INDIAN INSTITUTE OF SCIENCE

STOCHASTIC HYDROLOGY

Lecture -5

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Summary of the previous lecture

- Moments of a distribution
- Measures of central tendency, dispersion, symmetry and peakedness
- Introduction to Normal distribution

Normal Distribution

$$Z = \frac{X-\mu}{\sigma}$$

-- Linear function

Y = a + bX

Y ~ N(a+bμ, $b^2\sigma^2$)

$$a = \frac{-\mu}{\sigma}, b = \frac{1}{\sigma}$$

$$Z : N\left[\frac{-\mu}{\sigma}+\frac{\mu}{\sigma}, \frac{1}{\sigma^2}\times\sigma^2\right]$$

$$: N(0,1)$$

pdf of z

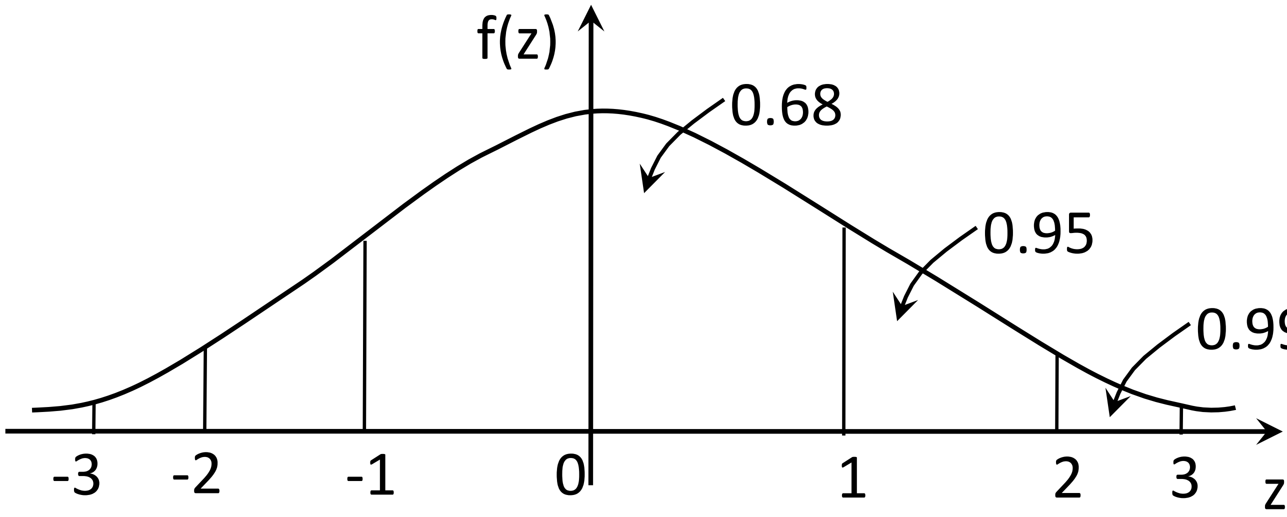
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < +\infty$$

cdf of z

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2} dz \qquad -\infty < z < +\infty$$

Normal Distribution

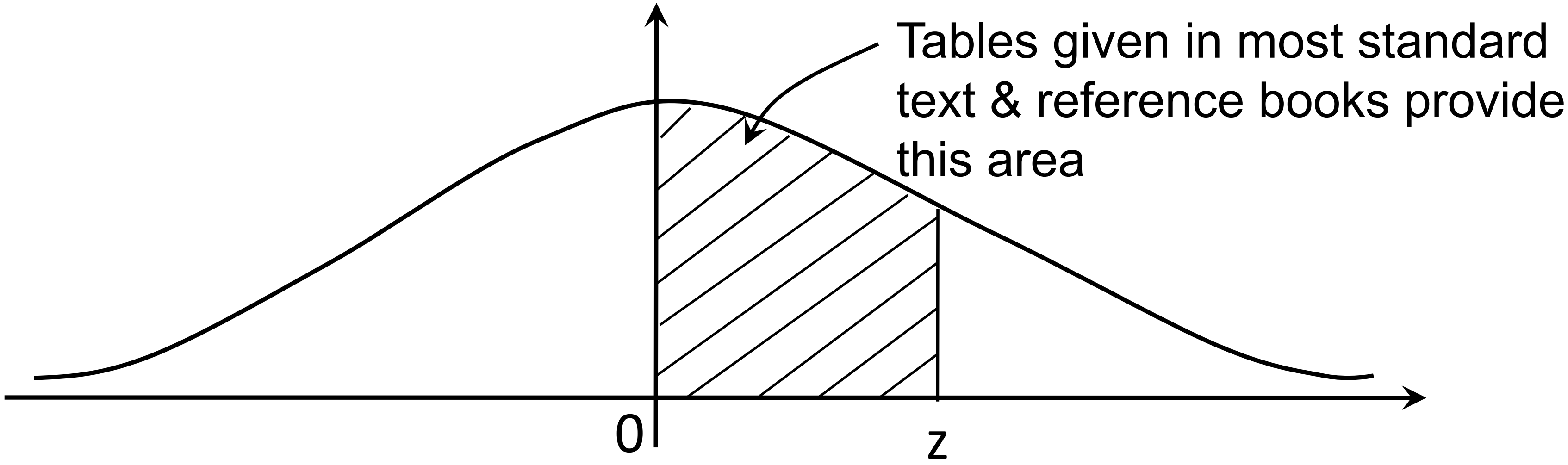
- f(z) is referred as standard normal density function
- The standard normal density curve is as shown
- 99% of area lies between $\pm 3\sigma$



- f(z) cannot be integrated analytically by ordinary means
- Methods of numerical integration used
- The values of F(z) are tabulated.

Normal Distribution

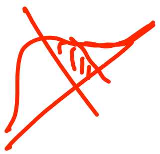
Obtaining standard variate ‘z’ using tables:



P[Z $\le$ z] = 0.5+Area from table

Normal Distribution Tables

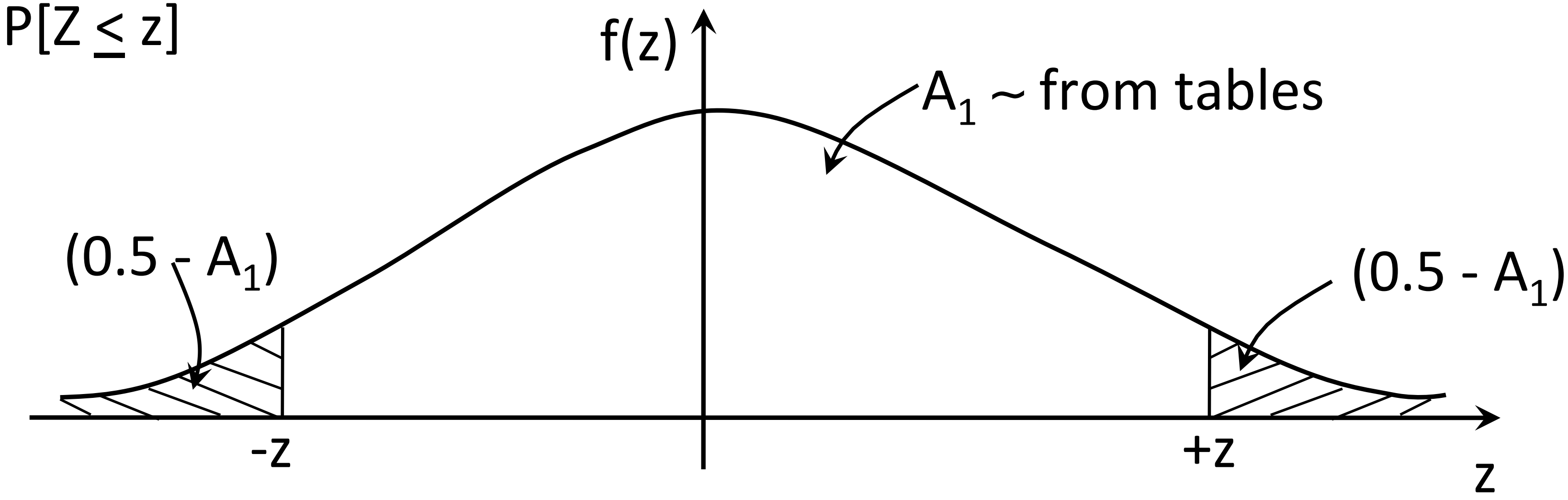
z	0	2	4	6	8
0	0	0.008	0.016	0.0239	0.0319
0.1	0.0398	0.0478	0.0557	0.0636	0.0714
0.2	0.0793	0.0871	0.0948	0.1026	0.1103
0.3	0.1179	0.1255	0.1331	0.1406	0.148
0.4	0.1554	0.1628	0.17	0.1772	0.1844
0.5	0.1915	0.1985	0.2054	0.2123	0.219
0.6	0.2257	0.2324	0.2389	0.2454	0.2517
0.7	0.258	0.2642	0.2704	0.2764	0.2823
0.8	0.2881	0.2939	0.2995	0.3051	0.3106
0.9	0.3159	0.3212	0.3264	0.3315	0.3365
1	0.3413	0.3461	0.3508	0.3554	0.3599



Normal Distribution Tables

z	0	2	4	6	8
3.1	0.499	0.4991	0.4992	0.4992	0.4993
3.2	0.4993	0.4994	0.4994	0.4994	0.4995
3.3	0.4995	0.4995	0.4996	0.4996	0.4996
3.4	0.4997	0.4997	0.4997	0.4997	0.4997
3.5	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5	0.5	0.5	0.5	0.5

Normal Distribution



from table

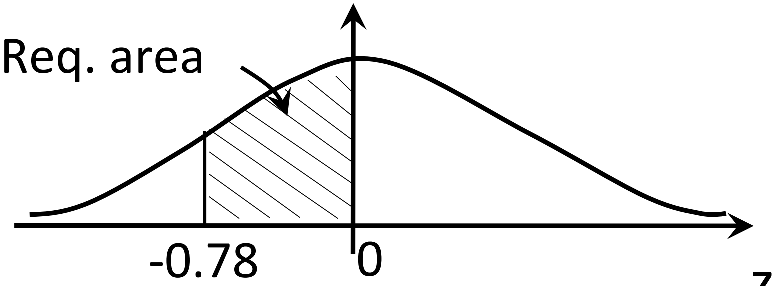
e.g., $P[Z \leq -0.7] = 0.5 - 0.258$

$= 0.242$

z	0
0.5	0.1915
0.6	0.2257
0.7	0.258

Example-1

Obtain the area under the standard normal curve between -0.78 and 0



$$P[-0.78 \le Z \le 0] = \frac{1}{\sqrt{2\pi}} \int_{-0.78}^{0} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{0.78} e^{-z^2/2} dz$$

$= 0.2823$

z	7	8	9
0.6	0.2486	0.2517	0.2549
0.7	0.2794	0.2823	0.2852
0.8	0.3078	0.3106	0.3133

From Tables:

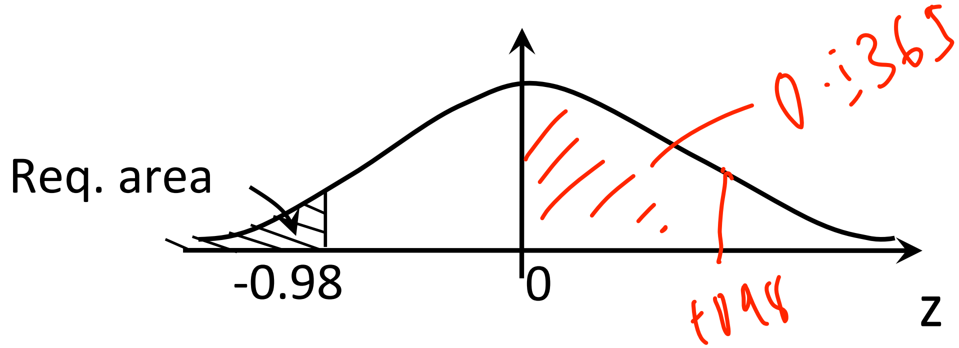
Req. area = area betn. 0 and +0.78

= 0.2823

Example-2

Obtain the area under the standard normal curve

$z \leq -0.98$



From tables:

Req. area = 0.5 – area between 0 and +0.98

= 0.5 – 0.3365

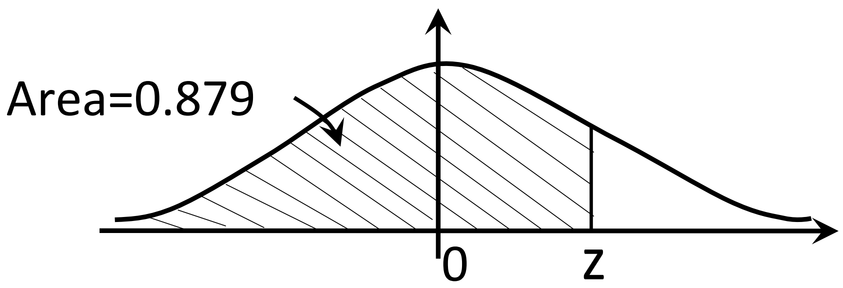
= 0.1635

z	7	8	9
0.8	0.3078	0.3106	0.3133
0.9	0.334	0.3365	0.3389
1	0.3577	0.3599	0.3621
1.1	0.379	0.381	0.383

Example-3

Obtain 'z' such that $P[Z \leq z]=0.879$

Since the probability of $P[Z \leq z]$ is greater than 0.5, 'z' must be +ve



z	6	7	8
1.1	0.377	0.379	0.381
1.2	0.3962	0.398	0.3997

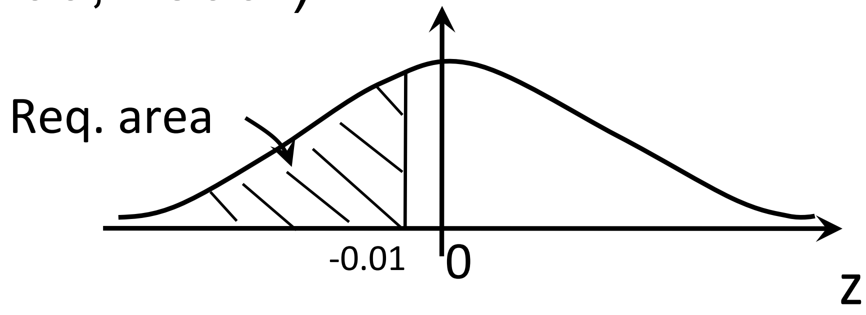
area between 0 to z = 0.879 – 0.5

= 0.379

From the table, for the area of 0.379, corresponding z = 1.17

Example-4

Obtain P[X ≤ 75] if N ~ (100, 2500^2)



$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{75-100}{2500}$$

$$= -0.01$$

From the table,

Req. area = 0.5 – area between 0 and +0.01

= 0.5 – 0.004

= 0.496

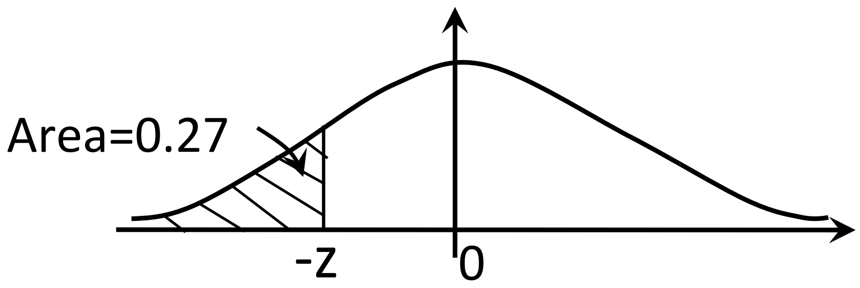
P[X ≤ 75] = 0.496

z	0	1	2
0.0	0	0.004	0.008
0.1	0.0398	0.0438	0.0478

Example-5

Obtain ‘x’ such that $P[X \geq x]=0.73$ if $\mu_x=650$; $\sigma_x = 200$

$P[X \leq x]=0.27$



$P[Z \leq z]=0.27$

z	1	2
0.5	0.195	0.1985
0.6	0.2291	0.2324

area between 0 to -z = 0.5 – 0.27

= 0.23

From the table, z = -0.613

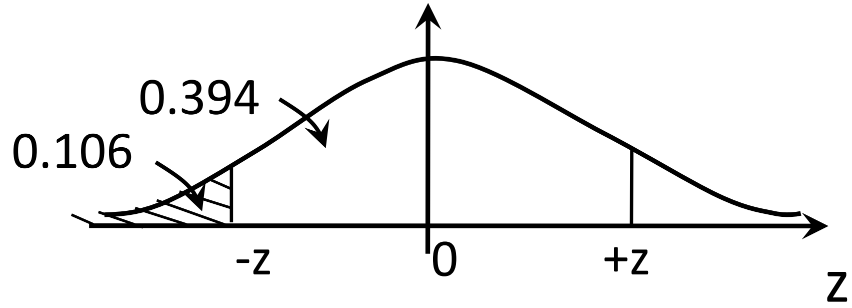
$$z = \frac{x-\mu}{\sigma}; \qquad -0.613 = \frac{x-650}{200} \quad ; \quad x = 527$$

Example-6

A rv 'X' is normally distributed with following probabilities:

P[X $\leq$ 50] = 0.106 and P[X $\leq$ 250] = 0.894

Obtain μ and σ of 'X'



P[X $\leq$ 50] = 0.106

P[Z $\leq$ z] = 0.106

Since the probability is less than 0.5, z is –ve.

From tables, for area of 0.394, -z=1.25

$$z = \frac{x-\mu}{\sigma}; \quad \frac{50-\mu}{\sigma} = -1.25$$

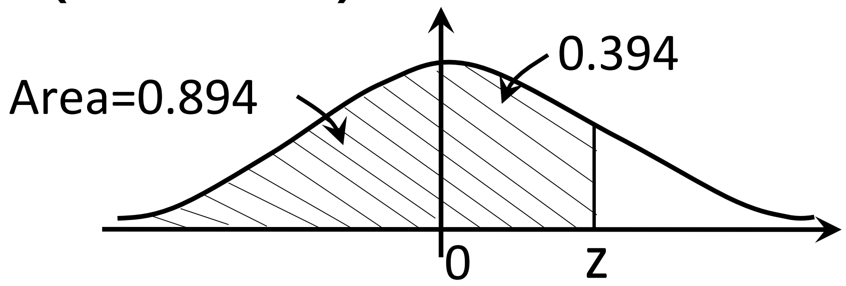
$$\mu = 50 + 1.25\sigma$$

z	4	5	6
1.1	0.3729	0.3749	0.377
1.2	0.3925	0.3944	0.3962
1.3	0.4099	0.4115	0.4131

Example-6 (contd.)

P[X $\le$ 250] = 0.894

P[Z $\le$ z] = 0.894



From tables, for area of 0.394, z=1.25

$$z = \frac{x-\mu}{\sigma}, \quad \frac{250-\mu}{\sigma} = 1.25$$

250 – μ =1.25 σ

250 – (50 + 1.25 σ) = 1.25 σ

200 = 2.5 σ

σ = 80,

μ = 150

Example-7

Annual rainfall ‘P’ is normally distributed over a basin with mean 1000mm and standard deviation 400mm. Annual runoff ‘R’ (in mm) from the basin is related to annual rainfall by

$R = 0.5P - 150.$

1. Obtain the mean and standard deviation of annual runoff.
2. Obtain the probability that the annual runoff will exceed 600mm

Example-7 (contd.)

$R = -150 + 0.5P$ – Linear function of ‘P’

Since $P \sim N(1000, 400^2)$,

$R \sim N(a+b\mu, b^2\sigma^2)$

$\sim N(-150+0.5 \times 1000, 0.5^2 \times 400^2)$

$\sim N(350, 200^2)$

Mean, μ = 350mm and

Standard deviation σ = 200mm

Example-7 (contd.)

P[R ≥ 600] = 1 – P[R ≤ 600]

Standard variate 'z' for R = 600

$$z = \frac{x-\mu}{\sigma}; \quad \frac{600-350}{200} = 1.25$$

From tables, P[Z ≤ 1.25] = 0.3944

P[R ≥ 600] = 1 – P[R ≤ 600]

= 1 – P[Z ≤ 1.25]

= 1 – 0.3944

= 0.6056

z	4	5	6
1.1	0.3729	0.3749	0.377
1.2	0.3925	0.3944	0.3962
1.3	0.4099	0.4115	0.4131

Central limit theorem

- If X_1, X_2,……….. are independent and identically distributed random variables with mean 'μ' and variance 'σ^2', then the sum

  $S_n = X_1+X_2+\ldots\ldots+X_n$

  approaches a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as $n \to \infty$

$$S_n : N\left(n\mu, n\sigma^2\right)$$

iid → independent & identically distributed

Central limit theorem

- For hydrological applications under most general conditions, if X_i's are all independent with $E[x_i] = \mu_i$ and $var(X_i) = \sigma_i^2$, then the sum

  $S_n = X_1 + X_2 + \ldots\ldots + X_n$ as $n \to \infty$

  approaches a normal distribution with

$$E\left[S_n\right] = \sum_{i=1}^{n} \mu_i \;\&$$

$$Var\left[S_n\right] = \sum_{i=1}^{n} \sigma_i^2$$

One condition for this generalised Central Limit Theorem is that each X_i has a negligible effect on the distribution of S_n (Statistical Methods in Hydrology, C.T.Haan, .Affiliated East-West Press Pvt Ltd, 1995, p. 89)

Log-Normal Distribution

- 'X' is said to be log-normally distributed if Y = ln X is normally distributed.
- The probability density function of the log normal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi}x\sigma_x} e^{-(\ln x - \mu_x)^2 / 2\sigma_x^2} \qquad 0 < x < \infty, 0 < \mu_x < \infty, \sigma_x > 0$$

- $\gamma_s = 3C_v + C_v^3$

  where C_v is the coefficient of variation of 'X'

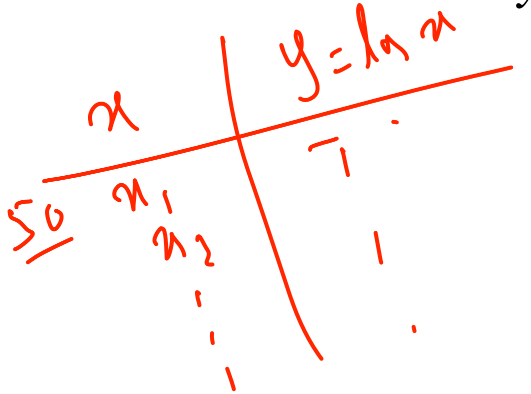
As C_v increases, the skewness, γ_s , increases

Log-Normal Distribution

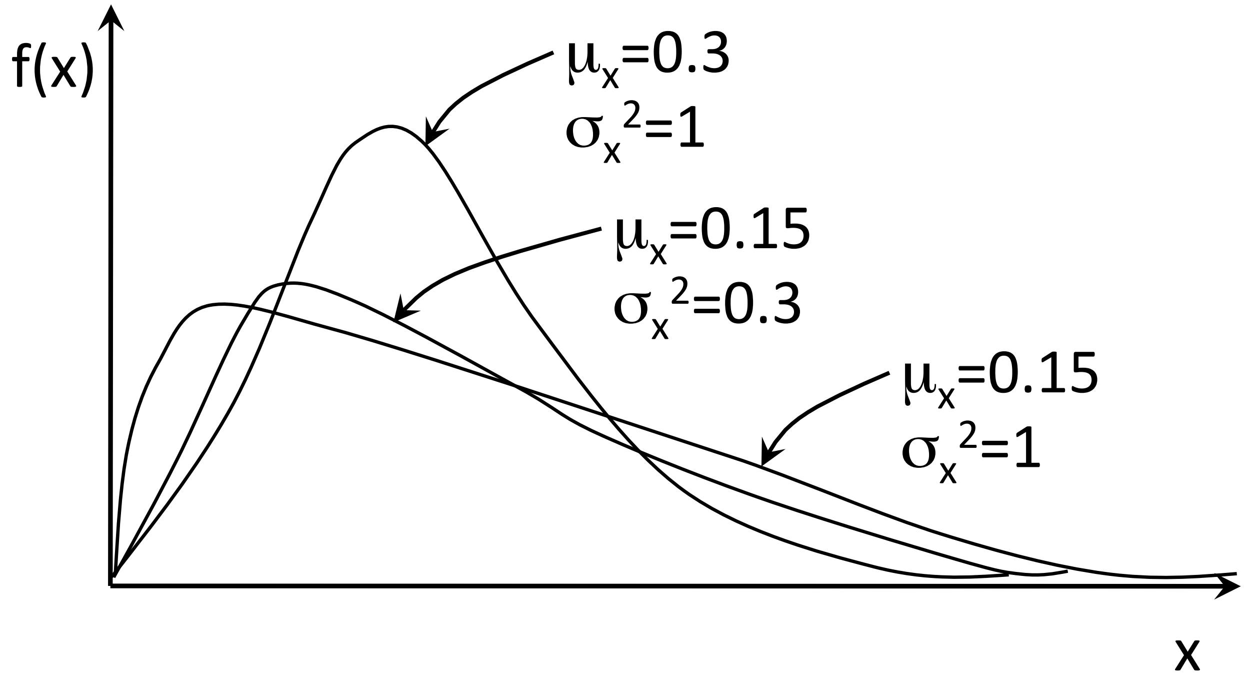
The parameters of Y= ln X may be estimated from

$$\mu_y = \frac{1}{2}\ln\left[\frac{\bar{x}^2}{1+C_v^2}\right]$$

$$\sigma_y^2 = \ln\left[1+C_v^2\right] \qquad where\, C_v = \frac{S_x}{\bar{x}}$$



Log-Normal Distribution



- Positively skewed – with long exponential tail on the right.
- Commonly used for monthly streamflow, monthly/ seasonal precipitation, evapotranspiration, hydraulic conductivity in a porous medium etc .

Example-8

Consider the annual peak runoff in a river - modeled by a lognormal distribution

$\mu = 5.00$ and $\sigma = 0.683$ || $y = \ln x$

Obtain the probability that annual runoff exceeds 300m^3/s

$$P[X > 300] = P[Z > (\ln 300 - 5.00)/0.683]$$
$$= P[Z > 1.03]$$
$$= 1 - P[Z \leq 1.03]$$
$$= 1 - 0.3485$$
$$= 0.6515$$

z	2	3	4
0.9	0.3212	0.3238	0.3264
1	0.3461	0.3485	0.3508
1.1	0.3686	0.3708	0.3729

Example-9

Consider

$\bar{x}$ = 135 Mm^3 ,S = 23.8 Mm^3 and C_v = 0.176

If X follows lognormal distribution

Obtain the P[X $\geq$ 150]

$$\bar{Y} = \frac{1}{2}\ln\left[\frac{\bar{X}^2}{C_v^2+1}\right]$$

$$= \frac{1}{2}\ln\left[\frac{135^2}{0.176^2+1}\right] = 4.89$$

$S_y^2 = \ln(C_v^2+1) = \ln(0.176^2+1) = 0.0305$

$S_y = 0.1747$

Example-9 (contd.)

Y = ln X follows log normal distribution

P[X $\geq$ 150] = P [Y $\geq$ ln150];

ln150 = 5.011

$$z = \frac{y - \overline{y}}{S_y}$$

$$= \frac{5.011 - 4.89}{0.1747}$$

$$= 0.693$$

P[Y $\geq$ ln150] = 1– P[Y $\leq$ ln150]

= 1– P[Z $\leq$ 0.693]

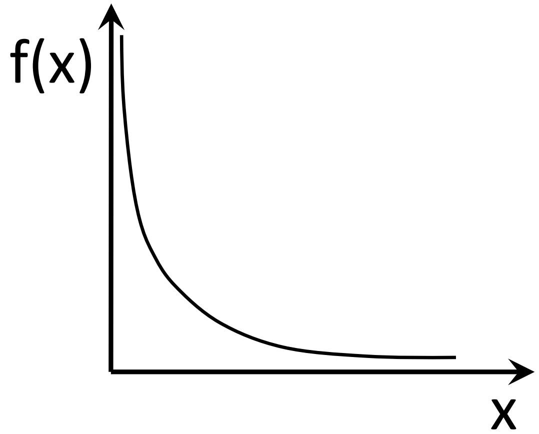
= 1 – (0.5+0.25583)

= 0.24117

Exponential Distribution

- The probability density function of the exponential distribution is given by

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0, \lambda > 0$$



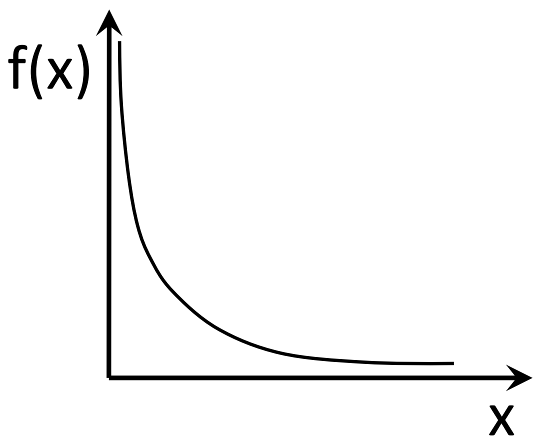
- E[X] = 1/λ
- λ = 1/μ
- Var(X) = $1/\lambda^2$

$$F(x) = \int_0^x f(x)dx = 1 - \lambda e^{-\lambda x} \qquad x > 0, \lambda > 0$$

Exponential Distribution

- $\gamma_s > 0$; positively skewed

- Used for expected time between two critical events (such as floods of a given magnitude), time to failure in hydrologic/water resources systems components



Example-10

The mean time between high intensity rainfall (rainfall intensity above a specified threshold) events occurring during a rainy season is 4 days. Assuming that the mean time follows an exponential distribution.

Obtain the probability of a high intensity rainfall repeating

1. within next 3 to 5 days.
2. within next 2 days

Mean time (μ) = 4

$\lambda = 1/\mu = 1/4$

Example-10 (contd.)

1. P[3 $\leq$ X $\leq$ 5] = F(5) - F(3)

   F(5) = $1-\frac{1}{4}e^{-5/4}$

   = 0.7135

   F(3) = $1-\frac{1}{4}e^{-3/4}$

   = 0.5276

   P[3 $\leq$ X $\leq$ 5] = 0.7135 – 0.5276 = 0.1859

2. P[X $\leq$ 2] = $1-\frac{1}{4}e^{-2/4}$ = 0.3935

$F(x) = 1 - e^{-\lambda x}$

Gamma Distribution

- The probability density function of the Gamma distribution is given by

$$f(x) = \frac{\lambda^{n} x^{\eta-1} e^{-\lambda x}}{\Gamma(\eta)} \qquad x, \lambda, \eta > 0$$

Two parameters λ & η

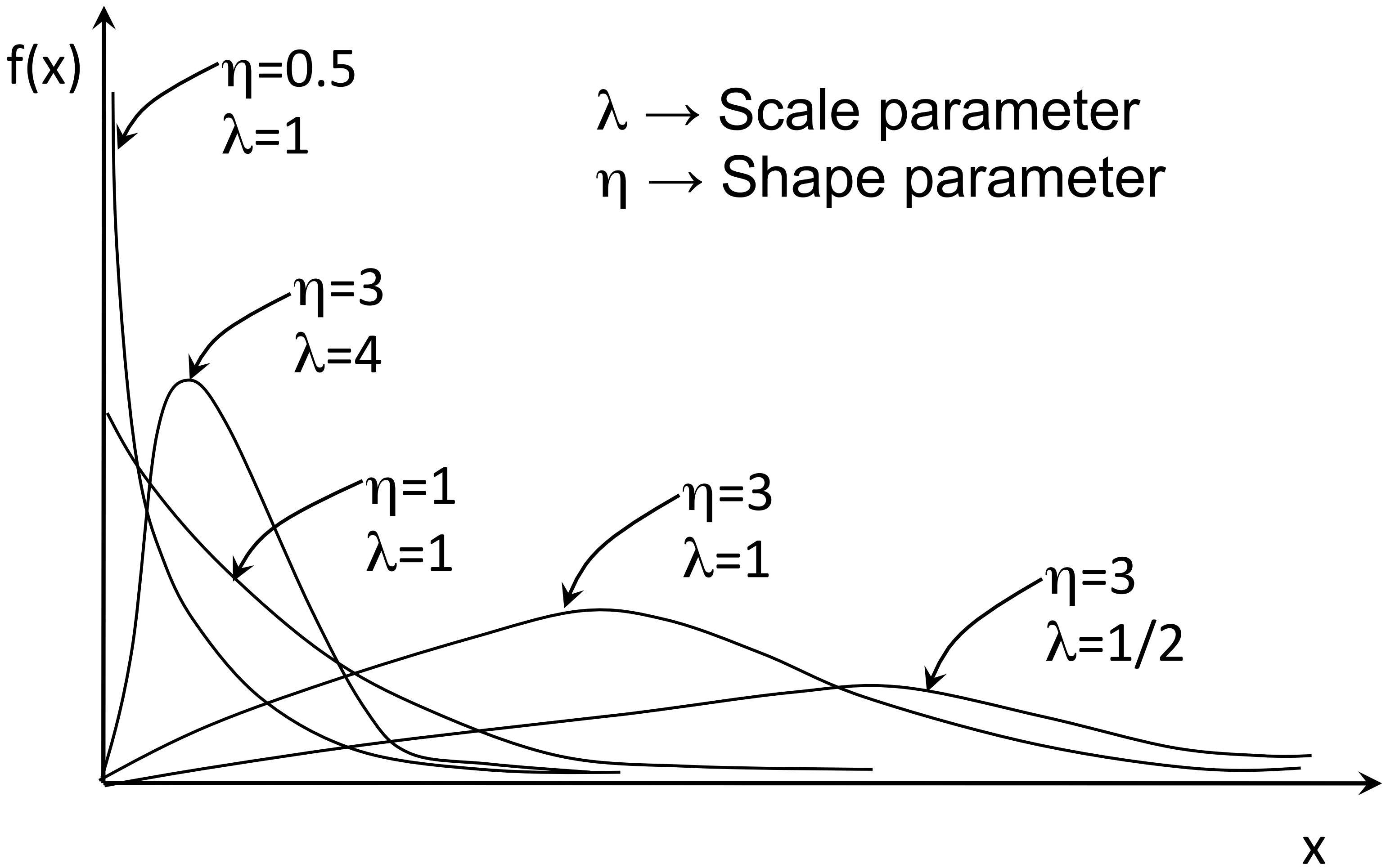
- $\Gamma(\eta)$ is a gamma function
- $\Gamma(\eta) = (\eta-1)!$, $\eta = 1,2,\ldots$ $\Gamma(1) = \Gamma(2) = 1$; $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(\eta+1) = \eta\sqrt{\eta}$ $\eta > 0$
- $\Gamma(\eta) = \int_0^{\infty} t^{\eta-1} e^{-t} dt$ $\eta > 0$
- Gamma distribution is in fact a family of distributions

Gamma Distribution

- Exponential distribution is a special case of gamma distribution with η=1
- $\lambda \rightarrow$ Scale parameter
- $\eta \rightarrow$ Shape parameter
- Mean = η/λ
- Variance = η/λ^2 $\rightarrow$ $\sigma = \sqrt{\eta}/\lambda$
- Skewness coefficient $\gamma = 2/\sqrt{\eta}$
- As γ decreases, η increases
- Cdf is given by

$$F(x) = 1 - e^{-\lambda x} \sum_{j=0}^{\eta-1} (\lambda x)^j / j! \qquad x, \lambda, \eta > 0$$

Gamma Distribution



Gamma Distribution

- If 'X' and 'Y' are two independent gamma rvs having parameters η_1, λ and η_2, λ respectively then U=X+Y is a gamma rv with parameters $\eta=\eta_1+\eta_2$ and λ
- This property can be extended to sum of 'n' number of independent gamma rvs.
- Gamma distribution is generally used for daily/ monthly/annual rainfall data
- Also used for annual runoff data

Example-11

During the month 1, the mean and standard deviation of the monthly rainfall are 7.5 and 4.33 cm resp. Assume monthly rainfall data can be approximated by Gamma distribution

1. Obtain the probability of receiving more than 3cm rain during month 1.

Given, μ= 7.5, σ= 4.33

Estimate the parameters λ, η

$\mu = \eta/\lambda \rightarrow \quad 7.5 = \eta/\lambda$

$\lambda = \eta/7.5$

Example-11 (contd.)

$\sigma = \sqrt{\eta}/\lambda \quad \rightarrow \quad 4.33 = \sqrt{\eta}/\lambda$

$\sqrt{\eta}/\eta = 4.33/7.5$

$\eta = 3$

$\lambda = 4$

$$f(x) = \frac{\lambda^{n} x^{\eta-1} e^{-\lambda x}}{\Gamma(\eta)} \qquad x, \lambda, \eta > 0$$

$$= \frac{4^{3} x^{3-1} e^{-4x}}{\Gamma(3)} \qquad \Gamma(3) = (3-1)! = 2!$$

$$= 32 x^{2} e^{-4x}$$

Example-11 (contd.)

P[X $\geq$ 3] = 1 – P[X $\leq$ 3]

$$= 1 - \int_0^3 32x^2 e^{-4x} dx$$

$$= 1 - \left(1 - \frac{85}{e^{12}}\right)$$

$$= 0.0005$$

Example-11 (contd.)

During the month 2, the mean and standard deviation of the monthly rainfall are 30 and 8.6 cm respectively.

1. Obtain the probability of receiving more than 3cm rain during month 2.
2. Obtain the probability of receiving more than 3cm rain during the two month period assuming that rainfalls during the two months are independent.

Given, μ= 30, σ= 8.66

The parameters λ, η are estimated.

$\mu = \eta/\lambda \rightarrow \quad 30 = \eta/\lambda$

$\lambda = \eta/30$

Example-11 (contd.)

$\sigma = \sqrt{\eta}/\lambda \;\rightarrow\; 8.66 = \sqrt{\eta}/\lambda$

$\sqrt{\eta}/\eta = 8.66/30$

$\eta = 12$

$\lambda = 4$

$$f(x) = \frac{\lambda^n x^{\eta-1} e^{-\lambda x}}{\Gamma(\eta)} \qquad x, \lambda, \eta > 0$$

$$= \frac{4^{12} x^{12-1} e^{-4x}}{\Gamma(12)} \qquad \Gamma(12) = (12-1)! = 11!$$

$$= 0.42 x^{11} e^{-4x}$$

Example-11 (contd.)

1. P[X ≥ 3] = 1 – P[X ≤ 3]

$$= 1 - \int_0^3 0.42 x^{11} e^{-4x} dx$$

$$= 1 - \left(0.993 - \frac{75073}{e^{12}} \right)$$

$$= 0.4683$$

2. Probability of receiving more than 3cm rain during the two month period:

   Since λ value is same for both the months and the rainfalls during the two months are independent,

Example-11 (contd.)

Then the combined distribution will have the parameters η , λ as

η = 3+12 = 15

λ = 4

Therefore $f(x) = \dfrac{\lambda^{n} x^{\eta-1} e^{-\lambda x}}{\Gamma(\eta)}$ $\qquad x, \lambda, \eta > 0$

$$= \frac{4^{15} x^{15-1} e^{-4x}}{\Gamma(15)}$$

$$= 0.0123 x^{14} e^{-4x}$$

Example-11 (contd.)

P[X ≥ 3] = 1 – P[X ≤ 3]

$$= 1 - \int_0^3 0.0123 x^{14} e^{-4x} dx$$

$$= 1 - \left(0.99865 - \frac{125481}{e^{12}} \right)$$

$$= 0.7723$$

The values of cumulative gamma distribution can be evaluated using tables with $\chi^2 = 2\lambda x$ and $\nu = 2\eta$

Extreme value Distributions

- Interest exists in extreme events. For example,
  - Annual peak discharge of a stream
  - Minimum daily flows (drought analysis)
- The extreme value of a set of random variables is also a random variable
- The probability of this extreme value depends on the sample size and parent distribution from which the sample was obtained
- Consider a random sample of size 'n' consisting of x_1, x_2,…….. x_n. Let 'Y' be the largest of the sample values.

Extreme value Distributions

- Let $F_y(y)$ be the prob($Y \leq y$) and $F_{x_i}(x)$ be the prob($X_i \leq x$)
- Let $f_y(y)$ and $f_{x_i}(x)$ be the corresponding pdfs.
- $F_y(y)$ = prob($Y \leq y$) = F(all of the x's $\leq$ y). If the x's are independently and identically distributed,

$$F_y(y) = F_{x_1}(y)\, F_{x_2}(y) \ldots\ldots\ F_{x_n}(y) = [F_x(y)]^n$$
$$f_y(y) = d\, F_y(y)\, /dy = n[F_x(y)]^{n-1}\, f_x(y)$$

- Therefore the probability distribution of the maximum of 'n' independently and identically distributed rvs depends on the sample size 'n' and parent distribution $F_x(x)$ of the sample

Extreme value Distributions

- Frequently the parent distribution from which the extreme is an observation is not known and cannot be determined.

- If the sample size is large, certain general asymptotic results that depend on limited assumptions concerning the parent distribution can be used to find the distribution of extreme values

Extreme value Distributions

- Three types of asymptotic distributions are developed
  - Type-I – parent distribution unbounded in direction of the desired extreme and all moments of the distribution exists. – Normal, log-normal, exponential
  - Type-II – parent distribution unbounded in direction of the desired extreme and all moments of the distribution do not exist. – Cauchy distribution
  - Type-III – parent distribution bounded in direction of the desired extreme. – Beta, Gamma, log-normal, exponential

Extreme value Type-I Distribution

- Referred as Gumbel's distribution
- Pdf is given by

$$f(x) = \exp\left\{ m(x - \beta)/\alpha - \exp\left[m(x - \beta)/\alpha \right] \right\} / \alpha$$

$$-\infty < x < \infty ; -\infty < \beta < \infty ; \alpha > 0$$

- '–' applies for maximum values and '+' for minimum values
- α and β are scale and location parameters
- β = mode of distribution
- Mean E[x] = α + 0.577 β (Maximum)

  = α - 0.577 β (Minimum)

Extreme value Type-I Distribution

- Variance Var(x) = 1.645 α^2
- Skewness coefficient γ = 1.1396 (maximum)

  = -1.1396 (minimum)
- Y = (X – β)/ $\alpha \rightarrow$ transformation
- Pdf becomes

$$f(y) = \exp\left\{\mathrm{m}y - \exp\left[\mathrm{m}y\right]\right\} \qquad -\infty < y < \infty$$

- Cdf – $F(y) = \exp\left\{-\exp\left(-y\right)\right\}$ (maximum)

$$= 1 - \exp\left\{-\exp\left(y\right)\right\} \quad \text{(minimum)}$$

$F_{min}(y) = 1 - F_{max}(-y)$

Extreme value Type-I Distribution

- The parameters α and β can be expressed in terms of mean and variance as (Lowery and Nash (1980))

$$\hat{\alpha} = \frac{\sigma}{1.283}$$

and $\hat{\beta} = \mu - 0.45\sigma \rightarrow$ (maximum)

$= \mu + 0.45\sigma \rightarrow$ (minimum)

Example on Gumbell's distribution

Consider the annual peak flood of a stream follows Gumbell's distribution with

$\mu = 9m^3/s$ and $\sigma = 4m^3/s$,

1. Obtain the probability that annual peak flood exceeds $18m^3/s$ and
2. Obtain the probability that it will be utmost $15m^3/s$

Example on Gumbell's distribution (contd.)

1. To obtain P[X $\geq$ 18000],

the parameters α and β are obtained initially

$\alpha = \sigma/1.283$

$= 4/1.283$

$= 3.118$

$\beta = \mu - 0.45\,\sigma$

$= 9-0.45*4$

$= 7.2$

$$P[X \geq 18] = 1 - P[X \leq 18]$$
$$= 1 - F(18)$$
$$= 1 - \exp\{-\exp(-y)\}$$

Example on Gumbell's distribution (contd.)

$$y = (x - \beta)/\alpha$$
$$= (18-7.2)/3.118$$
$$= 3.464$$

$$P[X \geq 18] = 1 - \exp\{-\exp(-y)\}$$
$$= 1 - \exp\{-\exp(-3.464)\}$$
$$= 1 - 0.9692$$
$$= 0.0308$$

Example on Gumbell's distribution (contd.)

2. To obtain $P[X \leq 15]$,

   $y = (x - \beta)/ \alpha$
   $= (15-7.2)/3.118$
   $= 2.502$

   $F(y) = \exp\{-\exp(-y)\}$
   $= \exp\{-\exp(-2.502)\}$
   $= 0.9213$

   $P[X \leq 15] = 0.9213$

Example-2

Consider the annual peak flood of a stream exceeds 2000m^3/s with a probability of 0.02 and exceeds 2250m^3/s with a probability of 0.01

1. Obtain the probability that annual peak flood exceeds 2500m^3/s

Initially the parameters α and β are obtained from the given data as follows

$P[X \geq 2000] = 0.02$

$P[X \leq 2000] = 0.98$

$\exp\{-\exp(-y)\} = 0.98$

Example-2 (contd.)

y = -ln{-ln(0.98)}

y = 3.902

$$\frac{2000-\beta}{\alpha} = 3.902 \quad \rightarrow \text{ Equation-1}$$

P[X $\geq$ 2250] = 0.01

P[X $\leq$ 2250] = 0.99

exp{-exp(-y)} = 0.99

y = -ln{-ln(0.99)}

y = 4.6

$$\frac{2250-\beta}{\alpha} = 4.6 \quad \rightarrow \text{ Equation-2}$$

Example-2 (contd.)

Solving both the equations,

α = 358 and β = 603

Now $P[X \geq 2500] = 1 - P[X \leq 2500]$

$= 1 - \exp\{-\exp(-y)\}$

$y = (x - \beta)/\alpha$

$= (2500-603)/358$

$= 5.299$

$P[X \geq 2500] = 1 - \exp\{-\exp(-5.299)\}$

$= 1 - 0.995$

$= 0.005$

Extreme value Type-III Distribution

- Referred as Weibull distribution
- Pdf is given by

$$f(x) = \alpha x^{\alpha-1} \beta^{-\alpha} \exp\left\{-\left(x/\beta\right)^{\alpha}\right\} \qquad x \geq 0; \alpha, \beta > 0$$

- Cdf is given by

$$F(x) = 1 - \exp\left\{-\left(x/\beta\right)^{\alpha}\right\} \qquad x \geq 0; \alpha, \beta > 0$$

- Mean and variance of the distribution are

  $E[X] = \beta\, \Gamma(1+1/\alpha)$

  $Var(X) = \beta^2 \{\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\}$

Extreme value Type-III Distribution

- The Weibull probability density function can range from a reverse-J with <1, to an exponential with =1 and to a nearly symmetrical distribution as increases
- If the lower bound on the parent distribution is not zero, a displacement must be added to the type III extreme distribution for minimums, then pdf is

$$f(x) = \alpha\left(x-\varepsilon\right)^{\alpha-1}\left(\beta-\varepsilon\right)^{-\alpha}\exp\left\{-\left[\left(x-\varepsilon\right)/\left(\beta-\varepsilon\right)\right]^{\alpha}\right\}$$

- known as 3-parameter Weibull distribution
- Cdf is

$$F(x) = 1-\exp\left\{-\left[\left(x-\varepsilon\right)/\left(\beta-\varepsilon\right)\right]^{\alpha}\right\}$$

Extreme value Type-III Distribution

- $Y = \{(X - \varepsilon)/ (\beta - \varepsilon)\}^{\alpha}$ → transformation
- Mean and variance of the 3-parameter Weibull distribution are

$$E[X] = \varepsilon + (\beta - \varepsilon)\, \Gamma(1+1/\alpha)$$

$$Var(X) = (\beta - \varepsilon)^2 \{\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)\}$$