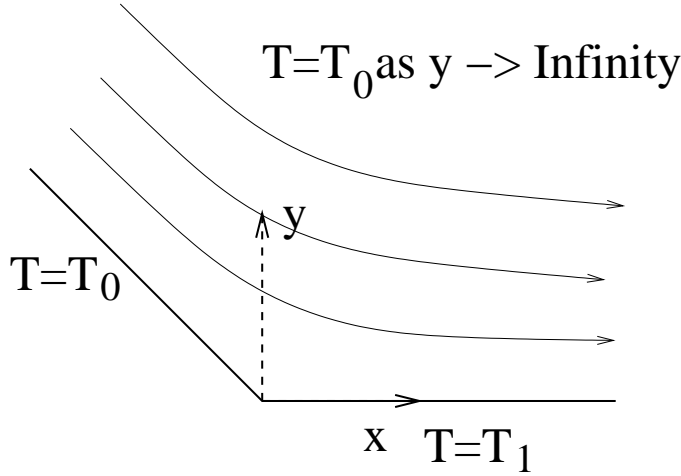


Figure 15: Heat transfer in flow around a corner.

- Determine the average heat flux and Nusselt number based on the length  $L$  of the heated section.
2. Consider the high Peclet number mass transfer in the pressure-driven flow of a ‘power-law’ fluid past a flat surface, as shown in figure 16. Use a Cartesian co-ordinate system where the surface is in the  $x - z$  plane, and the velocity is in the  $x$  direction. For a power-law fluid, the stress  $\tau_{xy}$  is related to the velocity gradient ( $du_x/dy$ ) by the relation,

$$\tau_{xy} = c \left( \frac{du_x}{dy} \right) \left| \frac{du_x}{dy} \right|^{n-1}$$

The velocity field close to the surface can be approximated as,

$$u_x = \left( -\frac{1}{c} \frac{dp}{dx} \right)^{1/n} \frac{n}{n+1} y^{((n+1)/n)}$$

Consider the flow of a power-law fluid past a flat plate with the above velocity profile. The temperature of the incoming fluid, as well as the plate temperature for  $x < 0$  is  $T_0$ , while the plate temperature is  $T_1$  for  $x > 0$ . If we neglect streamwise diffusion in comparison to convection, the convection-diffusion equation for the temperature field is,

$$u_x \frac{\partial T}{\partial x} = \alpha \left( \frac{\partial^2 T}{\partial y^2} \right)$$

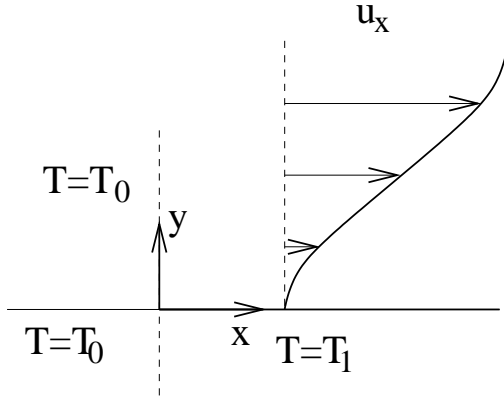


Figure 16: Flow of power law fluid across a surface.

- (a) Express the convection-diffusion equation and the boundary conditions in terms of the scaled temperature field  $T^* = (T - T_0)/(T_1 - T_0)$ . Insert the expression for the mean velocity into this equation. What are the boundary conditions?
  - (b) Define a scaled variable  $\eta = (y/g(x))$ , and express the conservation equation in terms of  $\eta$ . Substitute  $y = \eta g(x)$  in the resulting equation, to obtain an equation which does not contain the  $y$  coordinate.
  - (c) Determine the solution for  $g(x)$  required to obtain a similarity solution.
  - (d) What is the equation for the temperature field in terms of the similarity variable? **Do not attempt to solve the equation.**
3. Consider the flow around a cylinder in two dimensions as shown in figure 17. The velocity field in the radial co-ordinate system is given by,

$$\begin{aligned}
 u_r &= -U \cos(\theta) \left(1 - \frac{R^2}{r^2}\right) \\
 u_\theta &= U \sin(\theta) \left(1 + \frac{R^2}{r^2}\right)
 \end{aligned} \tag{15}$$

where  $R$  is the radius of the cylinder. The cylinder surface is at a temperature  $T_0$ , while the temperature far from the cylinder is  $T_\infty$ . The

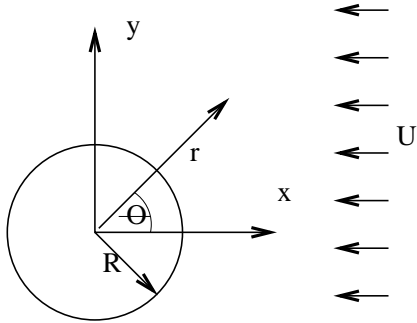


Figure 17: Flow past a cylinder.

equation for the temperature field is the convection diffusion equation in cylindrical co-ordinates,

$$u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r} \frac{\partial T}{\partial \theta} = \alpha \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right) \quad (16)$$

- (a) Insert the expression for the fluid velocity into the above equation, and scale the resulting equation to obtain a dimensionless equation for the temperature field. What is the Peclet number (ratio of convection and diffusion)?
  - (b) Consider the limit where the Peclet number is large. In this case, the temperature variation is expected to be confined to a thin boundary layer near the surface of the cylinder. Scale the distance from the surface of the cylinder by a boundary layer thickness, and simplify the equation. How is the boundary layer thickness related to the Peclet number?
  - (c) Use a similarity transform to express the convection-diffusion equation in terms of the ratio of the distance from the surface and a boundary layer thickness, where the boundary layer thickness is a function of the  $\theta$  co-ordinate. What is the equation for the variation for the boundary layer co-ordinate with  $\theta$ ?
  - (d) Solve the equation for the temperature in terms of the similarity variable.
4. Consider the flow past a curved surface with a slip boundary condition at the surface, as shown in figure 18. There is fluid flow across the sur-

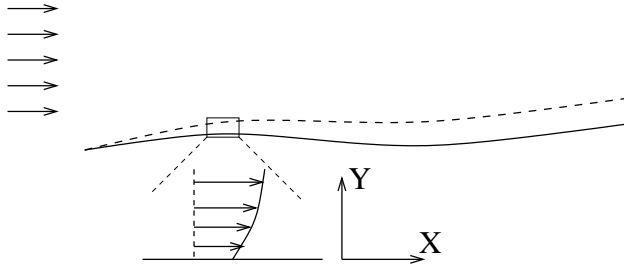


Figure 18: Flow with velocity slip across a surface.

face and conduction from the surface, and the Peclet number is considered to be high. Use a local orthogonal co-ordinate system,  $(X, Y)$ , at a point on the surface, and write down the convection-diffusion equation in a similar to that done for a general surface with no-slip boundary conditions, with the difference that the tangential velocity along the surface is not zero, though the normal velocity has to be zero. Assuming the flow is two-dimensional and that the velocity  $(u_X, u_Y)$  satisfies incompressibility. Write down the steady convection diffusion equation, and obtain a solution in terms of the similarity variable, as well as the boundary layer thickness.