INDIAN INSTITUTE OF SCIENCE

STOCHASTIC HYDROLOGY

Lecture -6

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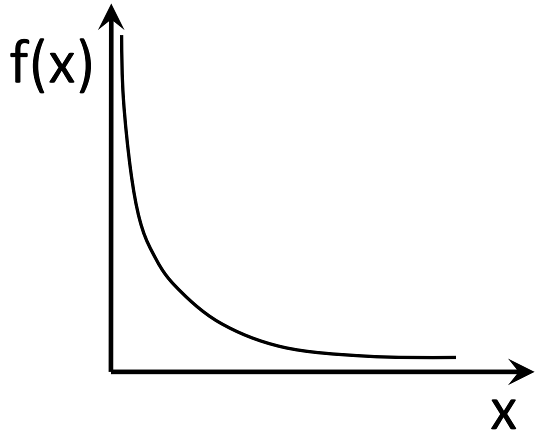
Summary of the previous lecture

- Normal distribution
  - Central limit theorem
- Log-normal distribution
- Exponential Distribution

Exponential Distribution

- The probability density function of the exponential distribution is given by

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0, \lambda > 0$$



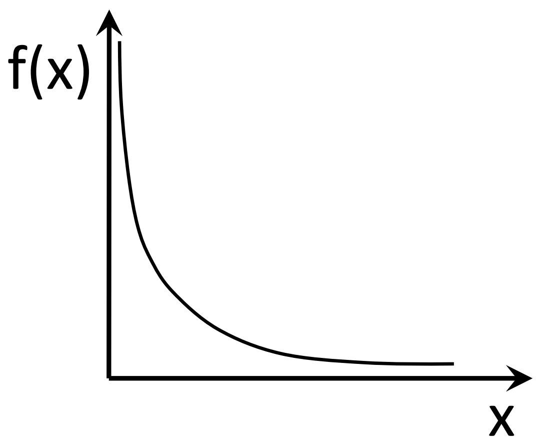
- E[X] = 1/λ
- λ = 1/μ
- Var(X) = $1/\lambda^2$

$$F(x) = \int_0^x f(x)dx = 1 - e^{-\lambda x} \qquad x > 0, \lambda > 0$$

Exponential Distribution

- $\gamma_s > 0$; positively skewed

- Used for expected time between critical events (such as floods of a given magnitude), time to failure in hydrologic/water resources systems components



Example-1

The mean time between high intensity rainfall (rainfall intensity above a specified threshold) events occurring during a rainy season is 4 days. Assuming that the mean time follows an exponential distribution.

Obtain the probability of a high intensity rainfall repeating

1. within next 3 to 5 days.
2. within next 2 days

Mean time (μ) = 4

$\lambda = 1/\mu = 1/4$

x

(1) $P[3 \le x \le 5]$

(2) $P[x \le 2]$

Example-1 (contd.)

1. $P[3 \leq X \leq 5] = F(5) - F(3)$

   $F(5) = 1 - e^{-5/4}$

   $= 0.7135$

   $F(3) = 1 - e^{-3/4}$

   $= 0.5276$

   $P[3 \leq X \leq 5] = 0.7135 - 0.5276 = 0.1859$

2. $P[X \leq 2] = 1 - e^{-2/4} = 0.3935$

$F(x) = 1 - e^{-\lambda x}$

$\lambda = \frac{1}{4}$

Gamma Distribution

- The probability density function of the Gamma distribution is given by

$$f(x) = \frac{\lambda^{\eta} x^{\eta-1} e^{-\lambda x}}{\Gamma(\eta)} \qquad x, \lambda, \eta > 0$$

Two parameters λ & η

- $\Gamma(\eta)$ is a gamma function
- $\Gamma(\eta) = (\eta-1)!$, $\eta = 1,2,\ldots$ $\quad \Gamma(1) = \Gamma(2) = 1$; $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(\eta+1) = \eta\sqrt{\eta} \quad \eta > 0$
- $\Gamma(\eta) = \int_0^{\infty} t^{\eta-1} e^{-t}\, dt \quad \eta > 0$

$\eta = 1$

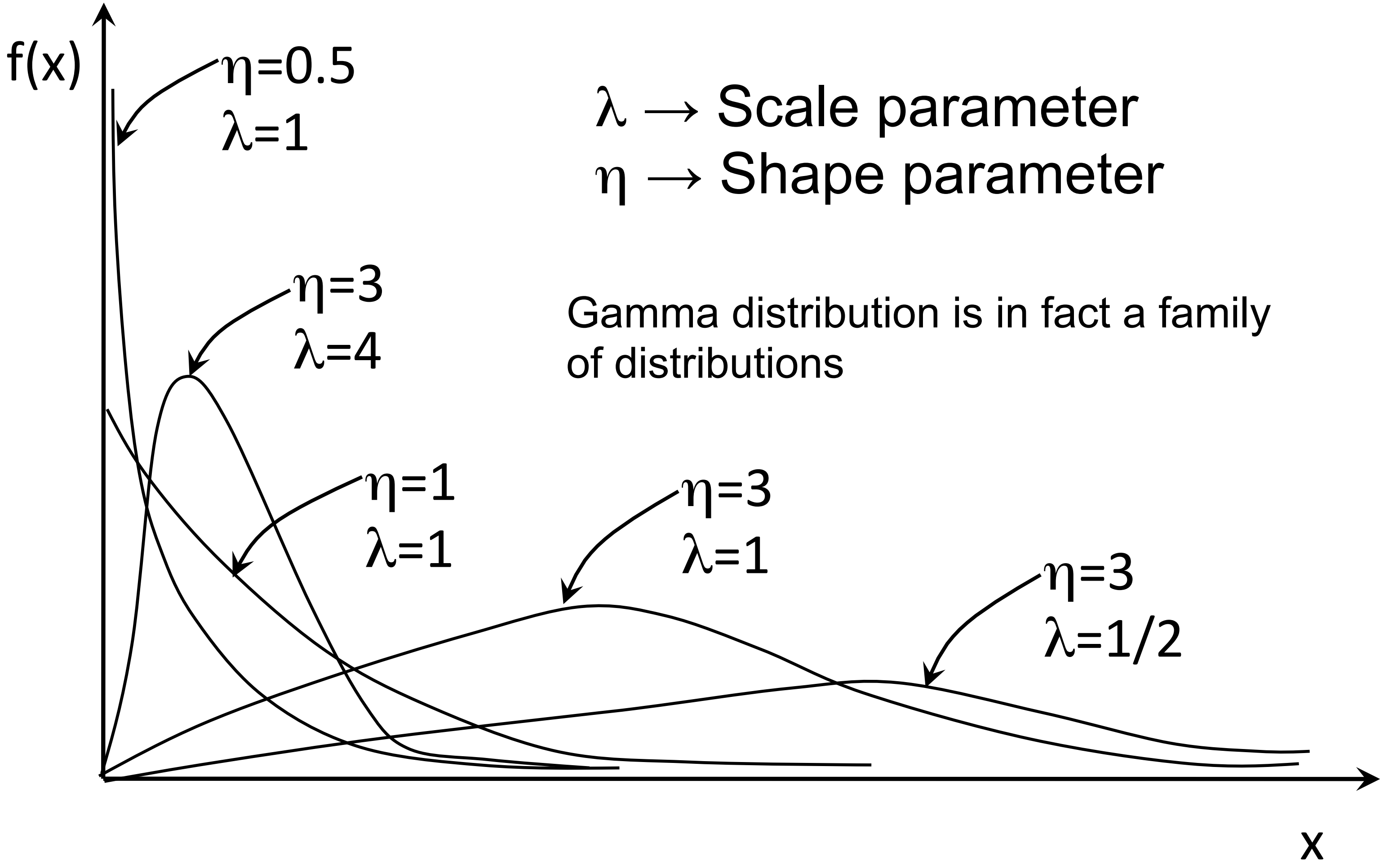
$f(x) = \lambda e^{-\lambda x}$ $\quad x > 0$, $\lambda > 0$

Exponential

Gamma Distribution

- Exponential distribution is a special case of gamma distribution with η=1
- $\lambda \rightarrow$ Scale parameter
- $\eta \rightarrow$ Shape parameter
- Mean = η/λ
- Variance = $\eta/\lambda^2 \quad \rightarrow \quad \sigma = \sqrt{\eta}\big/\lambda$
- Skewness coefficient γ = $2\big/\sqrt{\eta}$
- Gamma distribution is generally used for daily/ monthly/annual rainfall data

Gamma Distribution



λ → Scale parameter
η → Shape parameter

Gamma distribution is in fact a family of distributions

Gamma Distribution

- If 'X' and 'Y' are two independent gamma rvs having parameters η_1, λ and η_2, λ respectively then U=X+Y is a gamma rv with parameters $\eta=\eta_1+\eta_2$ and λ

- This property can be extended to sum of 'n' number of independent gamma rvs.

$\underbrace{X_1 + X_2 + \cdots + X_n}_{\sim \text{Gamma}(\sum \eta_i, \lambda)}$

$X_i \to \text{Gamma}(\eta_i, \lambda)$

Example-2

During the month 1, the mean and standard deviation of the monthly rainfall are 0.5 and 0.3535cm resp. Assume monthly rainfall data can be approximated by Gamma distribution

1.Obtain the probability of receiving more than 1cm rain during month 1.

Given, $\mu_1 = 0.5$, $\sigma_1 = 0.3535$

Estimate the parameters λ_1 and η_1

$$\mu_1 = \eta_1/\lambda_1 \rightarrow \quad 0.5 = \eta_1/\lambda_1$$

$$\lambda_1 = \eta_1/0.5$$

Example-2 (contd.)

$$\sigma_1 = \sqrt{\eta_1}/\lambda_1 \quad \rightarrow \quad 0.3535 = \sqrt{\eta_1}/\lambda_1 = \sqrt{\eta_1} \times 0.5/\eta_1$$

$$\eta_1^{1/2} = 1.4144$$

$\eta_1 = 2$

$\lambda_1 = \eta_1/0.5 = 4$

$$f_{X_1}(x) = \frac{\lambda_1^{\eta_1} x^{\eta_1 - 1} e^{-\lambda_1 x}}{\Gamma(\eta_1)} \qquad x, \lambda_1, \eta_1 > 0$$

$$= \frac{4^2 x^{2-1} e^{-4x}}{\Gamma(2)} \qquad \Gamma(2) = 1$$

$$= 16xe^{-4x}$$

Example-2 (contd.)

$P[X \geq 1] = 1 - P[X \leq 1]$

$$= 1 - \int_0^1 16xe^{-4x}\,dx$$

$$= 1 - \left(1 - \frac{5}{e^4}\right)$$

$$= 0.092$$

Example-2 (contd.)

During the month 2, the mean and standard deviation of the monthly rainfall are 1 and 0.5cm respectively.

1.Obtain the probability of receiving more than 1cm rain during month 2.

2.Obtain the probability of receiving more than 1cm rain during the two month period assuming that rainfalls during the two months are independent.

Given, $\mu_2 = 1$, $\sigma_2 = 0.5$

The parameters λ_2 and η_2 are estimated.

$$\mu_2 = \eta_2 / \lambda_2 \rightarrow \quad 1 = \eta_2 / \lambda_2$$

$$\lambda_2 = \eta_2$$

Example-2 (contd.)

$\sigma_2 = \sqrt{\eta_2}\big/\lambda_2 \;\rightarrow\; 0.5 = \sqrt{\eta_2}\big/\lambda_2 = \sqrt{\eta_2}\big/\eta_2$

$\eta_2^{1/2} = 2$

$\eta_2 = 4$

$\lambda_2 = \eta_2 = 4$

$$f_{X_2}(x) = \frac{\lambda_2^{\eta_2} x^{\eta_2 - 1} e^{-\lambda_2 x}}{\Gamma(\eta_2)} \qquad x, \lambda_2, \eta_2 > 0$$

$$= \frac{4^4 x^{4-1} e^{-4x}}{\Gamma(4)} \qquad \Gamma(4) = (4-1)! = 3! = 6$$

$$= 42.67 x^3 e^{-4x}$$

Example-2 (contd.)

1. P[X ≥ 1] = 1 – P[X ≤ 1]

$$= 1 - \int_0^1 42.67 x^3 e^{-4x} dx$$

$$= 1 - \left(1 - \frac{23.67}{e^4}\right)$$

$$= 0.4335$$

2. Probability of receiving more than 1cm rain during the two month period:

Since $\lambda_1 = \lambda_2$ and the rainfalls during the two months are independent,

Example-2 (contd.)

Since $\lambda_1 = \lambda_2$ and the rainfalls during the two months are independent, the distribution of (X_1+X_2) will have the parameters η , λ as

$\eta = \eta_1 + \eta_2 = 2+4 = 6$
$\lambda = 4$

Therefore $f_{X_1+X_2}(x) = \dfrac{\lambda^{\eta} x^{\eta-1} e^{-\lambda x}}{\Gamma(\eta)}$ $\quad x, \lambda, \eta > 0$

$$= \frac{4^6 x^{6-1} e^{-4x}}{\Gamma(6)}$$

$\Gamma(6) = (6-1)! = 5! = 120$

$$= 34.13 x^5 e^{-4x}$$

Example-2 (contd.)

$P[X_1+X_2 \geq 1] = 1 - P[X_1+X_2 \leq 1]$

$$= 1 - \int_0^1 34.13 x^5 e^{-4x} dx$$

$$= 1 - \left(0.9999 - \frac{42.862}{e^{12}} \right)$$

$$= 0.785$$

- The values of cumulative gamma distribution can be evaluated using tables with $\chi^2 = 2\lambda x$ and $\nu = 2\eta$

Extreme Value Distributions

- Extreme events:
  - Peak flood discharge in a stream
  - Maximum rainfall intensity
  - Minimum flow
- The extreme value of a set of random variables is also a random variable
- The probability of this extreme value depends on the sample size and parent distribution from which the sample is obtained

Extreme Value Distributions

- A random sample of size 'n' consisting of x_1, x_2,…….. x_n. Let 'Y' be the largest of the sample values.
- $F_Y(y) = P[Y \leq y]$ and $F_{X_i}(x) = P[X_i \leq x]$
- $f_Y(y)$ and $f_{X_i}(x)$ are corresponding pdfs.
- $F_y(y) = P[Y \leq y] = P[\text{all of the x's} \leq y]$.

If the x's are independently and identically distributed,

$$F_Y(y) = F_{X_1}(y)\, F_{X_2}(y) \ldots\ldots F_{X_n}(y) = [F_X(y)]^n$$

$$\begin{aligned} f_Y(y) &= d\,[F_Y(y)]\,/dy \\ &= d\,[F_X(y)]^n\,/dy \\ &= n[F_X(y)]^{n-1}\, d\,[F_X(y)]\,/dy \\ &= n[F_X(y)]^{n-1}\, f_X(y) \end{aligned}$$

Extreme Value Distributions

- Parent distribution from which the extreme is an observation may not be known and often cannot be determined.
- Three types of Extreme Value distributions are developed based on limited assumptions concerning parent distribution
  - Type-I – parent distribution unbounded in direction of the desired extreme and all moments of the distribution exist. – Normal, log-normal, exponential
  - Type-II – parent distribution unbounded in direction of the desired extreme and all moments of the distribution do not exist. – Cauchy distribution
  - Type-III – parent distribution bounded in direction of the desired extreme. – Beta, Gamma, log-normal, exponential

Extreme Value Type-I Distribution (Gumbel's Extreme Value Distribution)

- Referred as Gumbel's distribution
- Pdf is given by

$$f(x) = \exp\left\{\mp(x-\beta)/\alpha - \exp\left[\mp(x-\beta)/\alpha\right]\right\}/\alpha$$

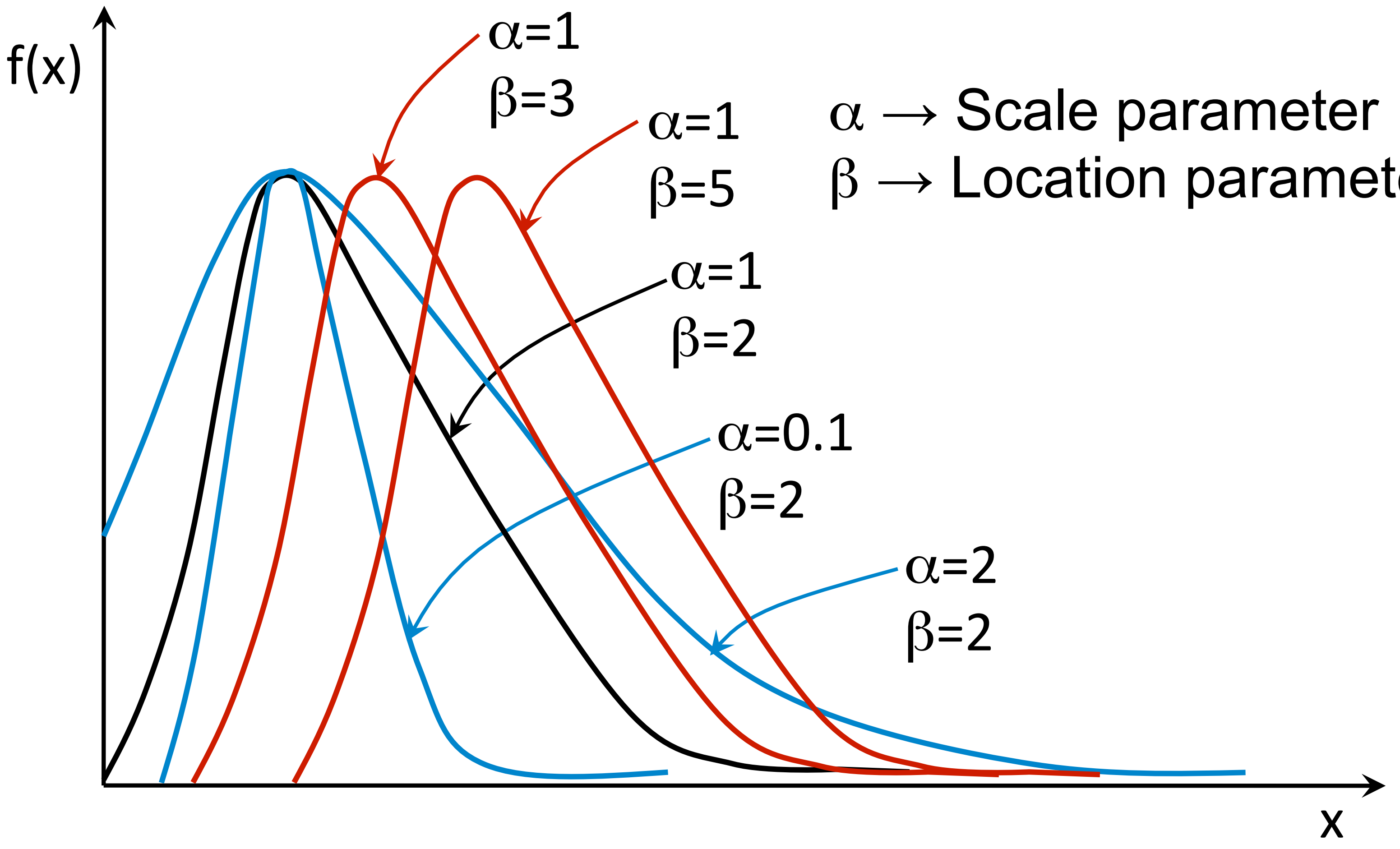
$$-\infty < x < \infty; -\infty < \beta < \infty; \alpha > 0$$

- '–' applies for maximum values and '+' for minimum values
- α and β are scale and location parameters
- β = mode of distribution
- Mean, E[x] = α + 0.577 β (Maximum)

  = α - 0.577 β (Minimum)

+

Extreme Value Type-I Distribution (Gumbel's Extreme Value Distribution)



α → Scale parameter

β → Location parameter

Extreme Value Type-I Distribution (Gumbel's Extreme Value Distribution)

- Variance Var(x) = 1.645 α^2
- Skewness coefficient γ = 1.1396 (maximum)
  = -1.1396 (minimum)
- Y = (X – β)/ α $\rightarrow$ transformation
- Pdf becomes

$$f(y) = \exp\left\{\mathrm{m}y - \exp\left[\mathrm{m}y\right]\right\} \qquad -\infty < y < \infty$$

$F(y) = e^{-e^{-y}}$

Double Exponential

- Cdf – $F(y) = \exp\left\{-\exp\left(-y\right)\right\}$ (maximum)

$$= 1 - \exp\left\{-\exp\left(y\right)\right\} \quad \text{(minimum)}$$

$F_{min}(y) = 1 - F_{max}(-y)$

Extreme Value Type-I Distribution (Gumbel's Extreme Value Distribution)

- The parameters α and β can be expressed in terms of mean and variance as

$$\hat{\alpha} = \frac{\sigma}{1.283}$$

and

$$\hat{\beta} = \mu - 0.45\sigma \qquad \rightarrow \text{(maximum)}$$

$$= \mu + 0.45\sigma \qquad \rightarrow \text{(minimum)}$$

Example-3

Consider the annual peak flood of a stream follows Gumbell's distribution with

μ= 9m^3/s and σ = 4m^3/s,

1. Obtain the probability that annual peak flood exceeds 18m^3/s and
2. Obtain the probability that it will be utmost 15m^3/s

Example-3 (contd.)

1. To obtain $P[X \geq 18\cancel{000}]$,

the parameters α and β are obtained initially

$\alpha = \sigma/1.283$

$= 4/1.283$

$= 3.118$

$\beta = \mu - 0.45\,\sigma$

$= 9-0.45*4$

$= 7.2$

$P[X \geq 18] = 1 - P[X \leq 18]$

$= 1 - F(18)$

$= 1 - \exp\{-\exp(-y)\}$

Example-3 (contd.)

$y = (x - \beta)/\ \alpha$

$= (18-7.2)/3.118$

$= 3.464$

$P[X \geq 18] = 1 - \exp\{-\exp(-y)\}$

$= 1 - \exp\{-\exp(-3.464)\}$

$= 1 - 0.9692$

$= 0.0308$

Example-3 (contd.)

2. To obtain $P[X \le 15]$,

$y = (x - \beta)/ \alpha$

$= (15-7.2)/3.118$

$= 2.502$

$F(y) = \exp\{-\exp(-y)\}$

$= \exp\{-\exp(-2.502)\}$

$= 0.9213$

$P[X \le 15] = 0.9213$