INDIAN INSTITUTE OF SCIENCE

STOCHASTIC HYDROLOGY

Lecture -18

Course Instructor : Prof. P. P. MUJUMDAR

Department of Civil Engg., IISc.

Summary of the previous lecture

- Model selection
  - Mean square error
- Model testing/validation
  - Significance of residual mean
  - Significance of periodicities
    - Cumulative periodogram test or Bartlett's test
  - White noise test
    - Whittle's test
    - Portmanteau test

ARIMA Models

Data Generation:

Consider AR(1) model,

$$X_t = \phi_1 X_{t-1} + e_t$$

e.g., $\phi_1 = 0.5$: AR(1) model is

$X_t = 0.5X_{t-1} + e_t$ → zero mean; uncorrelated

ARIMA Models

$X_1 = 3.0$ (assumed)

$X_2 = 0.5*3.0 + 0.335$
$= 1.835$

$X_3 = 0.5*1.835 + 1.226$
$= 2.14$

And so on…

ARIMA Models

Consider ARMA (1, 1) model,

$$X_t = \phi_1 X_{t-1} + \theta_1 e_{t-1} + e_t$$

e.g., $\phi_1 = 0.5$, $\theta_1 = 0.4$: ARMA(1, 1) model is written as

$$X_t = 0.5X_{t-1} + 0.4e_{t-1} + e_t$$

ARIMA Models

Say $X_1 = 3.0$

$X_2 = 0.5*3.0 + 0.4*0 + 0.667$
$= 2.167$

$X_3 = 0.5*2.167 + 0.4*0.667 + 1.04$
$= 2.39$

$X_4 = 0.5*2.39 + 0.4*1.04 + 2.156$
$= 3.767$ and so on...

ARIMA Models

Data Forecasting:

Consider AR(1) model,

$X_t = \phi_1 X_{t-1} + e_t$

Expected value is considered.

$$E\left[X_t\right] = \phi_1 E\left[X_{t-1}\right] + E\left[e_t\right]$$

$$\hat{X}_t = \phi_1 X_{t-1}$$

Expected value of e_t is zero

ARIMA Models

Consider ARMA(1, 1) model,

$$X_t = \phi_1 X_{t-1} + \theta_1 e_{t-1} + e_t$$

$$E[X_t] = \phi_1 X_{t-1} + \theta_1 e_{t-1} + 0$$

(e_{t-1}:) Error in forecast in the previous period

e.g., $\phi_1 = 0.5$, $\theta_1 = 0.4$: Forecast model is written as

$$X_t = 0.5X_{t-1} + 0.4e_{t-1}$$

ARIMA Models

Say $X_1 = 3.0$

Initial error assumed to be zero

$$\hat{X}_2 = 0.5 \times 3.0 + 0.4 \times 0$$

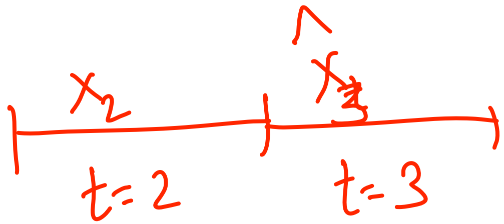
$$= 1.5$$

Forecasted

Observed Value

$X_2 = 2.8$

Error $e_2 = 2.8 - 1.5 = 1.3$



$$\hat{X}_3 = 0.5 \times 2.8 + 0.4 \times 1.3$$

$$= 1.92$$

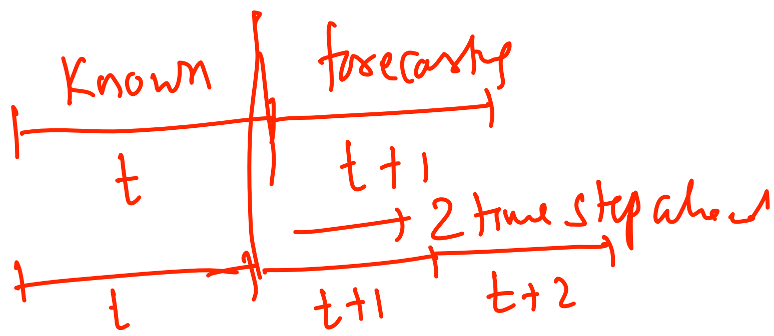
Actual value to be used

ARIMA Models

$X_3 = 1.8$

Error $e_3 = 1.8 – 1.92 = -0.12$

$$\hat{X}_4 = 0.5 \times 1.8 + 0.4 \times (-0.12)$$
$$= 0.852$$



and so on.

CASE STUDIES

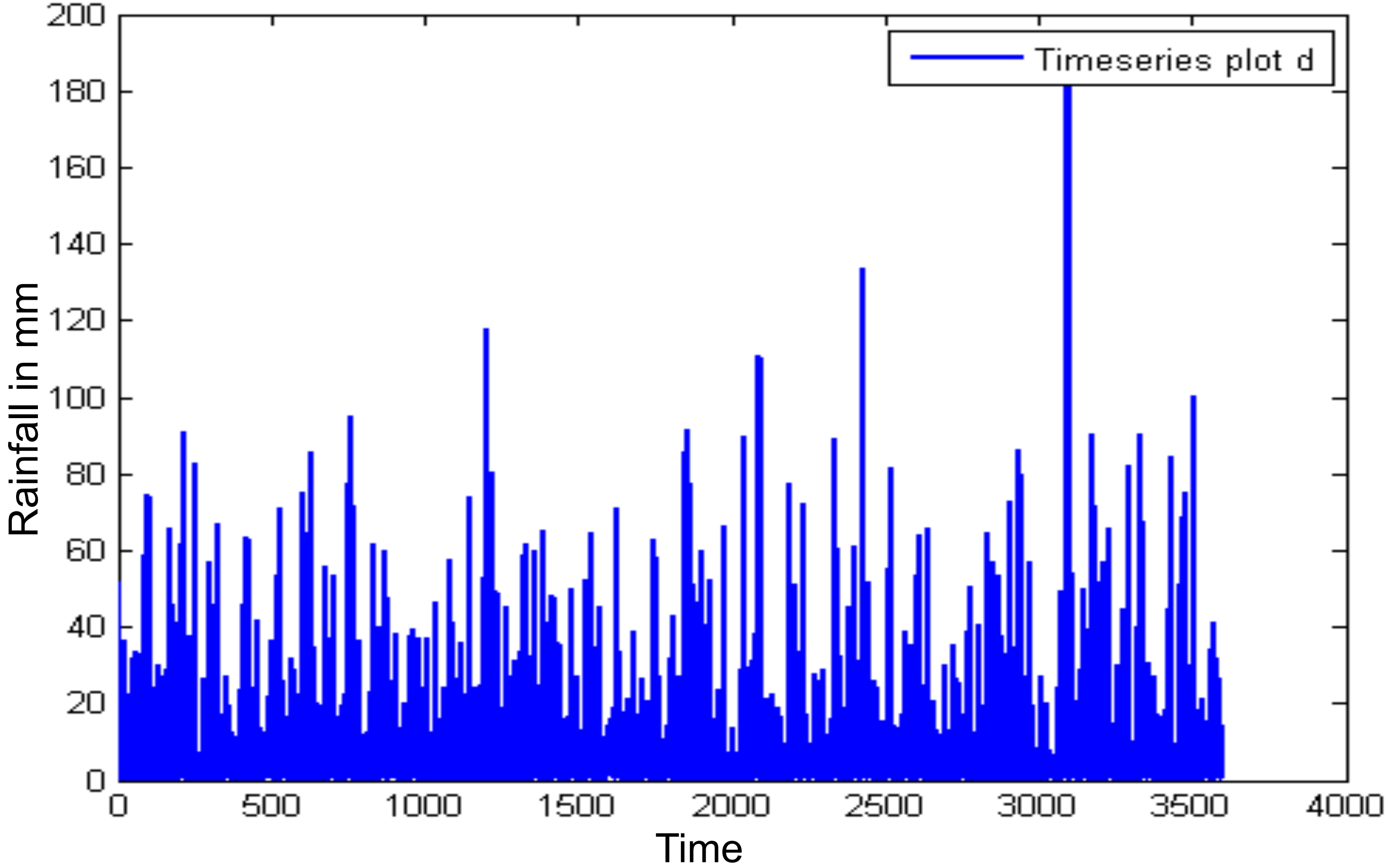
Case study – 1

Rainfall data for Bangalore city is considered.

•Time series plot, auto correlation function, partial auto correlation function and power spectrum area plotted for daily, monthly and yearly data.

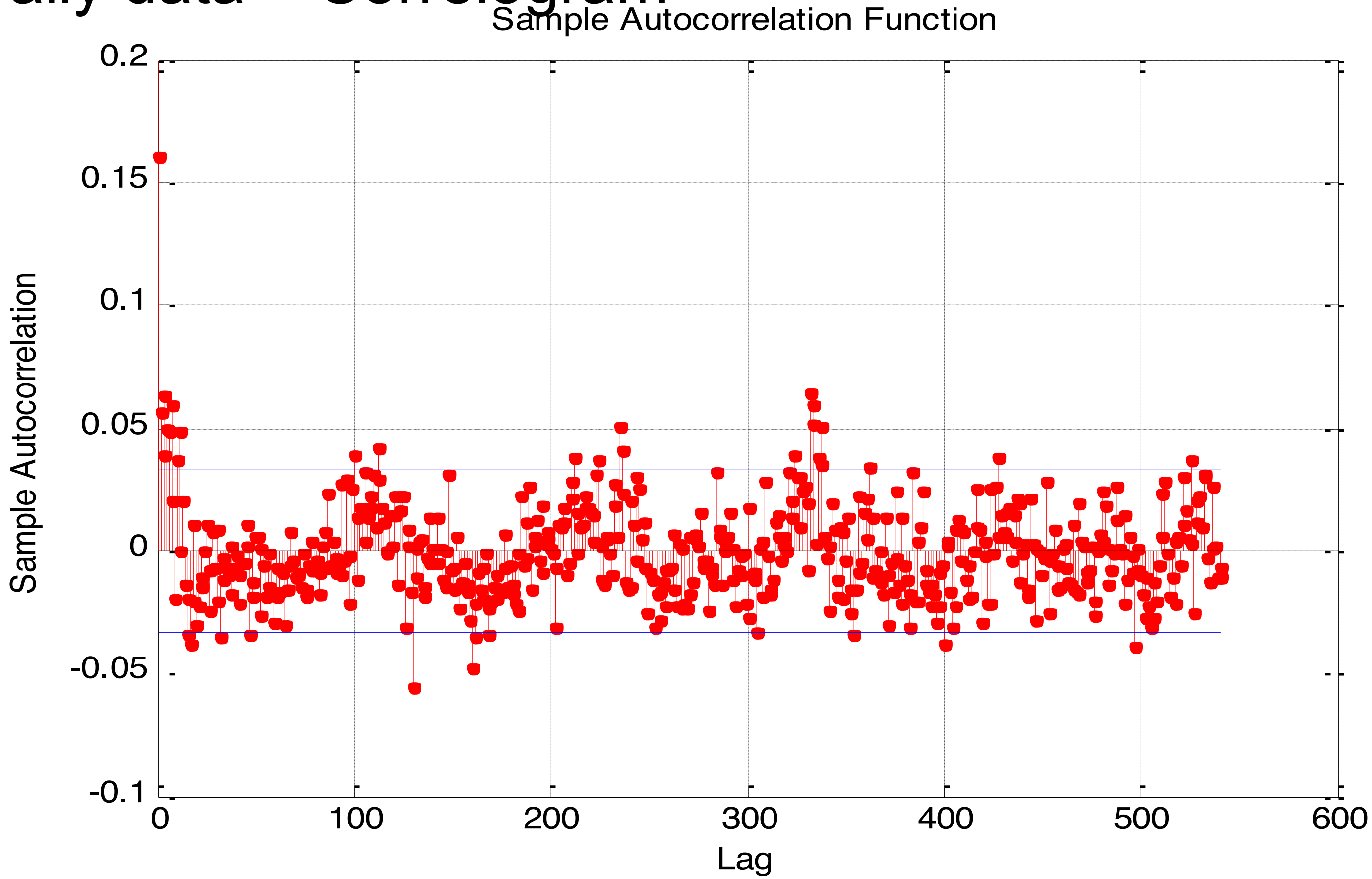
Case study – 1 (Contd.)

Daily data – Time series plot



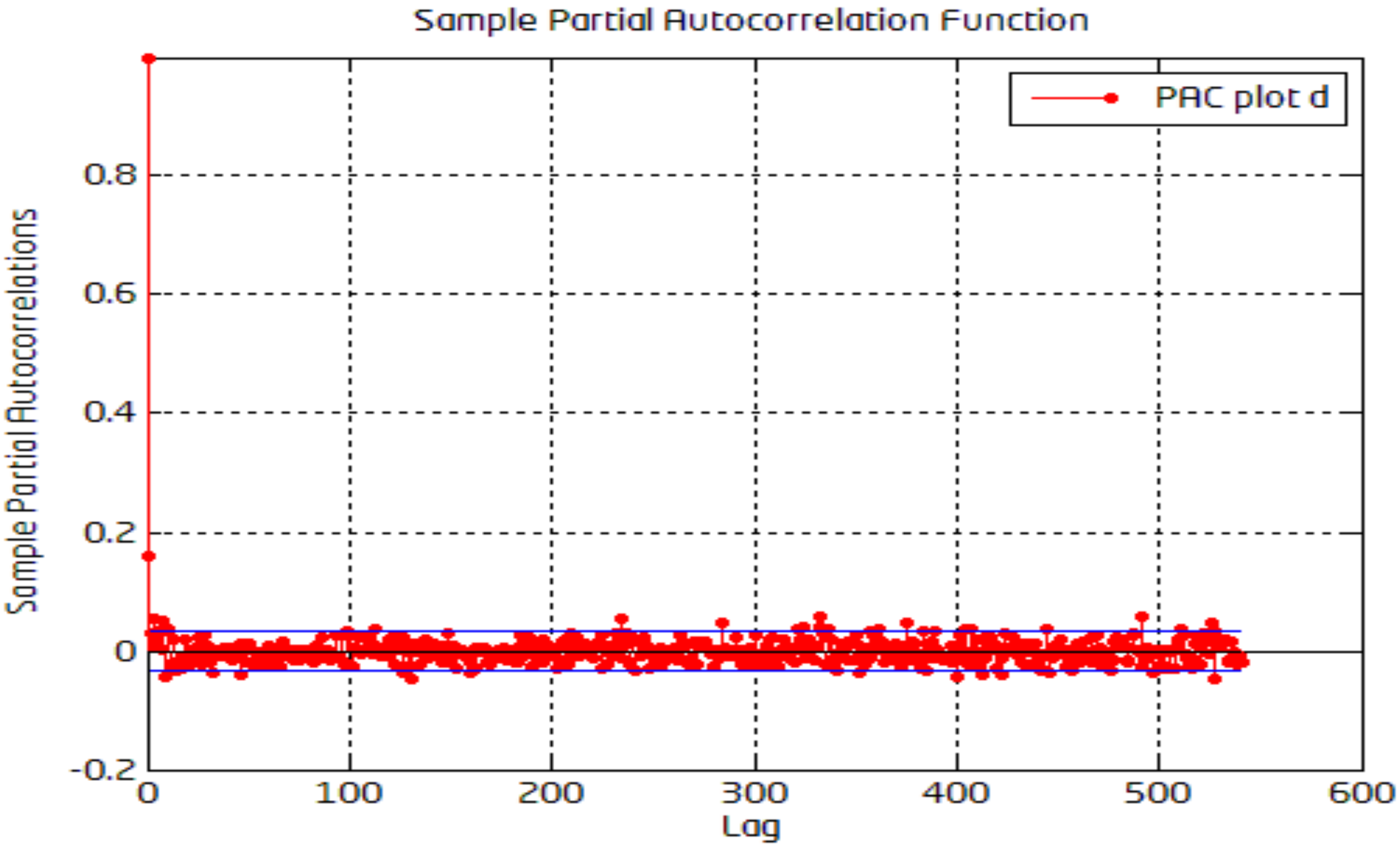
Case study – 1 (Contd.)

Daily data – Correlogram



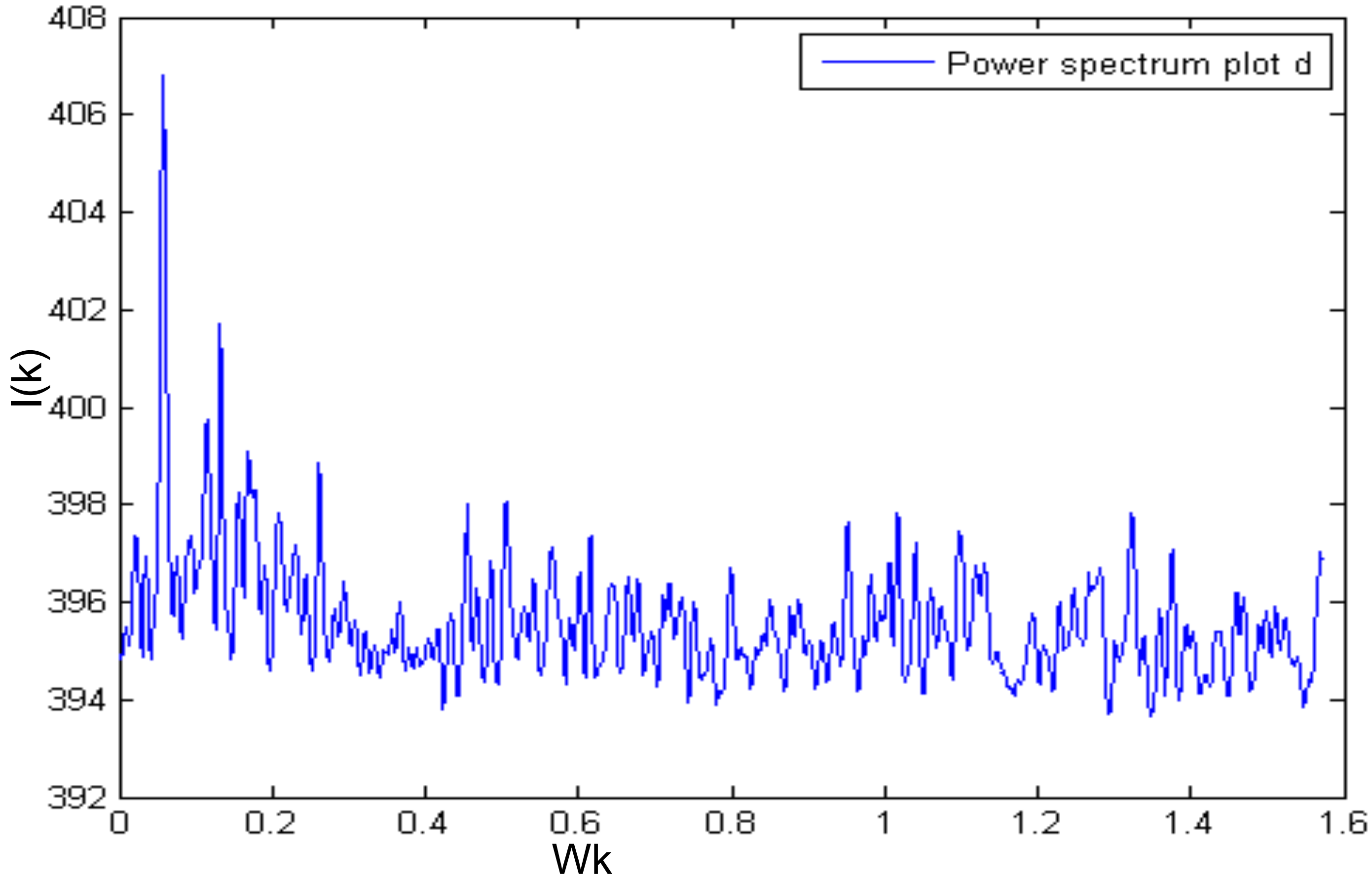
Case study – 1 (Contd.)

Daily data – Partial auto correlation function



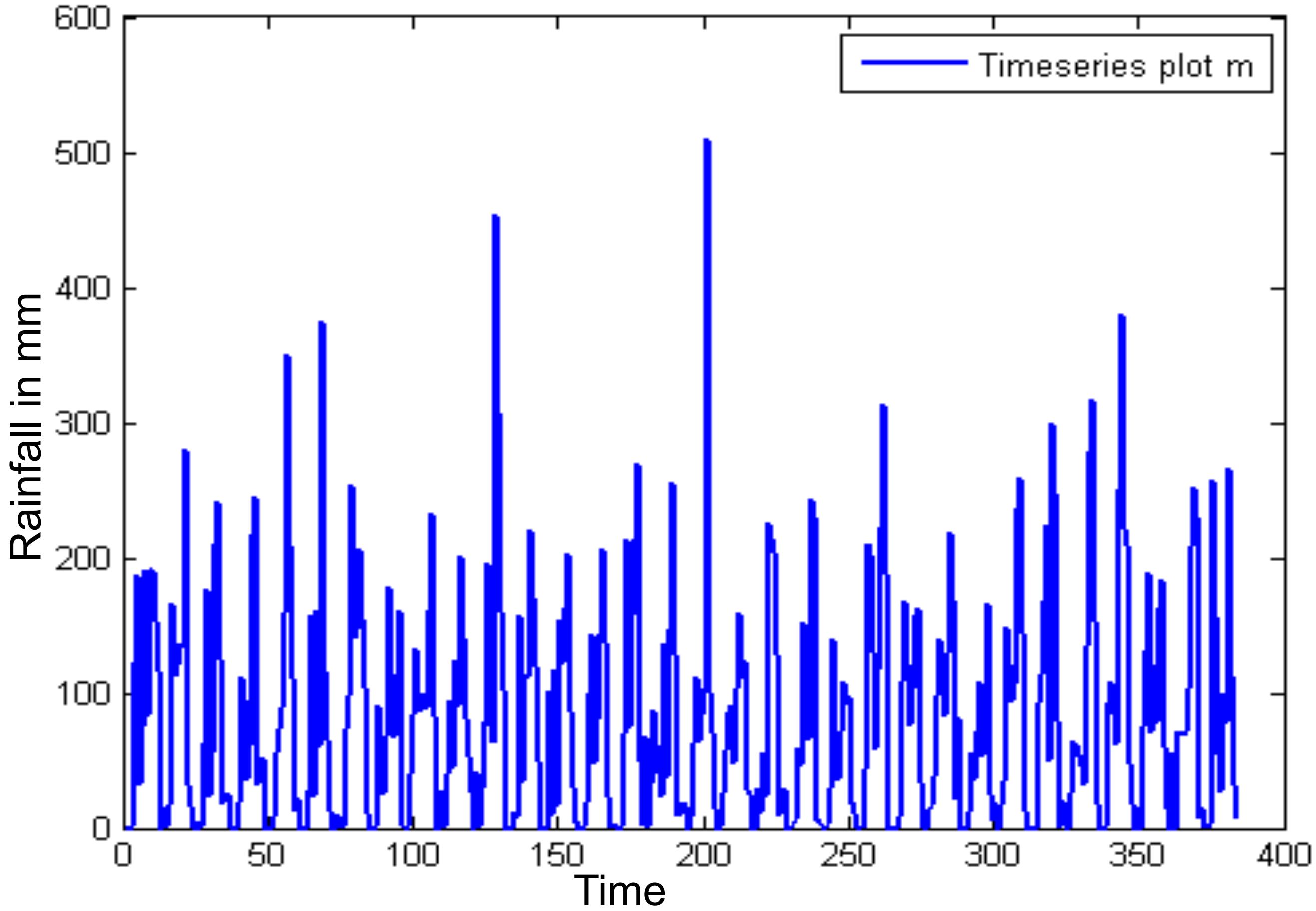
Case study – 1 (Contd.)

Daily data – Power spectrum



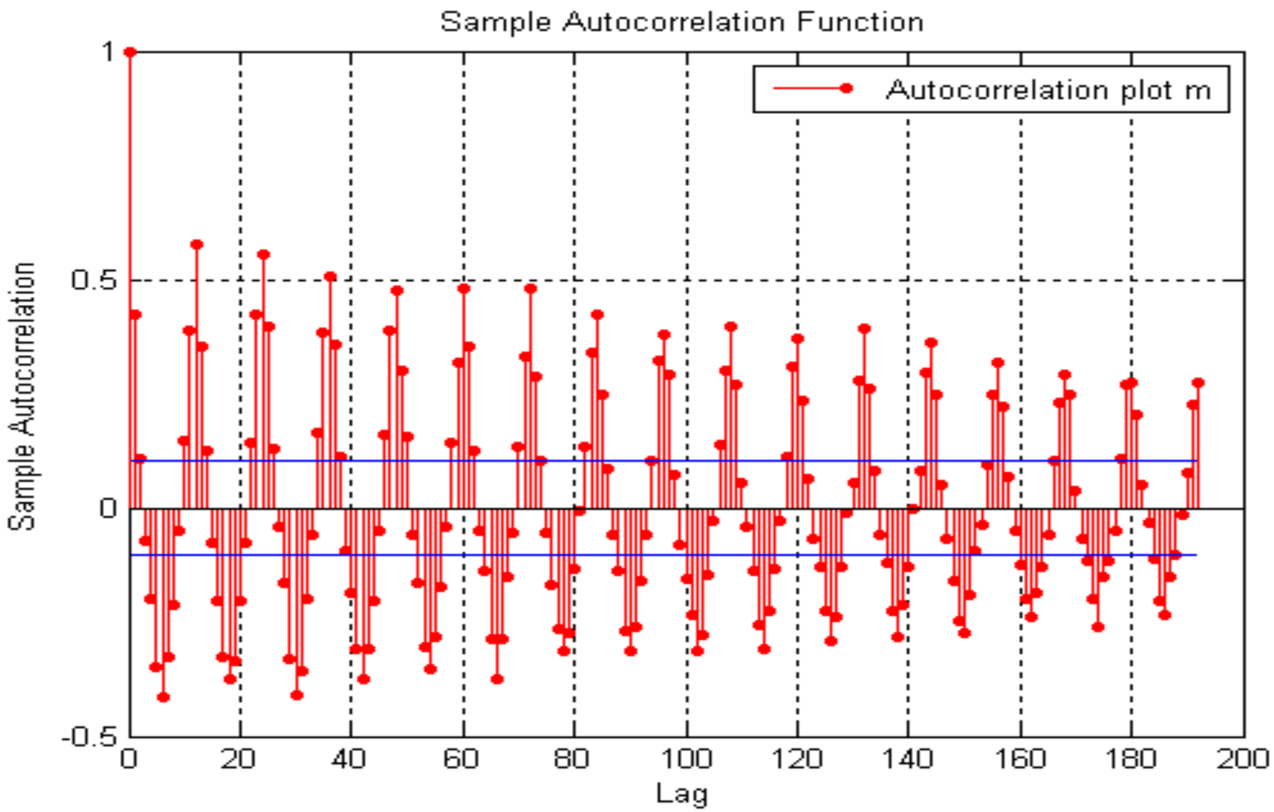
Case study – 1 (Contd.)

Monthly data – Time series plot



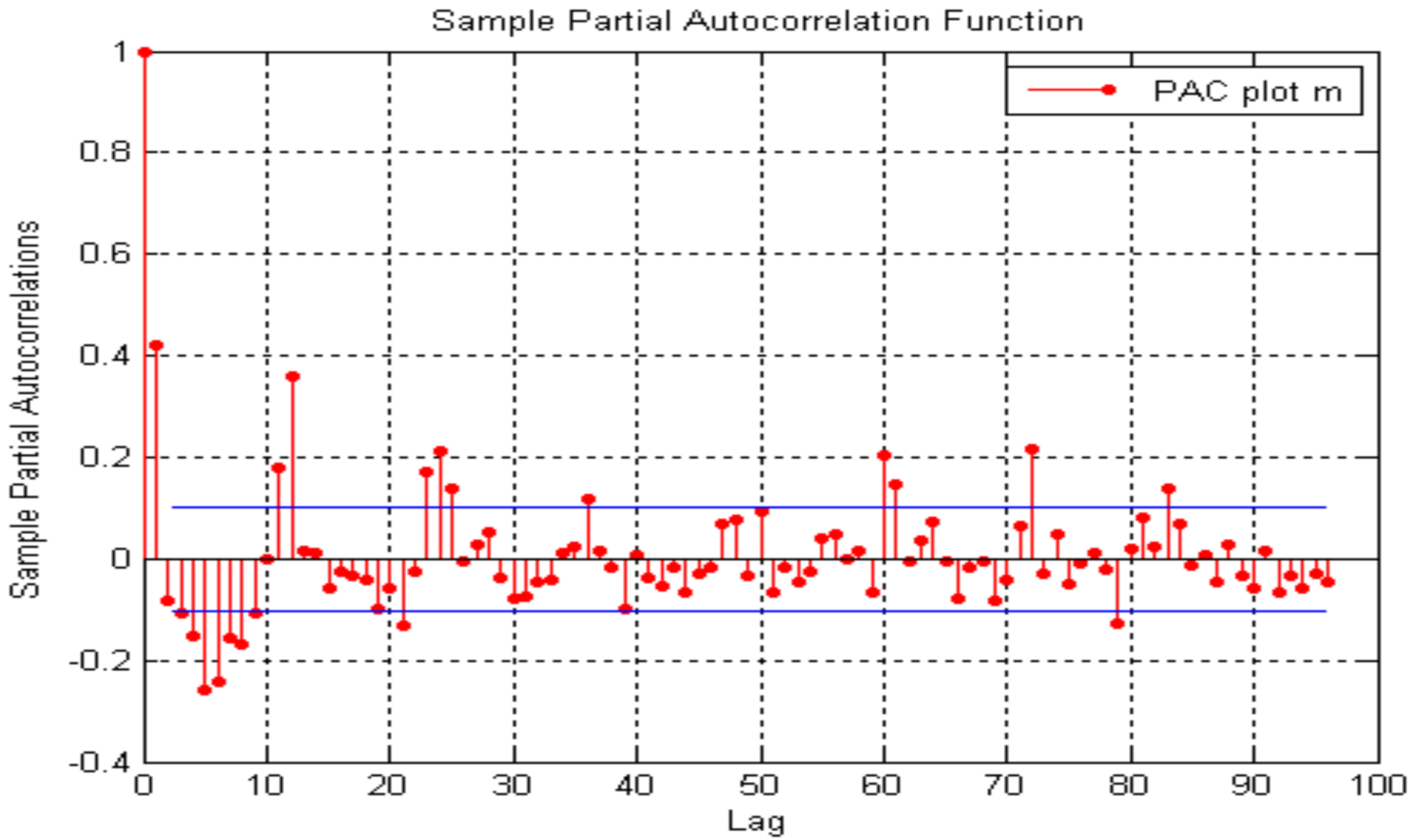
Case study – 1 (Contd.)

Monthly data – Correlogram



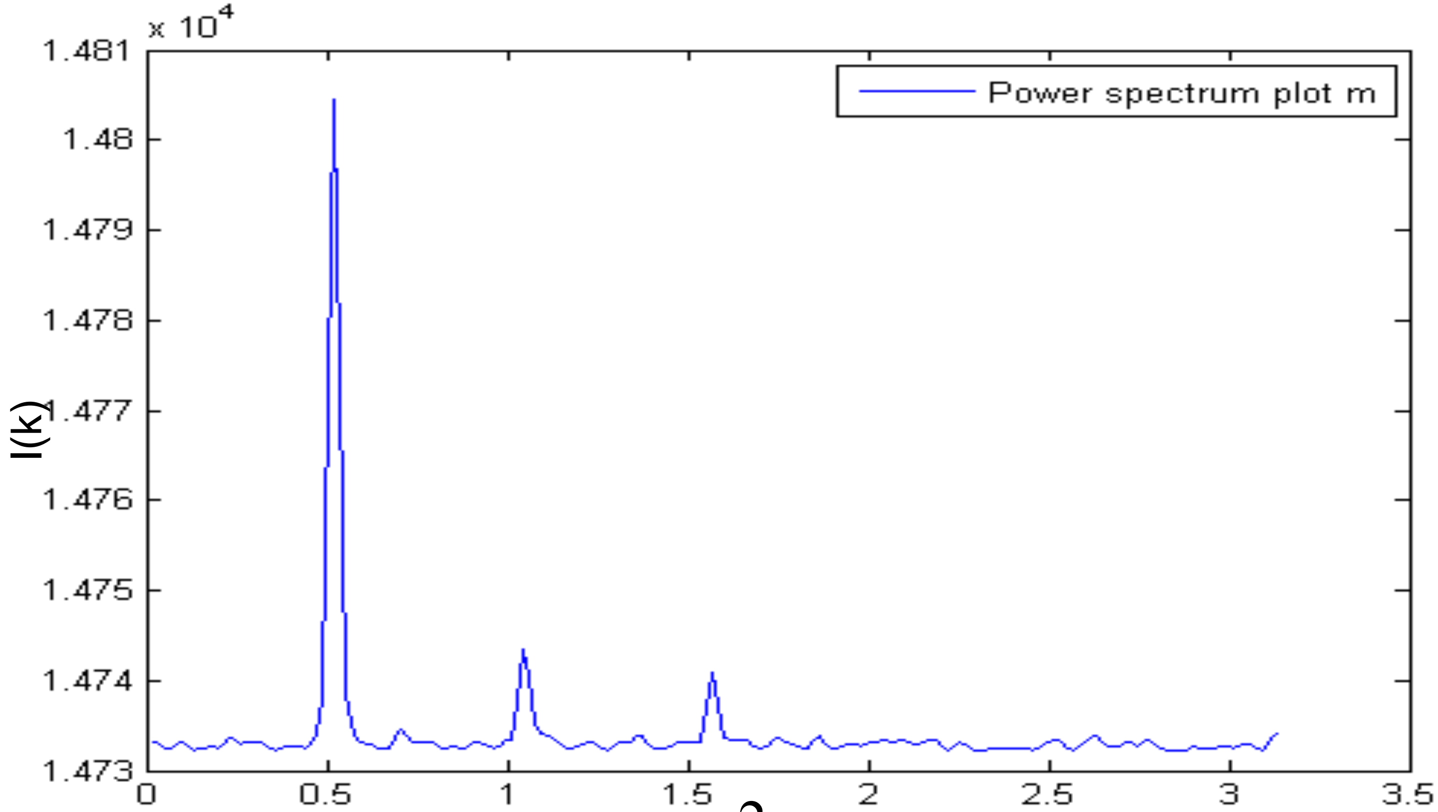
Case study – 1 (Contd.)

Monthly data – Partial auto correlation function



Case study – 1 (Contd.)

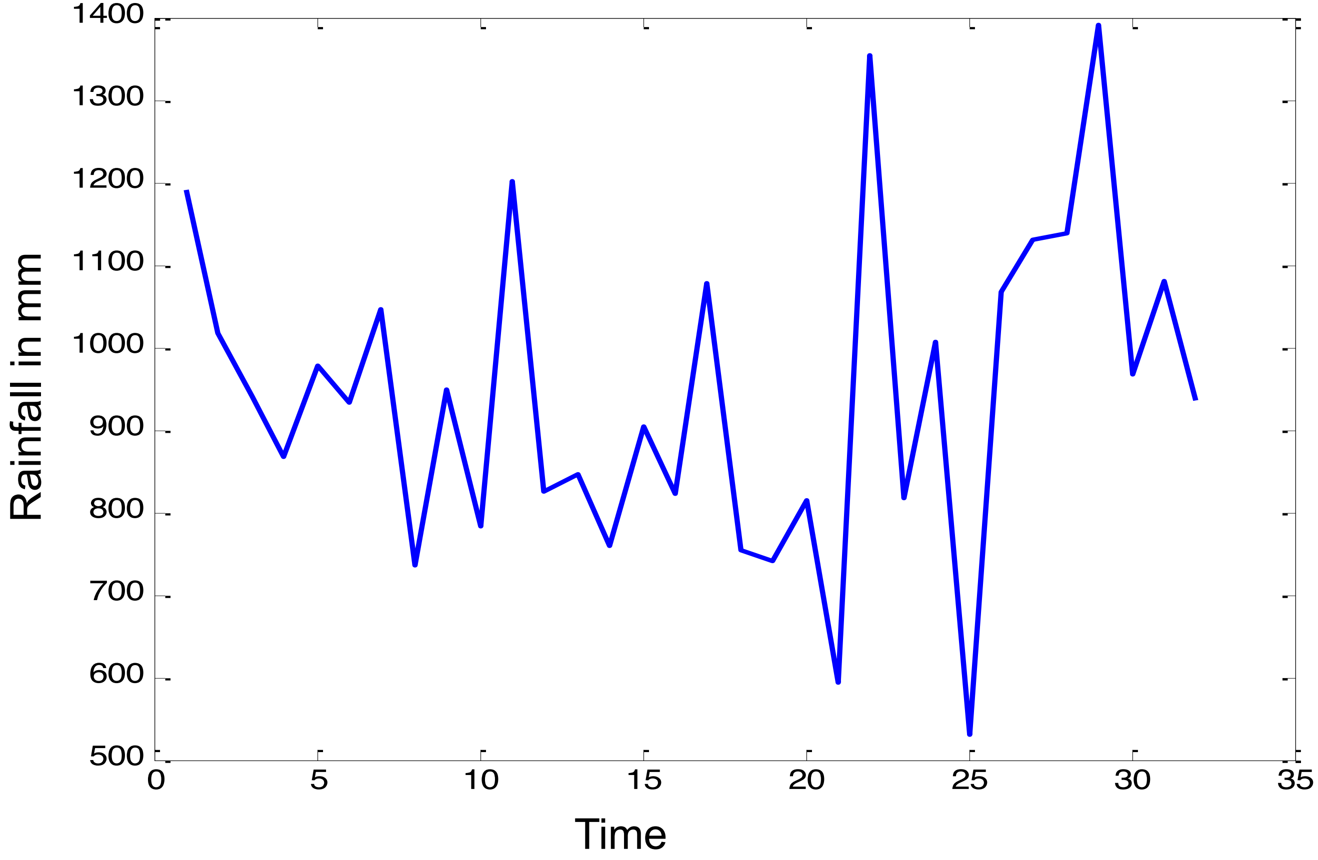
Monthly data – Power spectrum



$$\omega_k = \frac{2\pi}{P}$$

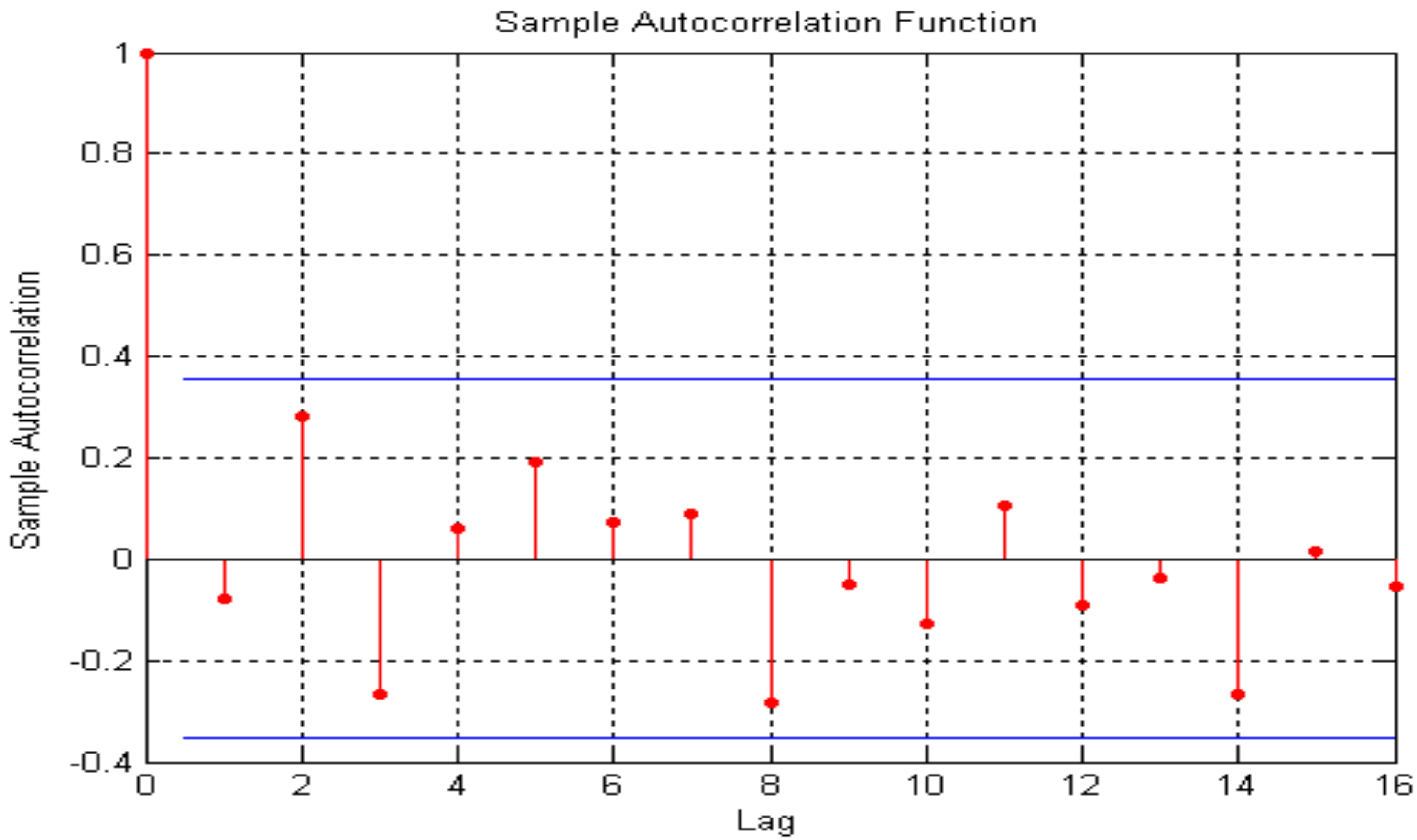
Case study – 1 (Contd.)

Yearly data – Time series plot



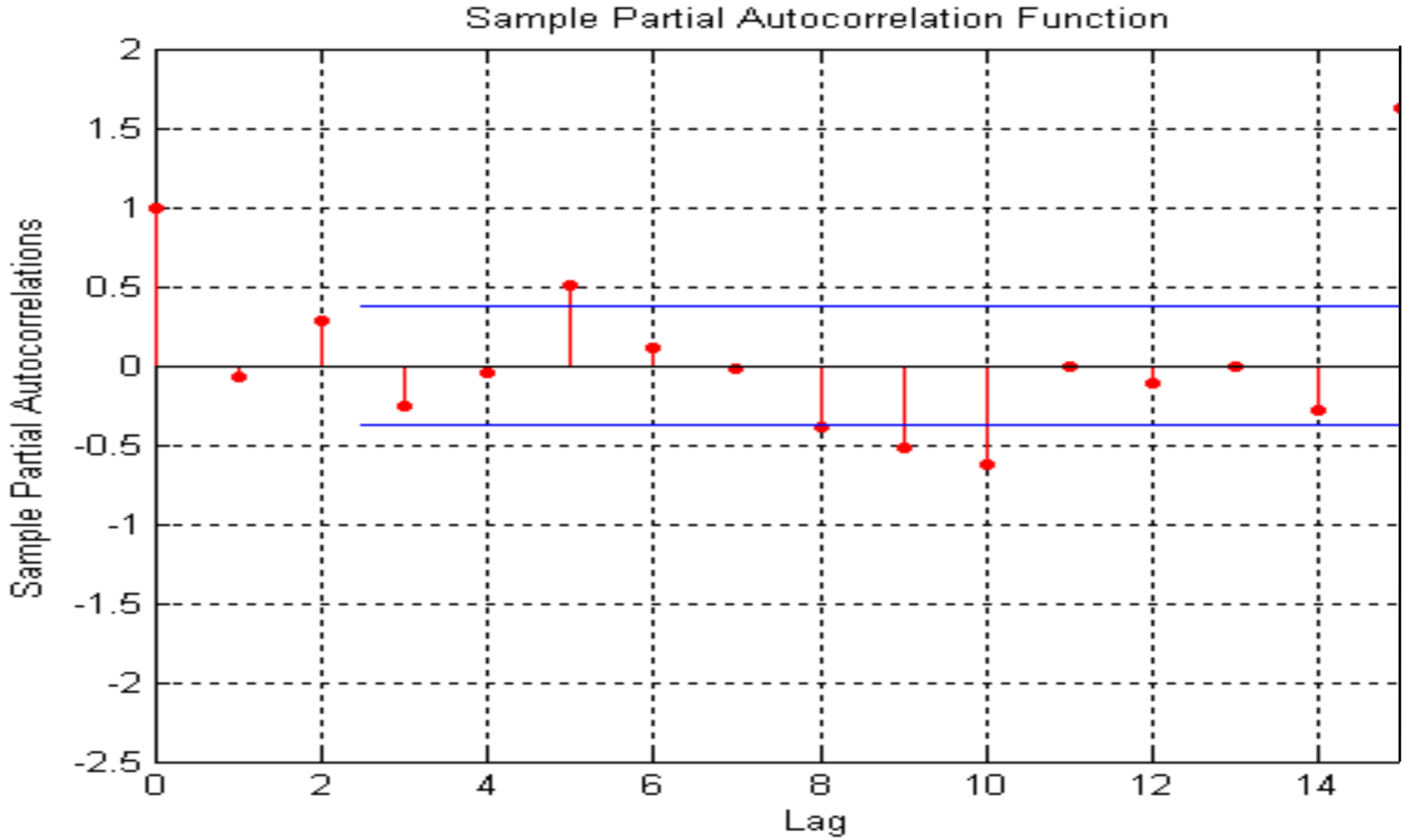
Case study – 1 (Contd.)

Yearly data – Correlogram



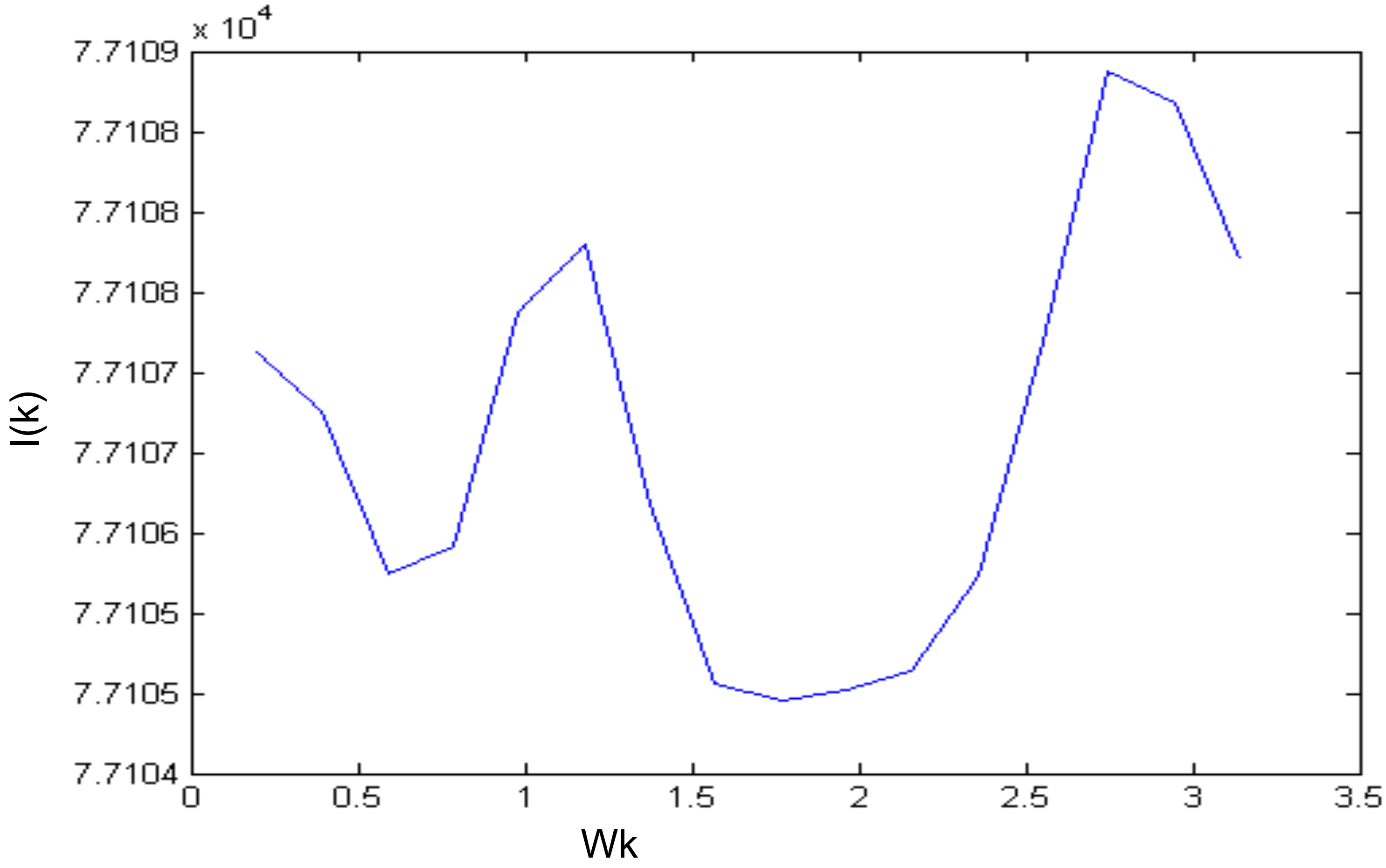
Case study – 1 (Contd.)

Yearly data – Partial auto correlation function



Case study – 1 (Contd.)

Yearly data – Power spectrum



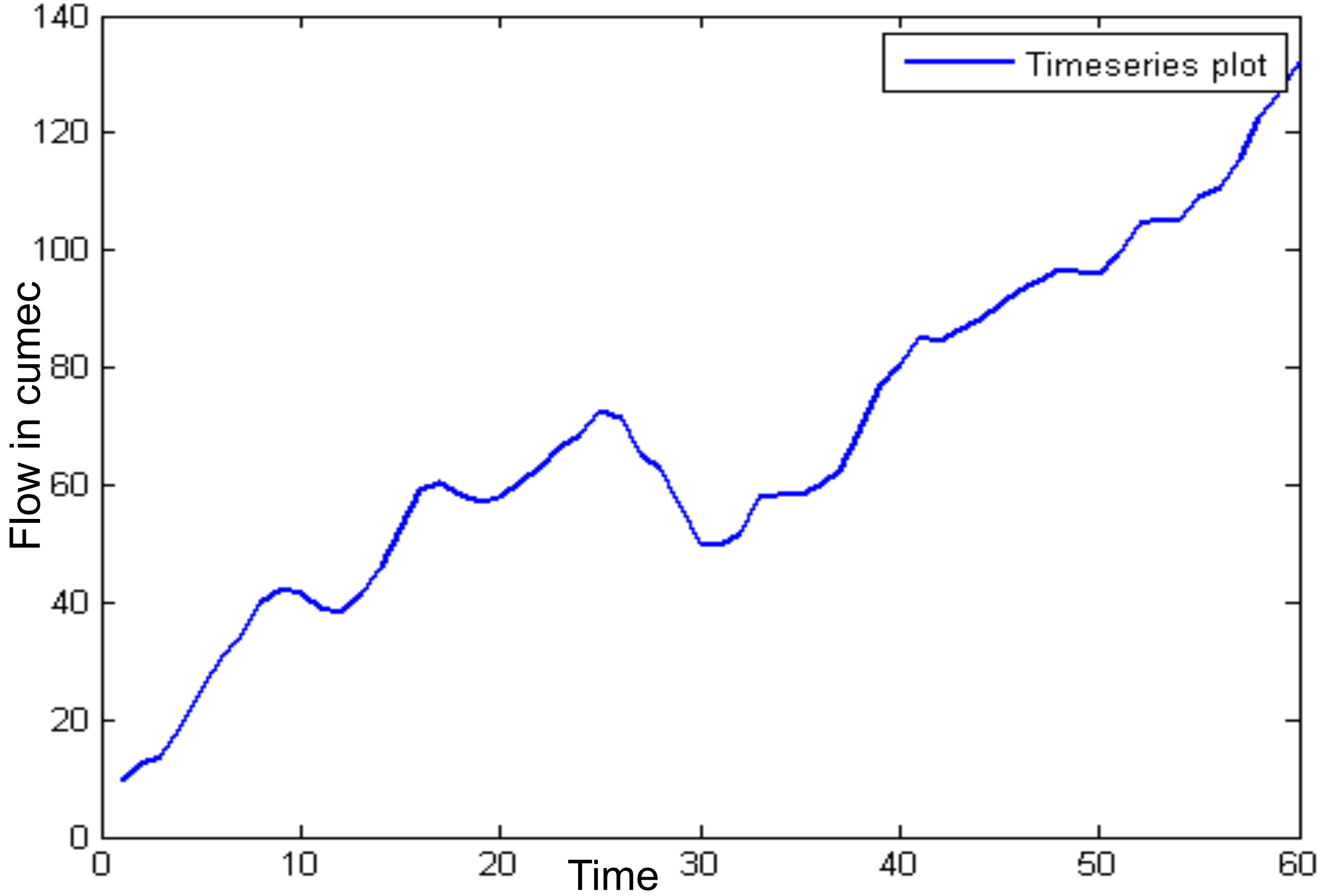
Case study – 2

Stream flow data for a river is considered in the case study.

- Time series plot, auto correlation function, partial auto correlation function and power spectrum area plotted for the original data and the first differenced data.

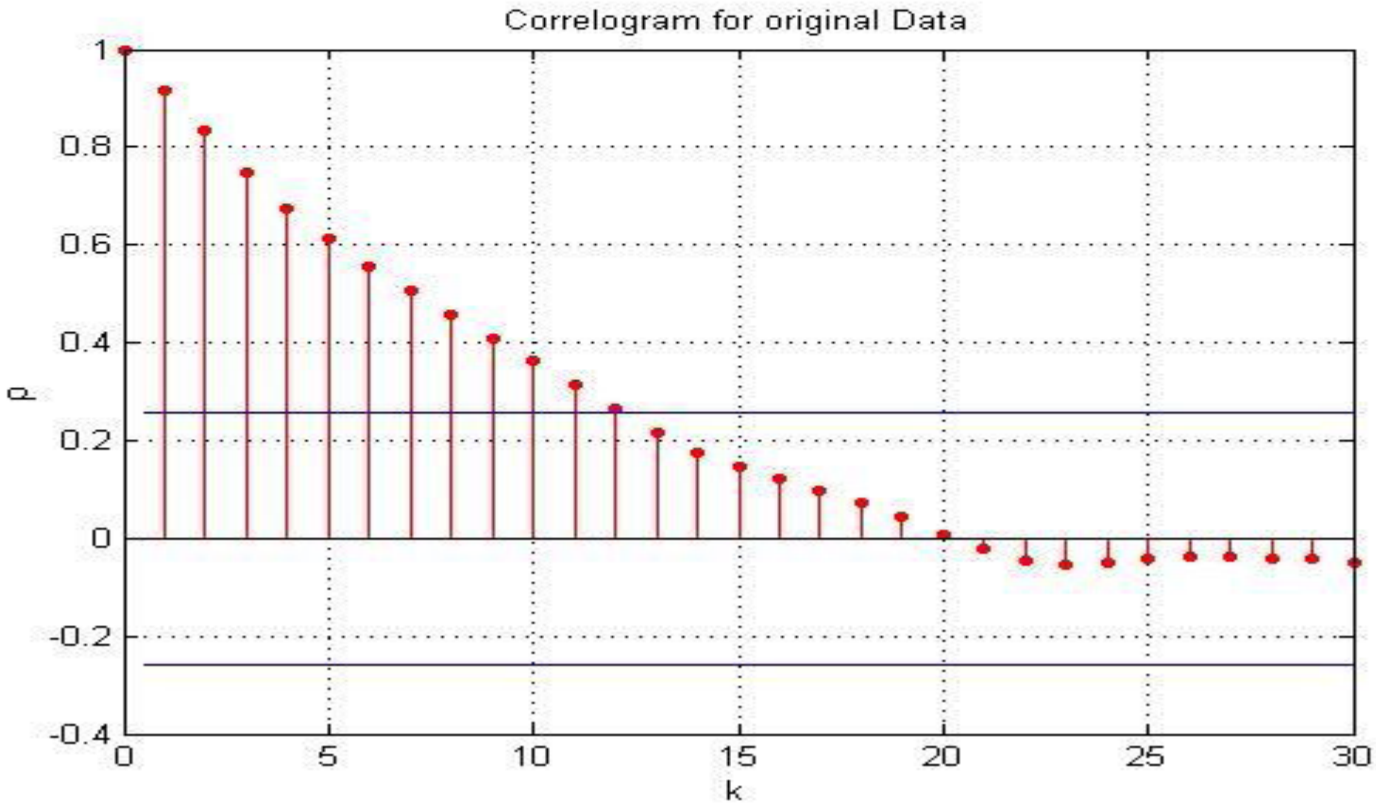
Case study – 2 (Contd.)

Time series plot shows an increasing trend



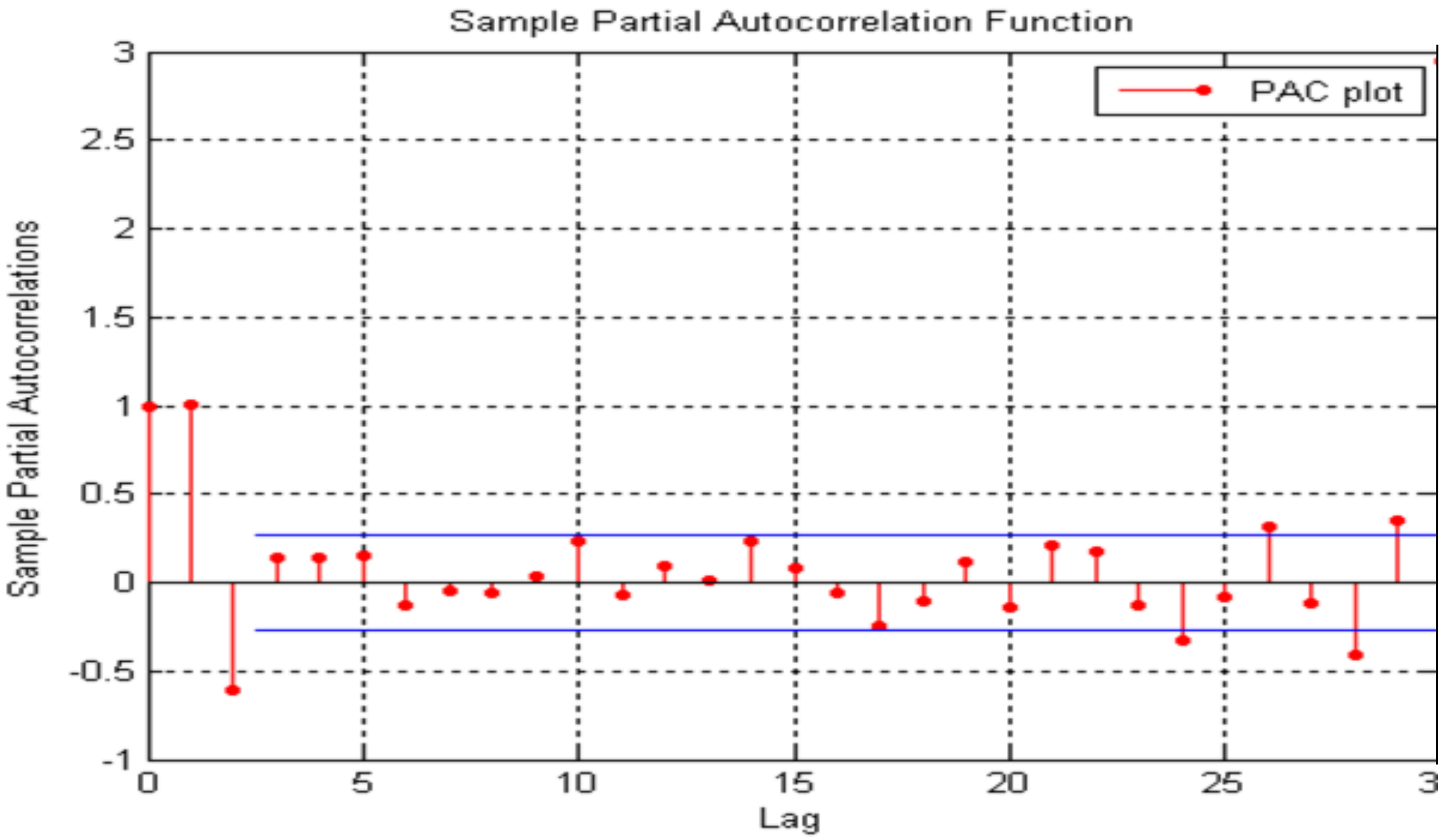
Case study – 2 (Contd.)

Correlogram:



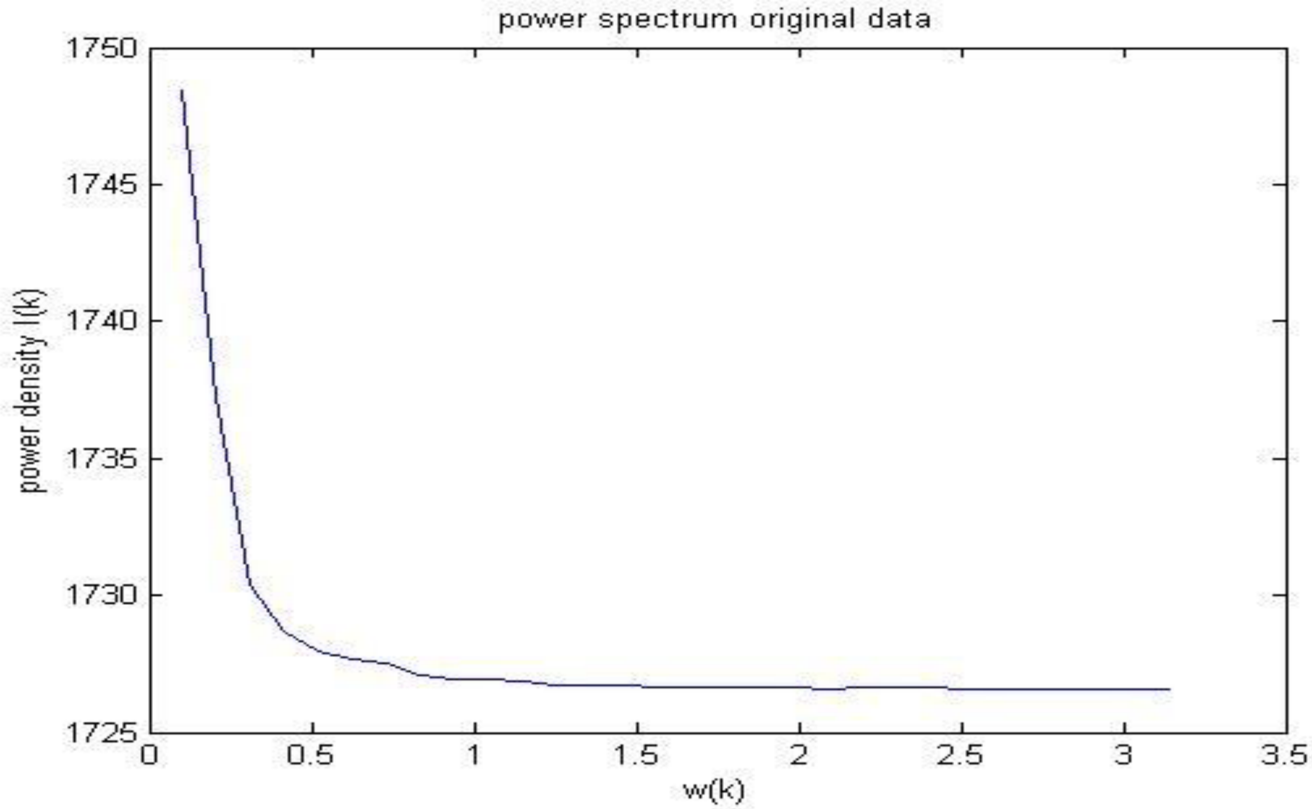
Case study – 2 (Contd.)

Partial auto correlation function



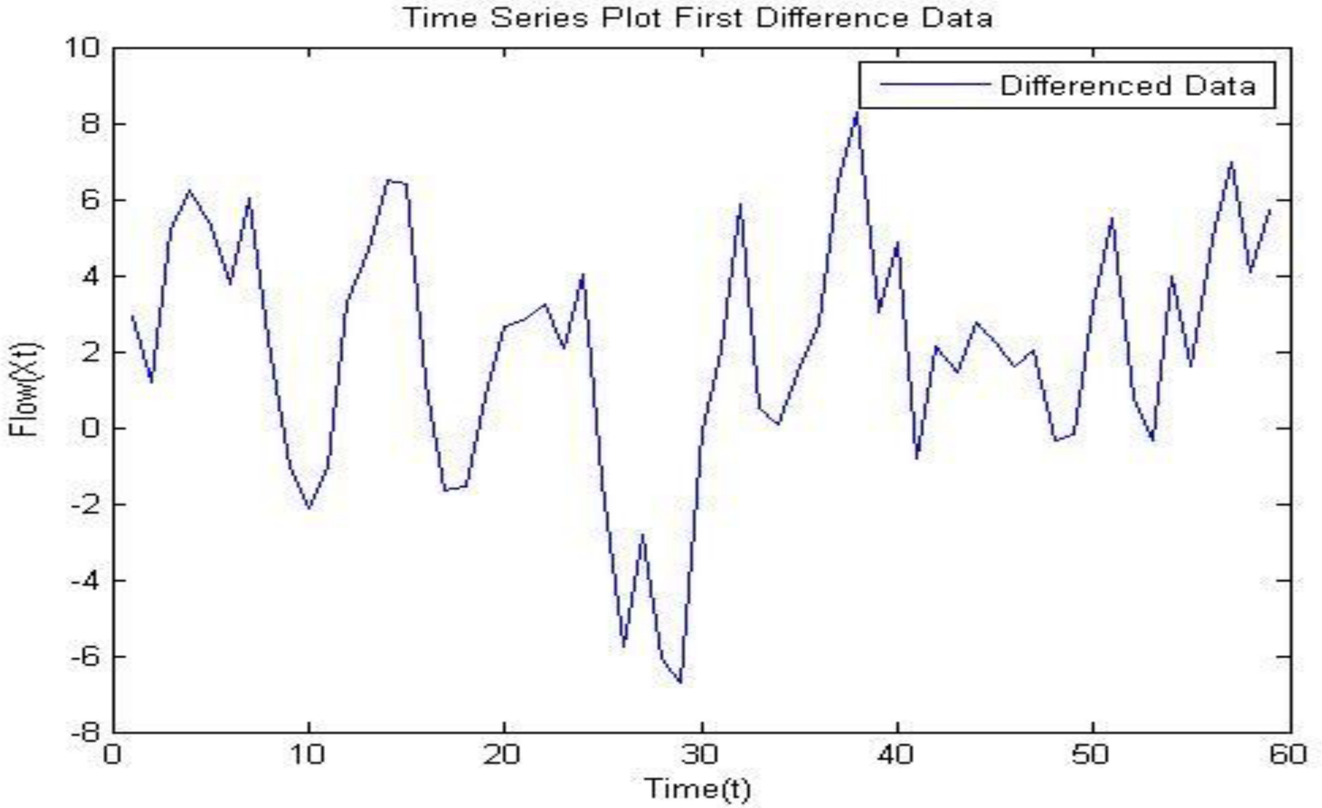
Case study – 2 (Contd.)

Power spectrum



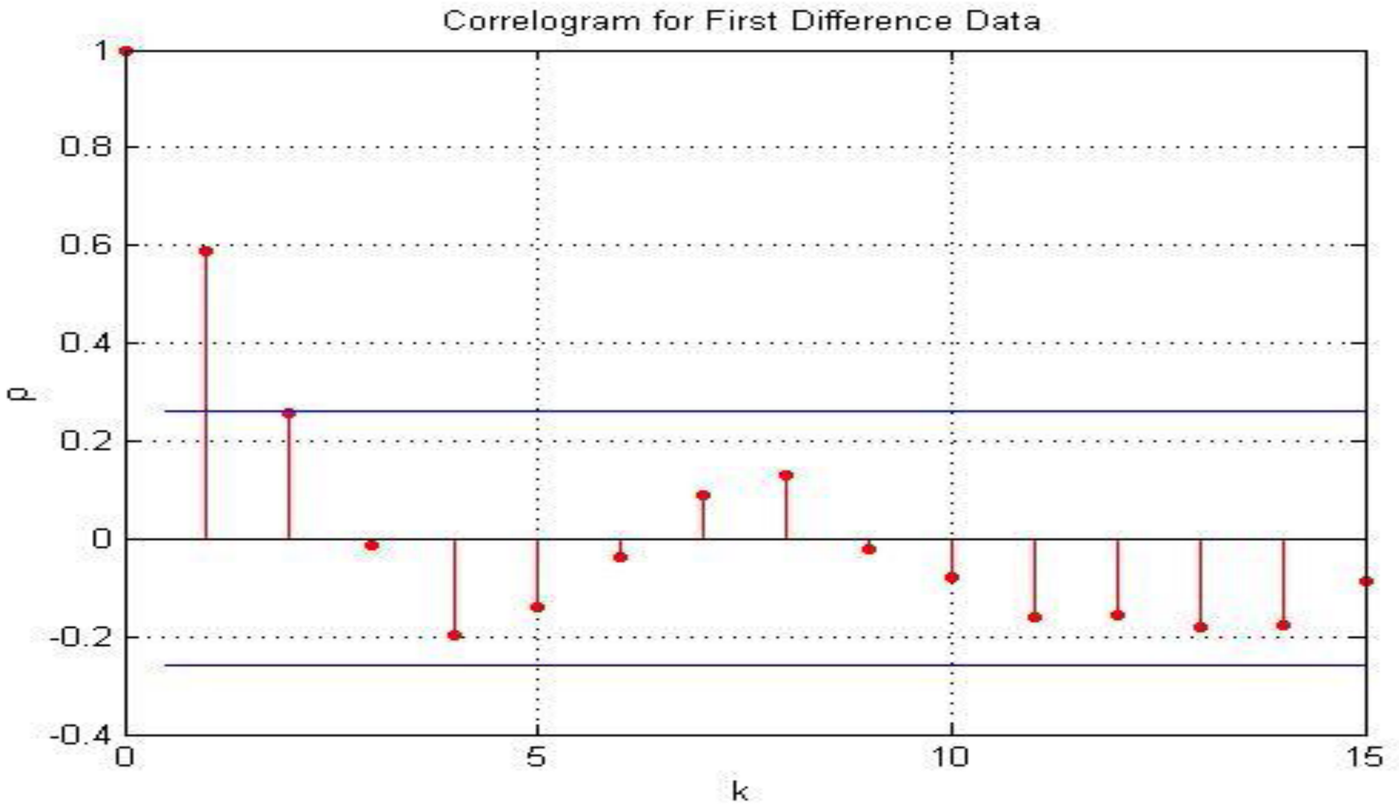
Case study – 2 (Contd.)

Time series plot – first differenced data : No trend



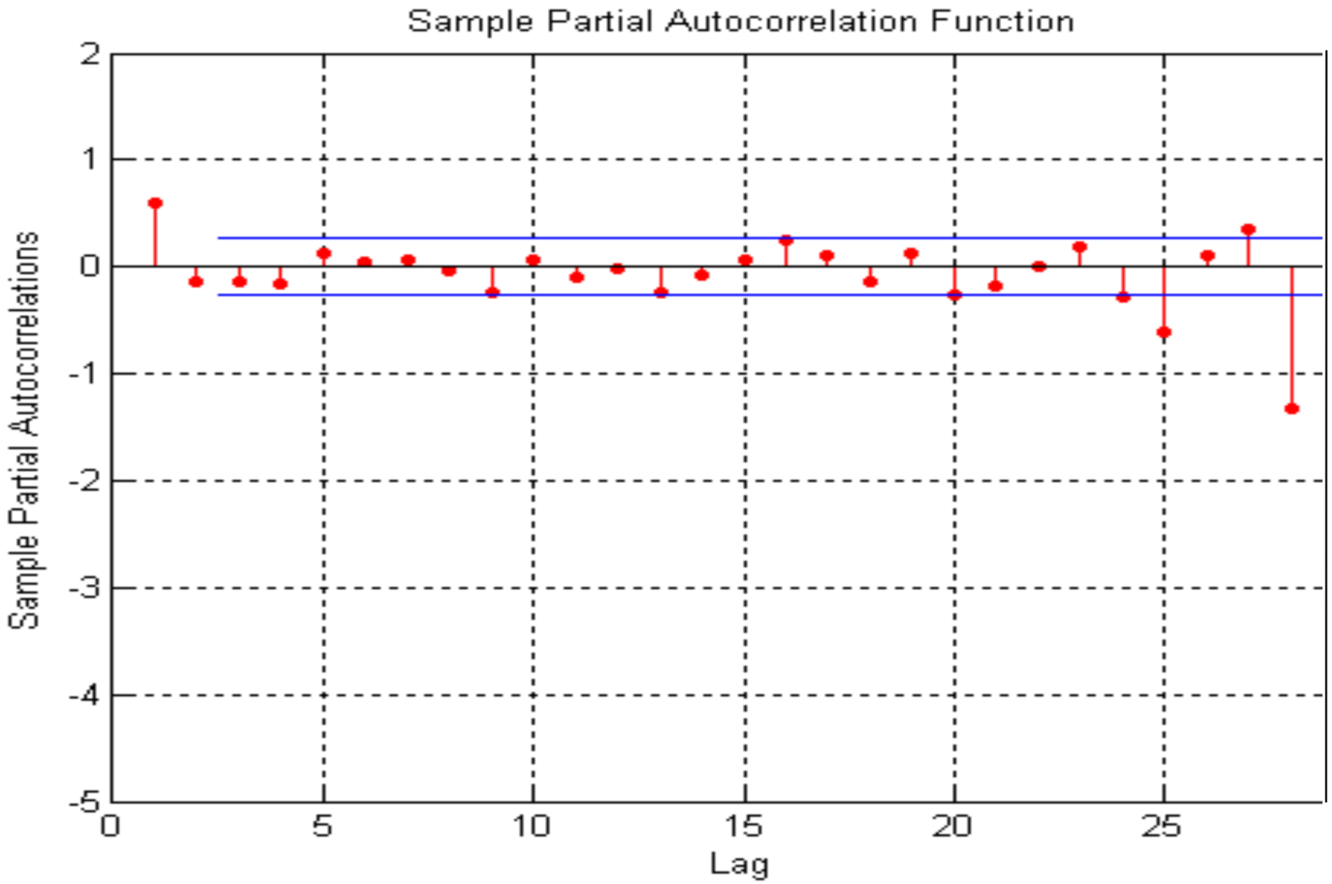
Case study – 2 (Contd.)

Correlogram – differenced data: First two correlations significant



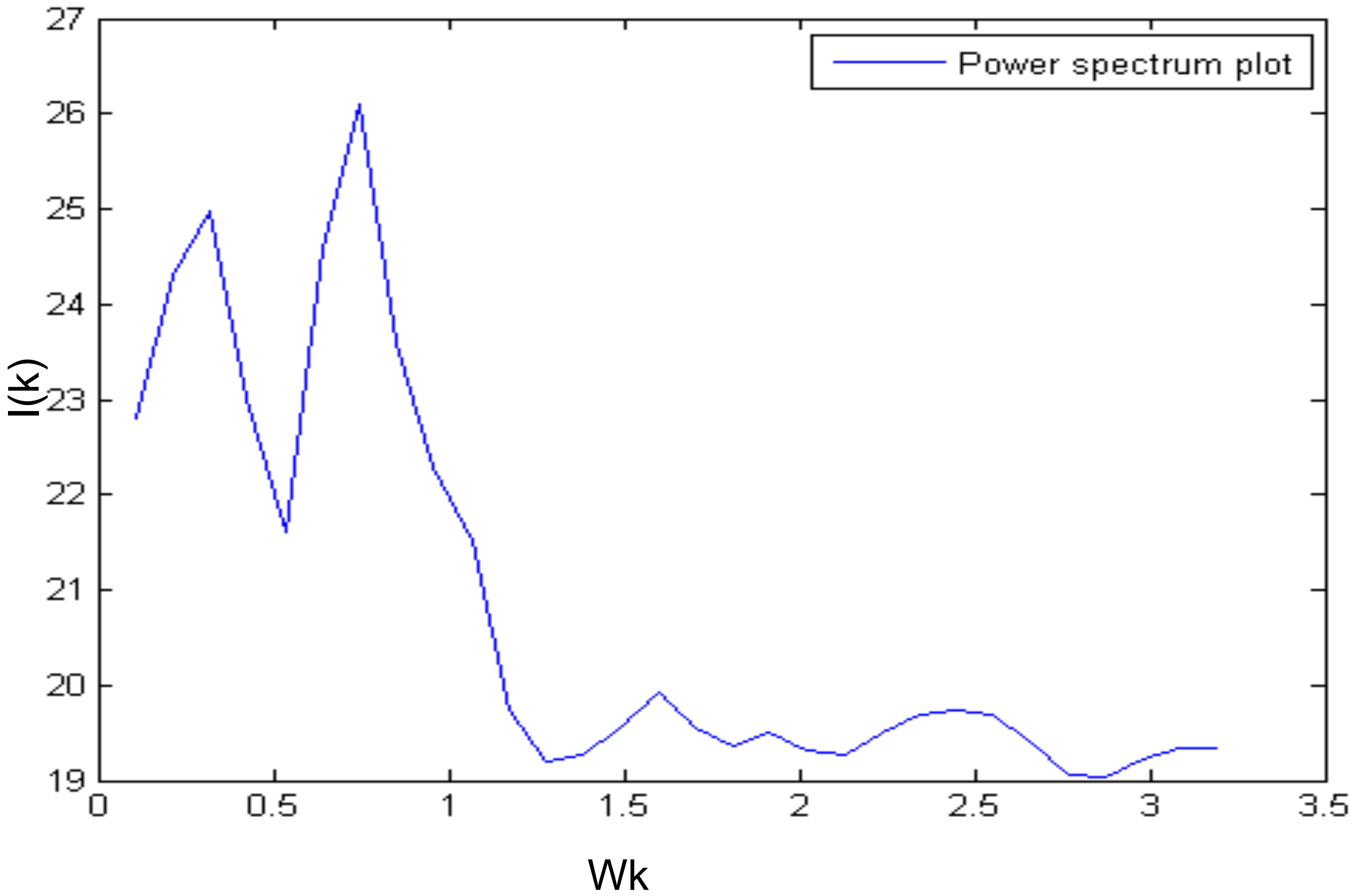
Case study – 2 (Contd.)

Partial auto correlation function – differenced data



Case study – 2 (Contd.)

Power spectrum – differenced data



CASE STUDIES ON ARMA MODELS

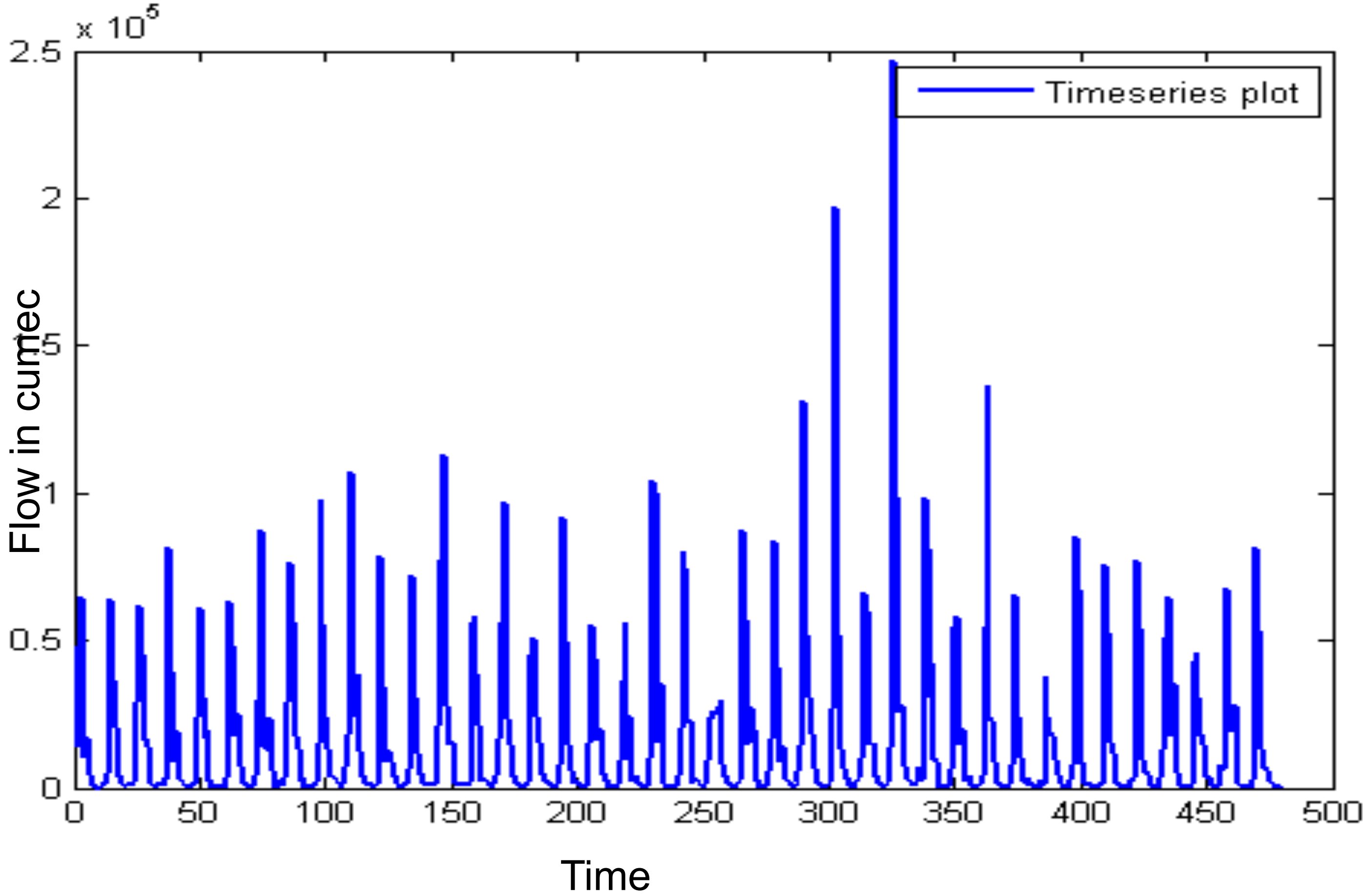
Case study – 3

Monthly Stream flow (cum/sec) Statistics(1934 -1974) for Cauvery River at Krishna Raja Sagar Reservoir is considered in the case study.

- Time series of the data, auto correlation function, partial auto correlation function and the power spectrum are plotted.
- The series indicates presence of periodicities.
- The series is standardized to remove the periodicities.

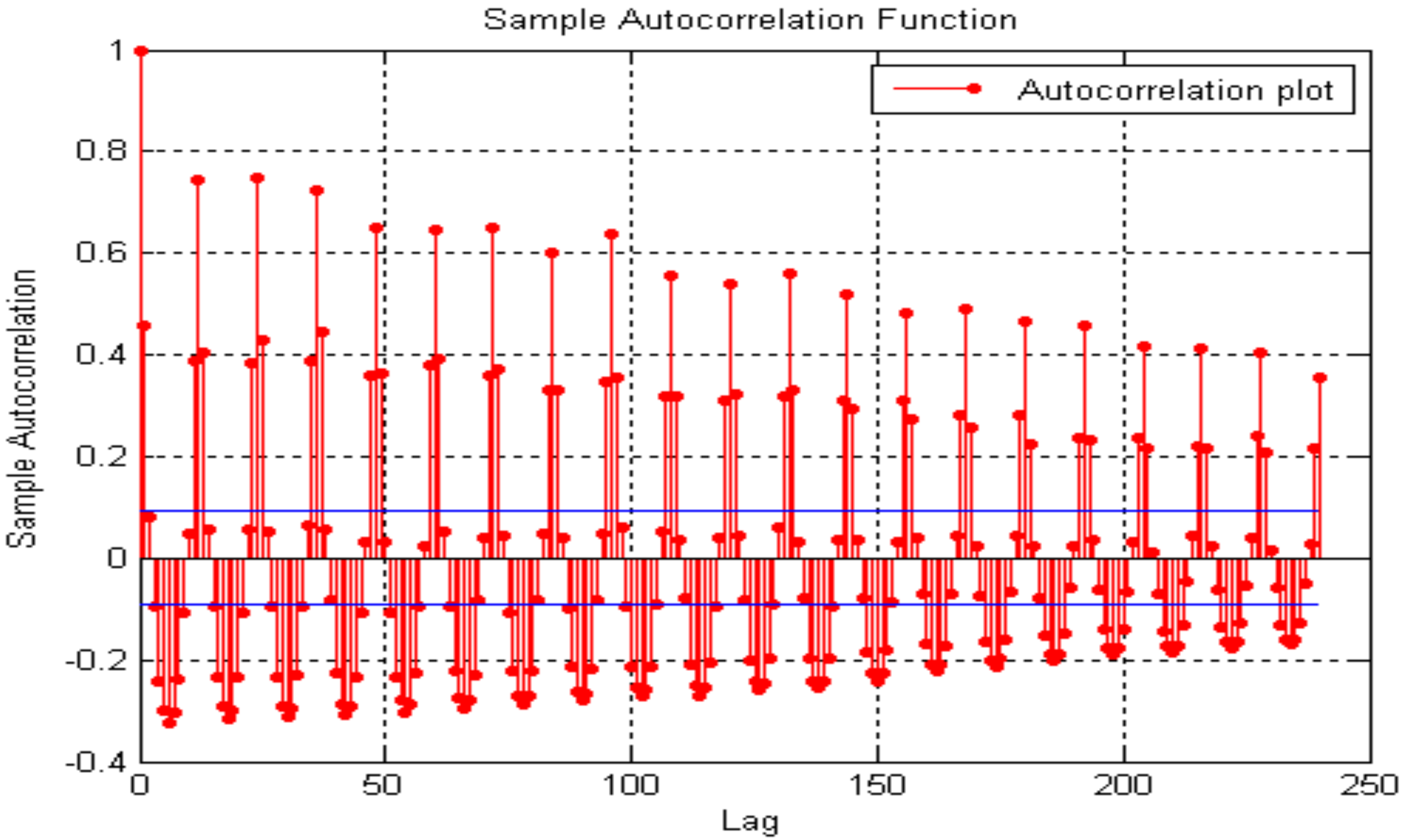
Case study – 3 (Contd.)

Time series plot – original series



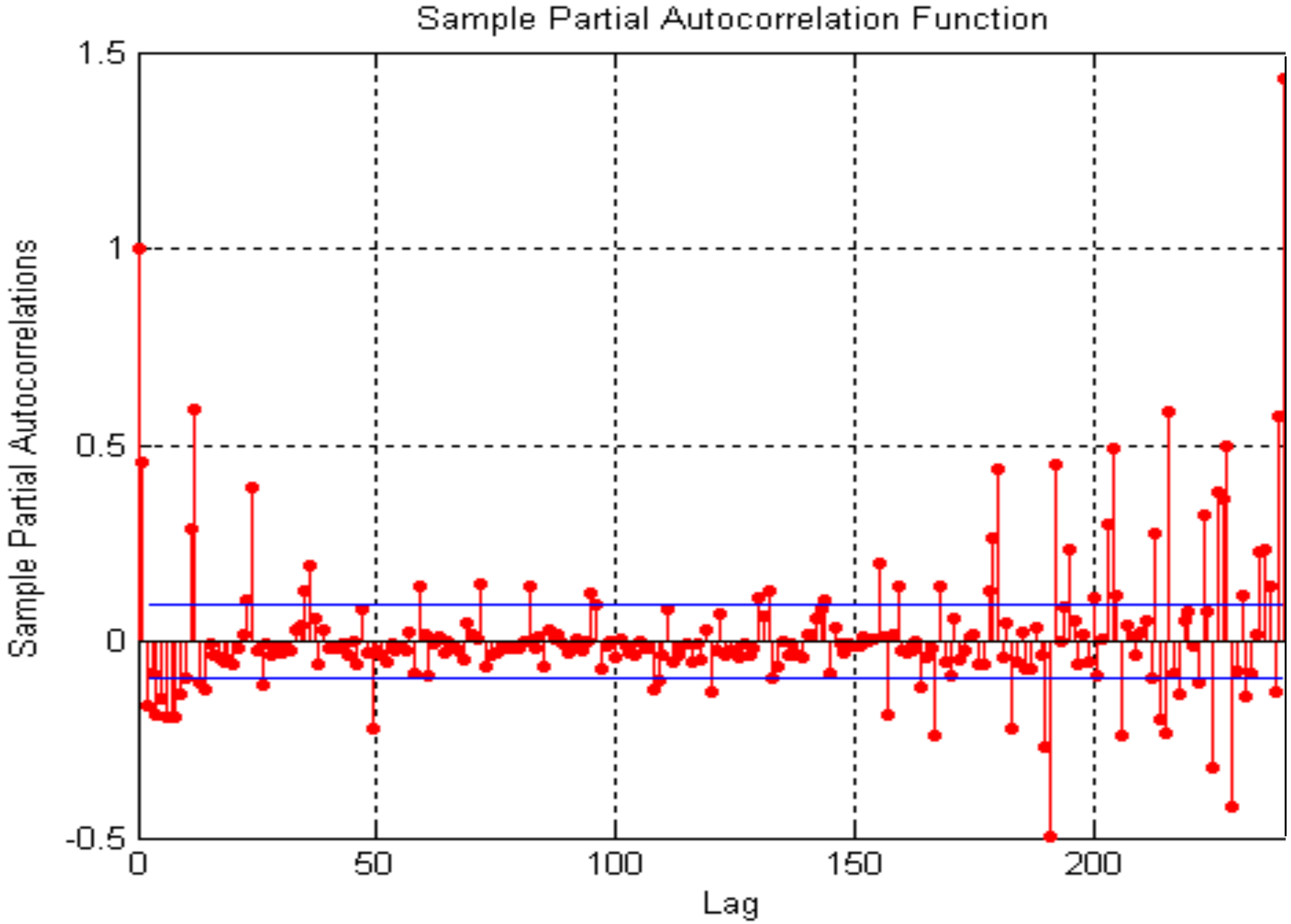
Case study – 3 (Contd.)

Correlogram – original series



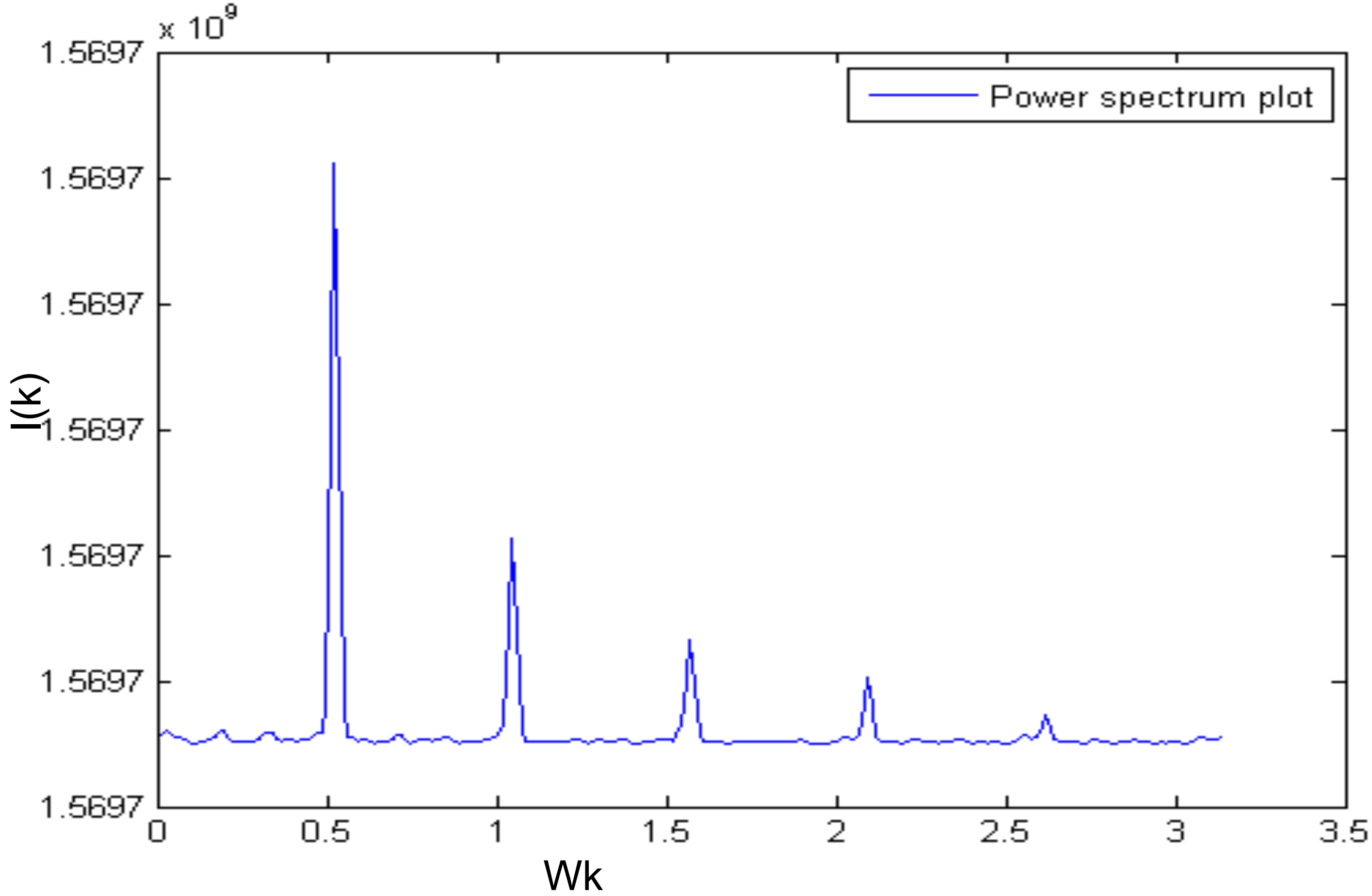
Case study – 3 (Contd.)

Partial auto correlation function – original series



Case study – 3 (Contd.)

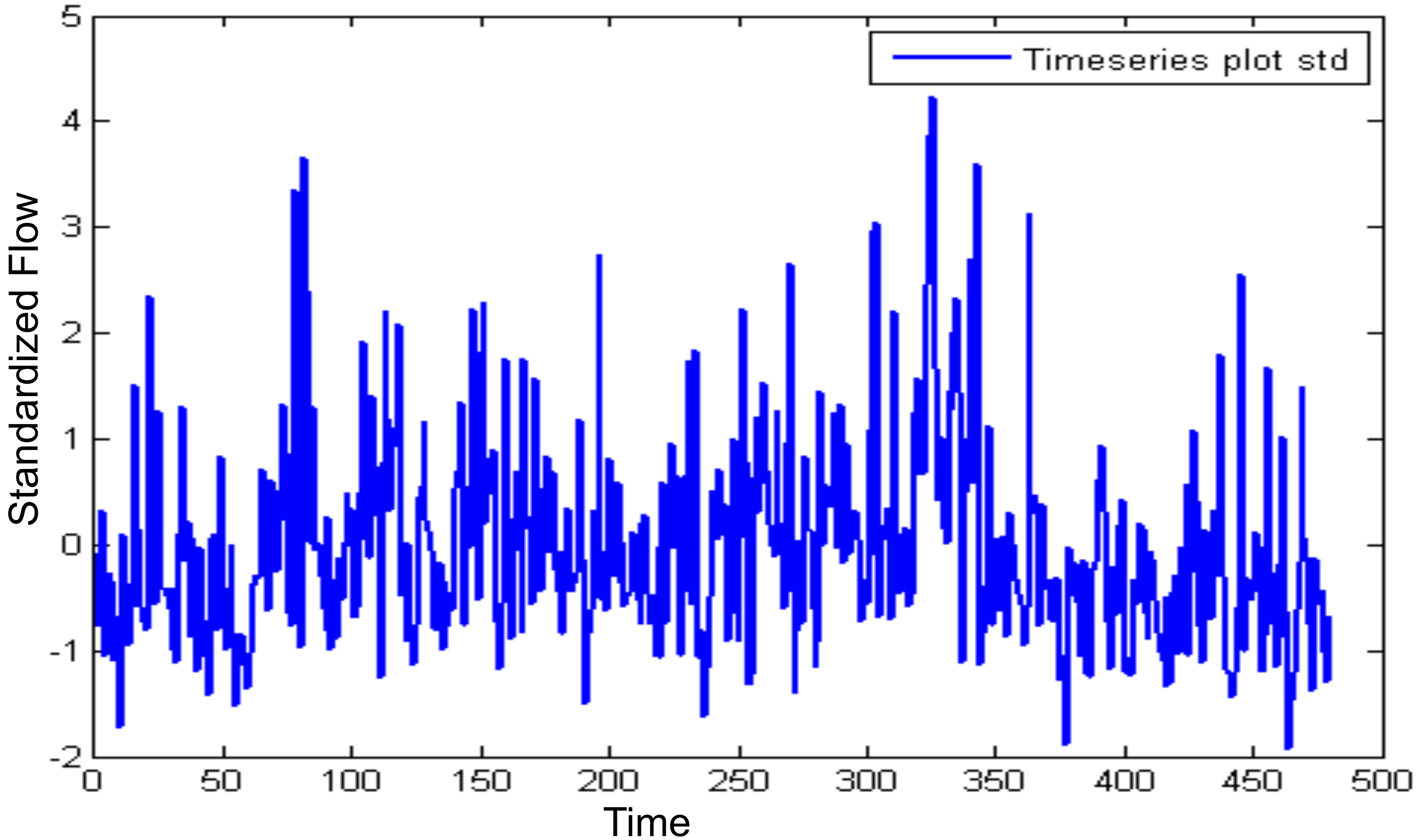
Power spectrum – original series



Case study – 3 (Contd.)

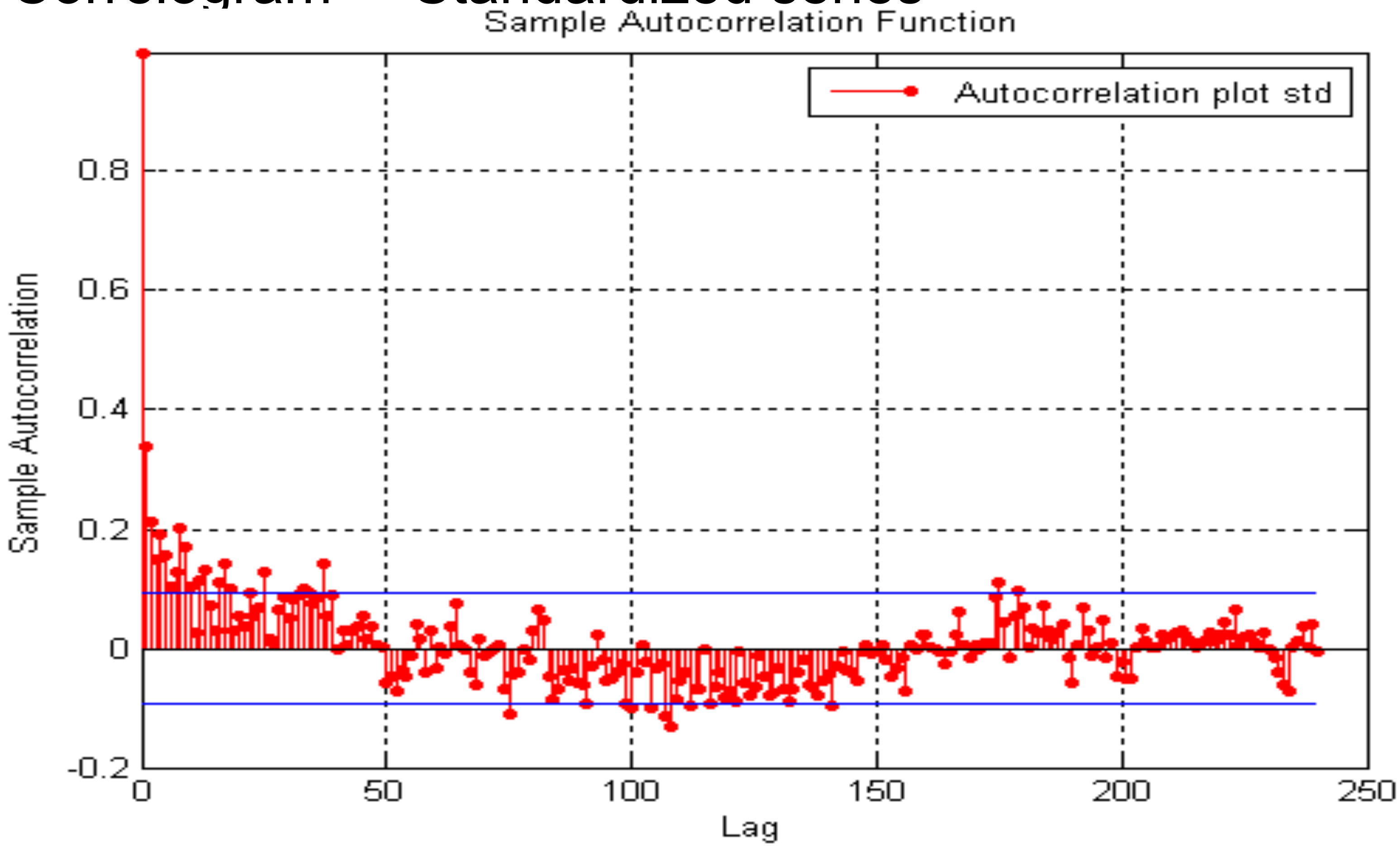
$Z_i = \frac{x_i - \bar{x}_i}{S_i}$

Time series plot – Standardized series



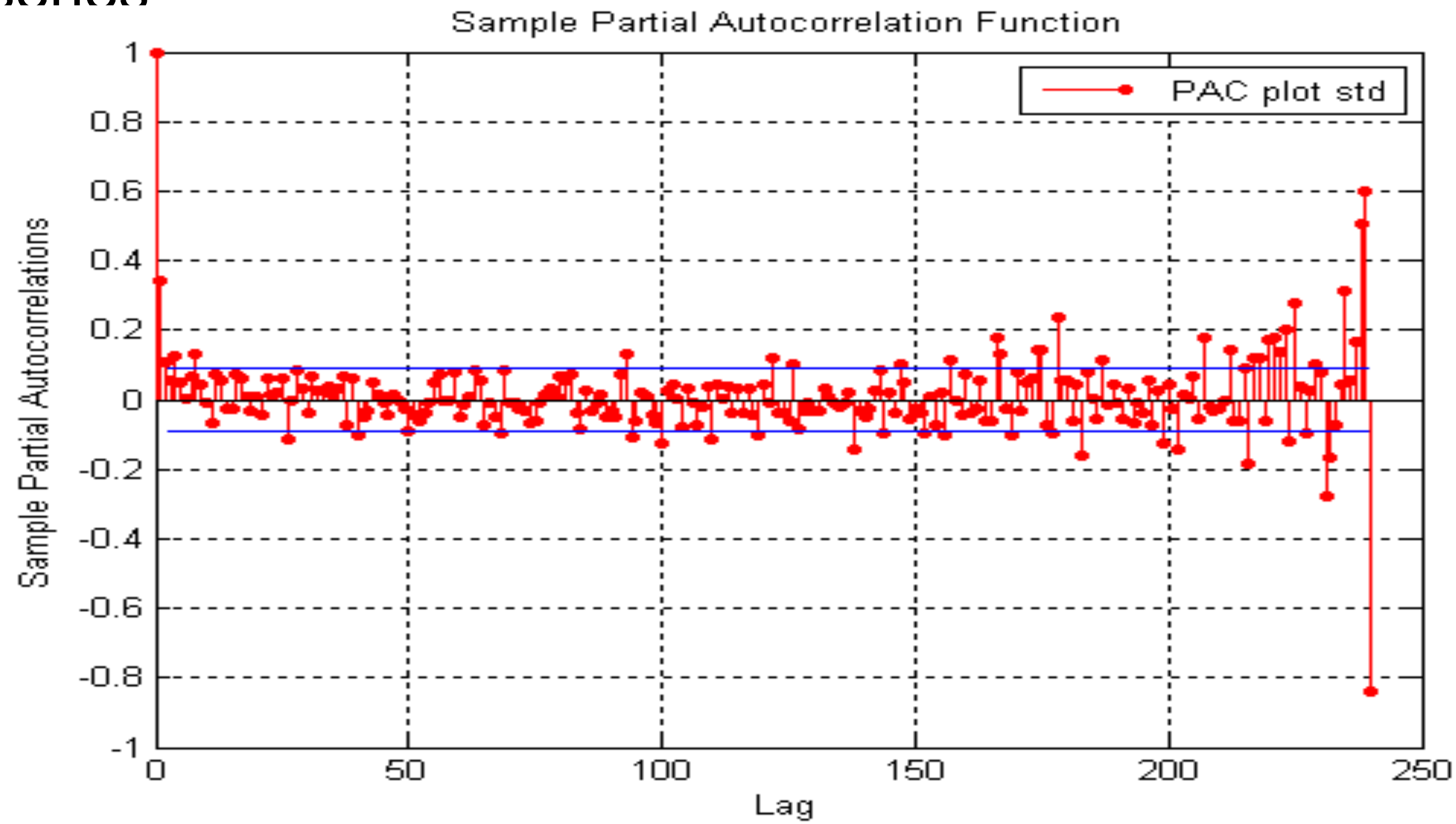
Case study – 3 (Contd.)

Correlogram – Standardized series



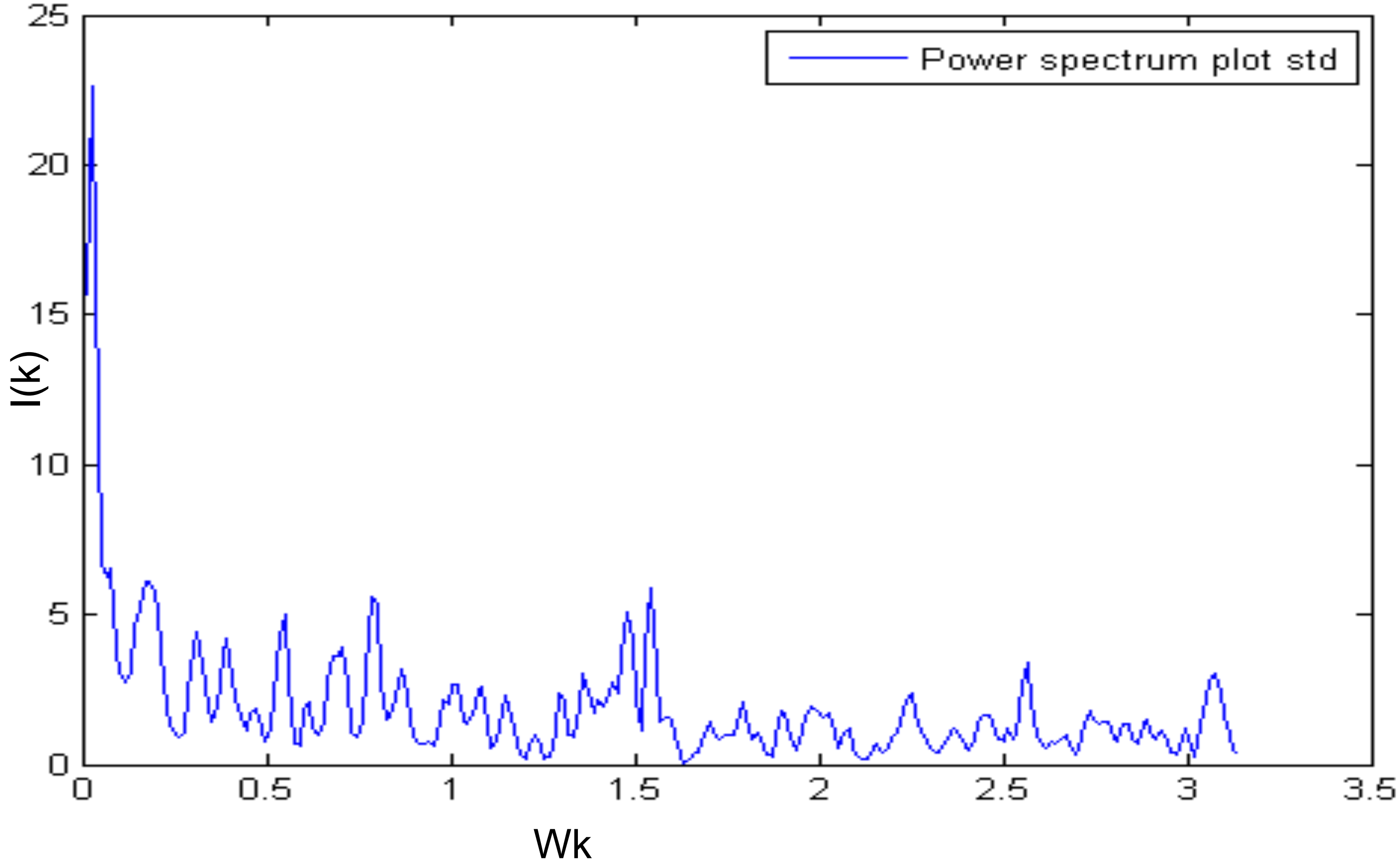
Case study – 3 (Contd.)

Partial auto correlation function – Standardized series



Case study – 3 (Contd.)

Power spectrum – Standardized series



Case study – 3 (Contd.)

- Standardized series is considered for fitting the ARMA models
- Total length of the data set N = 480
- Half the data set (240 values) is used to construct the model and other half is used for validation.
- Both contiguous and non-contiguous models are studied
- Non-contiguous models consider the most significant AR and MA terms leaving out the intermediate terms

Case study – 3 (Contd.)

- For example, a non-contiguous AR(3), with significant dependence at lags 1, 4 and 12, the model is written as

$$X_t = \phi_1 X_{t-1} + \phi_4 X_{t-4} + \phi_{12} X_{t-12} + e_t$$

- Similarly the moving average terms are also considered and a non-contiguous ARMA(3, 3) is written as

$$X_t = \phi_1 X_{t-1} + \phi_4 X_{t-4} + \phi_{12} X_{t-12} +$$
$$\theta_1 e_{t-1} + \theta_4 e_{t-4} + \theta_{12} e_{t-12} + e_t$$

Case study – 3 (Contd.)

- The advantage of non-contiguous models is the reduction of number of AR and MA parameters to be estimated.
- Exactly which terms to include is to be decided based on the correlogram and spectral analysis of the series under consideration.
- For a given series, the choice of contiguous or a non-contiguous model is decided by the relative likelihood values for the two models

Case study – 3 (Contd.)

Contiguous models:

$$L_i = -\frac{N}{2}\ln\left(\sigma_i\right) - n_i$$

Sl. No	Model	Likelihood values
1	ARMA(1,0)	29.33
2	ARMA(2,0)	28.91
3	ARMA(3,0)	28.96
4	ARMA(4,0)	31.63
5	ARMA(5,0)	30.71
6	ARMA(6,0)	29.90
7	ARMA(1,1)	30.58
8	ARMA(1,2)	29.83
9	ARMA(2,1)	29.83
10	ARMA(2,2)	28.80
11	ARMA(3,1)	29.45

Case study – 3 (Contd.)

Non-contiguous models:

Sl. No	Model	Likelihood values
1	ARMA(2,0)	28.52
2	ARMA(3,0)	28.12
3	ARMA(4,0)	28.21
4	ARMA(5,0)	30.85
5	ARMA(6,0)	29.84
6	ARMA(7,0)	29.12
7	ARMA(2,2)	29.81
8	ARMA(2,3)	28.82
9	ARMA(3,2)	28.48
10	ARMA(3,3)	28.06
11	ARMA(4,2)	28.65

Case study – 3 (Contd.)

- For this time series, the likelihood values for
  - contiguous model = 31.63
  - non-contiguous model = 30.85
- Hence contiguous ARMA(4,0) can be used.
- The parameters for the selected model are as follows

  ϕ_1 = 0.2137

  ϕ_2 = 0.0398

  ϕ_3 = 0.054

  ϕ_4 = 0.1762

  Constant = -0.0157

Case study – 3 (Contd.)
Forecasting Models

Contiguous models:

Sl. No	Model	Mean square error values
1	ARMA(1,0)	0.97
2	ARMA(2,0)	1.92
3	ARMA(3,0)	2.87
4	ARMA(4,0)	3.82
5	ARMA(5,0)	4.78
6	ARMA(6,0)	5.74
7	ARMA(1,1)	2.49
8	ARMA(1,2)	2.17
9	ARMA(2,1)	3.44
10	ARMA(2,2)	4.29
11	ARMA(3,1)	1.89

Case study – 3 (Contd.)

Non-contiguous models:

Sl. No	Model	Mean square error values
1	ARMA(2,0)	0.96
2	ARMA(3,0)	1.89
3	ARMA(4,0)	2.84
4	ARMA(5,0)	3.79
5	ARMA(6,0)	4.74
6	ARMA(7,0)	5.7
7	ARMA(2,2)	2.42
8	ARMA(2,3)	1.99
9	ARMA(3,2)	2.52
10	ARMA(3,3)	1.15
11	ARMA(4,2)	1.71

Case study – 3 (Contd.)

- The simplest model AR(1) results in the least value of the MSE
- For one step forecasting, quite often the simplest model is appropriate
- Also as the number of parameters increases, the MSE increases which is contrary to the common belief that models with large number of parameters give better forecasts.
- AR(1) model is recommended for forecasting the series and the parameters are as follows

  $\phi_1 = 0.2557$ and $C = -0.009$

Case study – 3 (Contd.)

- Validation tests on the residual series
  - Significance of residual mean
  - Significance of periodicities
  - Cumulative periodogram test or Bartlett's test
  - White noise test
    - Whittle's test
    - Portmanteau test

- Residuals, $$e_t = X_t - \underbrace{\left(\sum_{j=1}^{m_1} \phi_j X_{t-j} + \sum_{j=1}^{m_2} \theta_j e_{t-j} \right)}_{\text{Simulated from the model}}$$

  e_t: Residual; X_t: Data

Case study – 3 (Contd.)

Significance of residual mean:

Sl. No	Model	$\eta(e)$	$t_{0.95}(239)$
1	ARMA(1,0)	0.002	1.645
2	ARMA(2,0)	0.006	1.645
3	ARMA(3,0)	0.008	1.645
4	ARMA(4,0)	0.025	1.645
5	ARMA(5,0)	0.023	1.645
6	ARMA(6,0)	0.018	1.645
7	ARMA(1,1)	0.033	1.645
8	ARMA(1,2)	0.104	1.645
9	ARMA(2,1)	0.106	1.645
10	ARMA(2,2)	0.028	1.645

$$\eta(e) = \frac{N^{1/2}\overline{e}}{\hat{\rho}^{1/2}}$$

$\eta(e) \leq t(0.95, 240-1)$; All models pass the test

Case study – 3 (Contd.)

Significance of periodicities:

$$\eta(e) = \frac{\gamma_k^2 (N-2)}{4\hat{\rho}_1}$$

$\gamma_k^2 = \alpha_k^2 + \beta_k^2$

$$\hat{\rho}_1 = \frac{1}{N}\left[\sum_{t=1}^{N}\left\{e_t - \hat{\alpha}\cos(\omega_k t) - \hat{\beta}\sin(\omega_k t)\right\}^2\right]$$

$$\alpha_k = \frac{2}{N}\sum_{t=1}^{n} e_t \cos(\omega_k t)$$

Case study – 3 (Contd.)

$$\beta_k = \frac{2}{N}\sum_{t=1}^{n} e_t \sin(\omega_k t)$$

$2\pi/\omega_k$ is the periodicity for which test is being carried out.

$\eta(e) \leq F_{\alpha}(2, N-2)$ – Model passes the test

Case study – 3 (Contd.)

Significance of periodicities:

Sl. No	Model	η Value for the periodicity				$F_{0.95}(2,238)$
		1st	2nd	3rd	4th	
1	ARMA(1,0)	0.527	1.092	0.364	0.065	3.00
2	ARMA(2,0)	1.027	2.458	0.813	0.129	3.00
3	ARMA(3,0)	1.705	4.319	1.096	0.16	3.00
4	ARMA(4,0)	3.228	6.078	0.948	0.277	3.00
5	ARMA(5,0)	3.769	7.805	1.149	0.345	3.00
6	ARMA(6,0)	4.19	10.13	1.262	0.441	3.00
7	ARMA(1,1)	4.737	10.09	2.668	0.392	3.00
8	ARMA(1,2)	6.786	10.67	2.621	0.372	3.00
9	ARMA(2,1)	7.704	12.12	2.976	0.422	3.00
10	ARMA(2,2)	6.857	13.22	3.718	0.597	3.00

Case study – 3 (Contd.)

Significance of periodicities by Bartlett's test :
(Cumulative periodogram test)

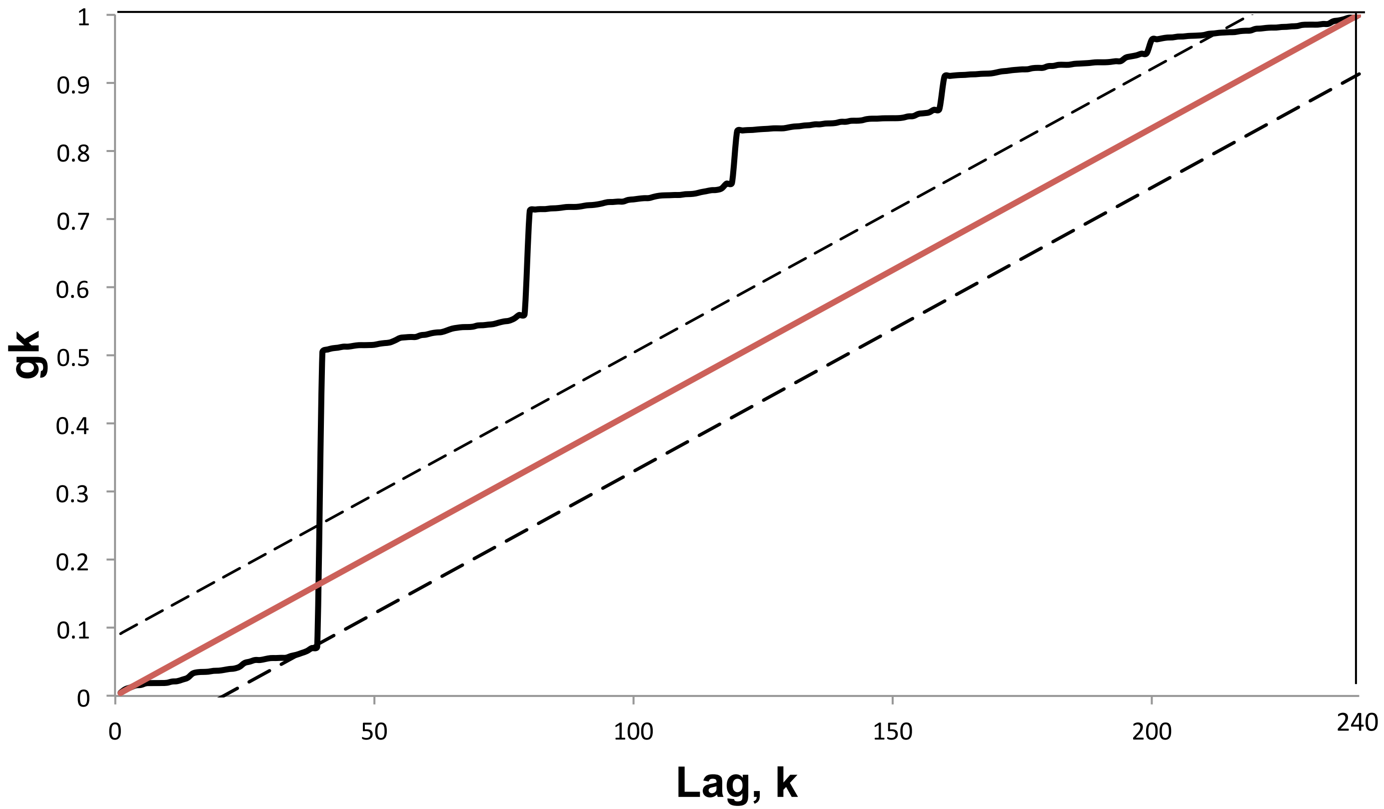
$$\gamma_k^2 = \left\{ \frac{2}{N} \sum_{t=1}^{N} e_t \cos\left(\omega_k t\right) \right\}^2 + \left\{ \frac{2}{N} \sum_{t=1}^{N} e_t \sin\left(\omega_k t\right) \right\}^2$$

k = 1,2,……N/2

$$g_k = \frac{\sum_{j=1}^{k} \gamma_j^2}{\sum_{k=1}^{N/2} \gamma_k^2}$$

Case study – 3 (Contd.)

Cumulative periodogram for the original series without standardizing

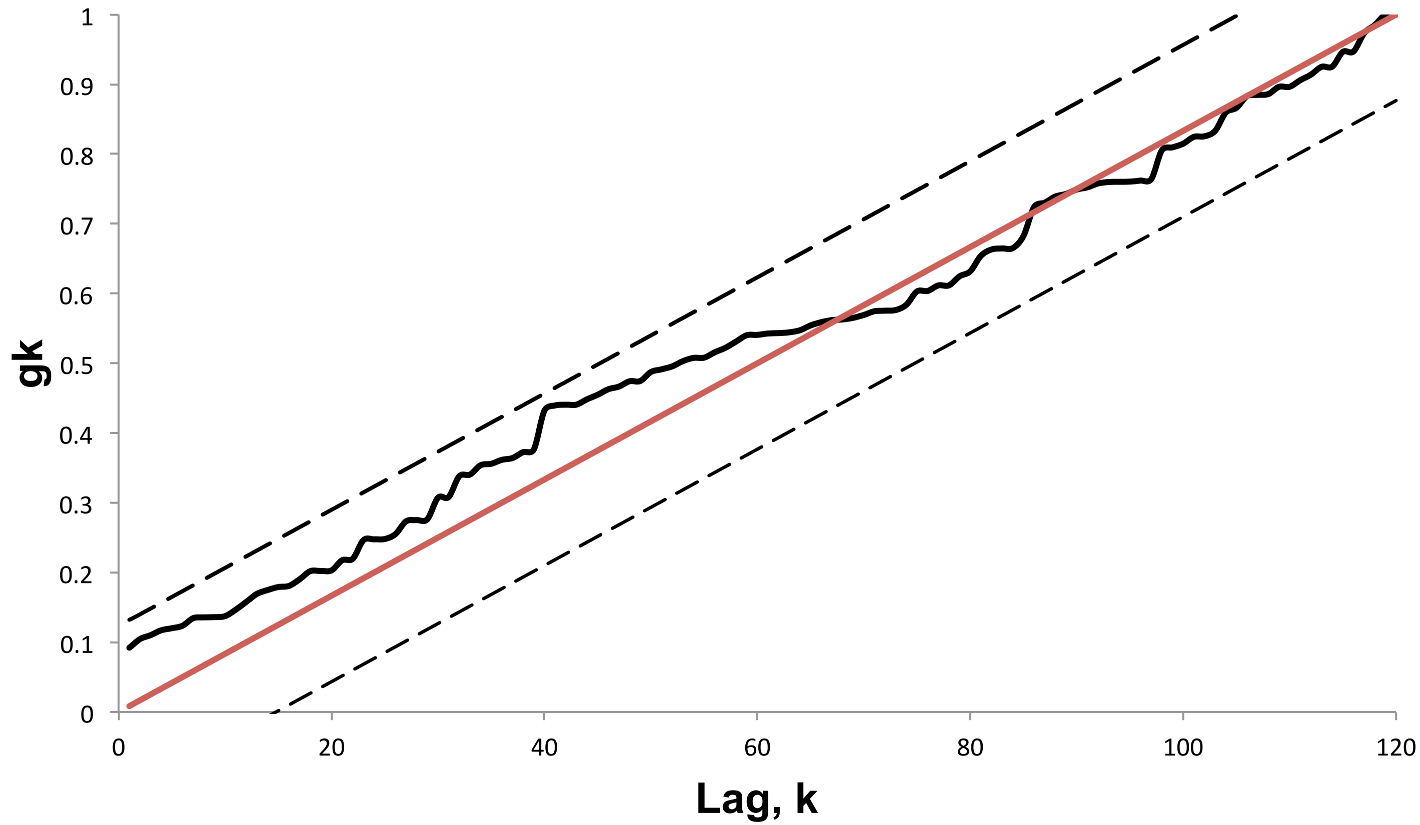


Case study – 3 (Contd.)

- The confidence limits ($\pm 1.35/(N/2)^{1/2} = \pm\ 0.087$) are plotted for 95% confidence.
- Cumulative periodogram lies outside the significance bands confirming the presence of periodicity in the data.
- for k=40, sudden increase in the graph is seen indicating the significant periodicity
- This 'k' corresponds to a periodicity of 12 months (480/40)
- k=80, corresponds to a periodicity of 6 months

Case study – 3 (Contd.)

Cumulative periodogram for the residual series of ARMA(4, 0) model



Case study – 3 (Contd.)

- The confidence limits ($\pm 1.35/(N/2)^{1/2} = \pm 0.123$) are plotted for 95% confidence.
- Cumulative periodogram lies within the significance bands confirming that no significant periodicity present in the residual series.
- The model pass the test.

Case study – 3 (Contd.)

Whittle's test for white noise: $\eta(e) = \frac{N}{n1-1}\left(\frac{\hat{\rho}_0}{\hat{\rho}_1} - 1\right)$

Model \ $F_{0.95}(2,239)$	n1 = 73 1.29 η	n1 = 49 1.39 η	n1 = 25 1.52 η
ARMA(1,0)	0.642	0.917	0.891
ARMA(2,0)	0.628	0.898	0.861
ARMA(3,0)	0.606	0.868	0.791
ARMA(4,0)	0.528	0.743	0.516
ARMA(5,0)	0.526	0.739	0.516
ARMA(6,0)	0.522	0.728	0.493
ARMA(1,1)	0.595	0.854	0.755
ARMA(1,2)	0.851	1.256	1.581
ARMA(2,1)	0.851	1.256	1.581
ARMA(2,2)	0.589	0.845	0.737

(shaded) model fails

Case study – 3 (Contd.)

$$\eta(e) = (N - n1)\sum_{k=1}^{n1}\left(\frac{r_k}{r_0}\right)^2$$

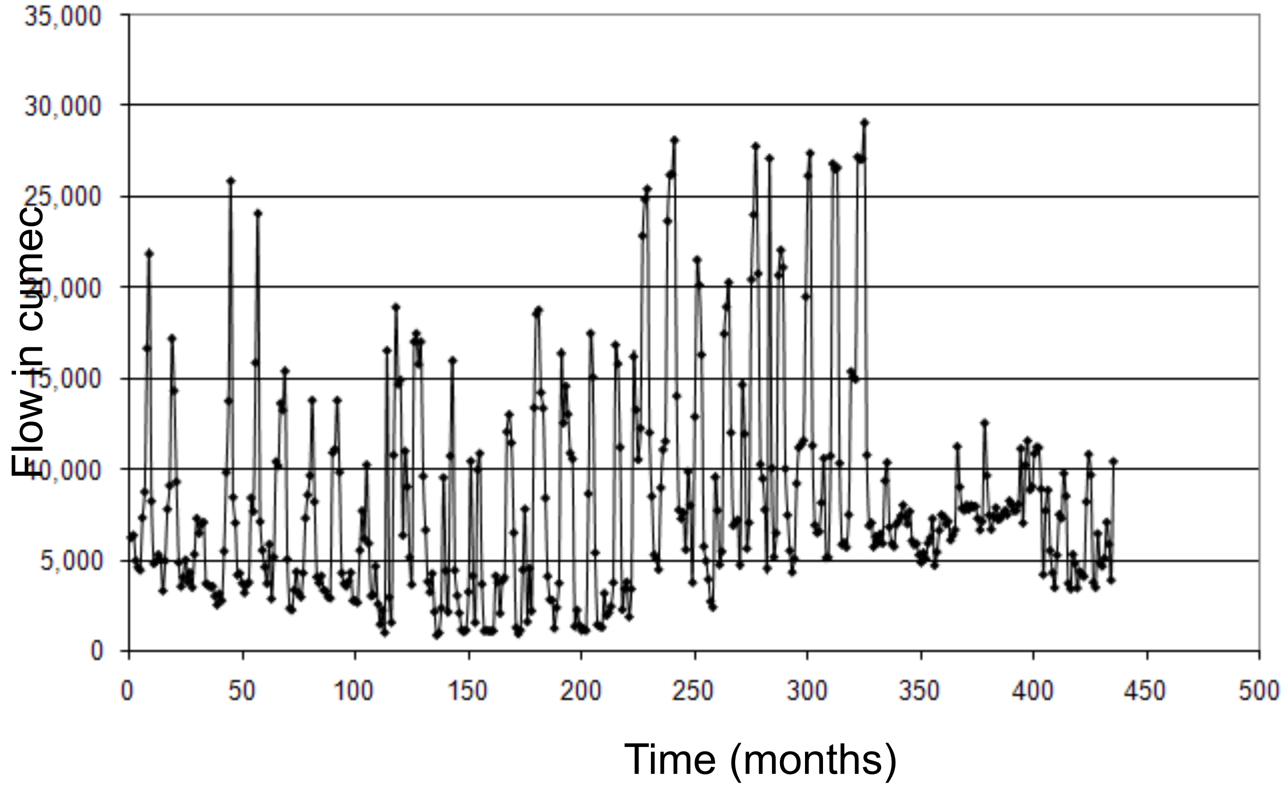
Portmanteau test for white noise:

Model \ $\chi^2_{0.95}(k_{max})$	kmax = 48 65.0 η	kmax = 36 50.8 η	kmax = 24 36.4 η	kmax = 12 21.0 η
ARMA(1,0)	31.44	33.41	23.02	14.8
ARMA(2,0)	32.03	34.03	24.47	15.17
ARMA(3,0)	30.17	32.05	21.61	13.12
ARMA(4,0)	20.22	21.49	11.85	4.31
ARMA(5,0)	19.84	21.08	11.75	4.14
ARMA(6,0)	19.64	20.87	11.48	3.79
ARMA(1,1)	29.89	31.76	22.24	12.76
ARMA(1,2)	55.88	59.38	48.37	39.85
ARMA(2,1)	55.88	59.38	48.37	38.85
ARMA(2,2)	28.62	30.41	20.39	11.25

model fails (shaded cells: ARMA(1,2) and ARMA(2,1) at kmax = 36, 24, 12)

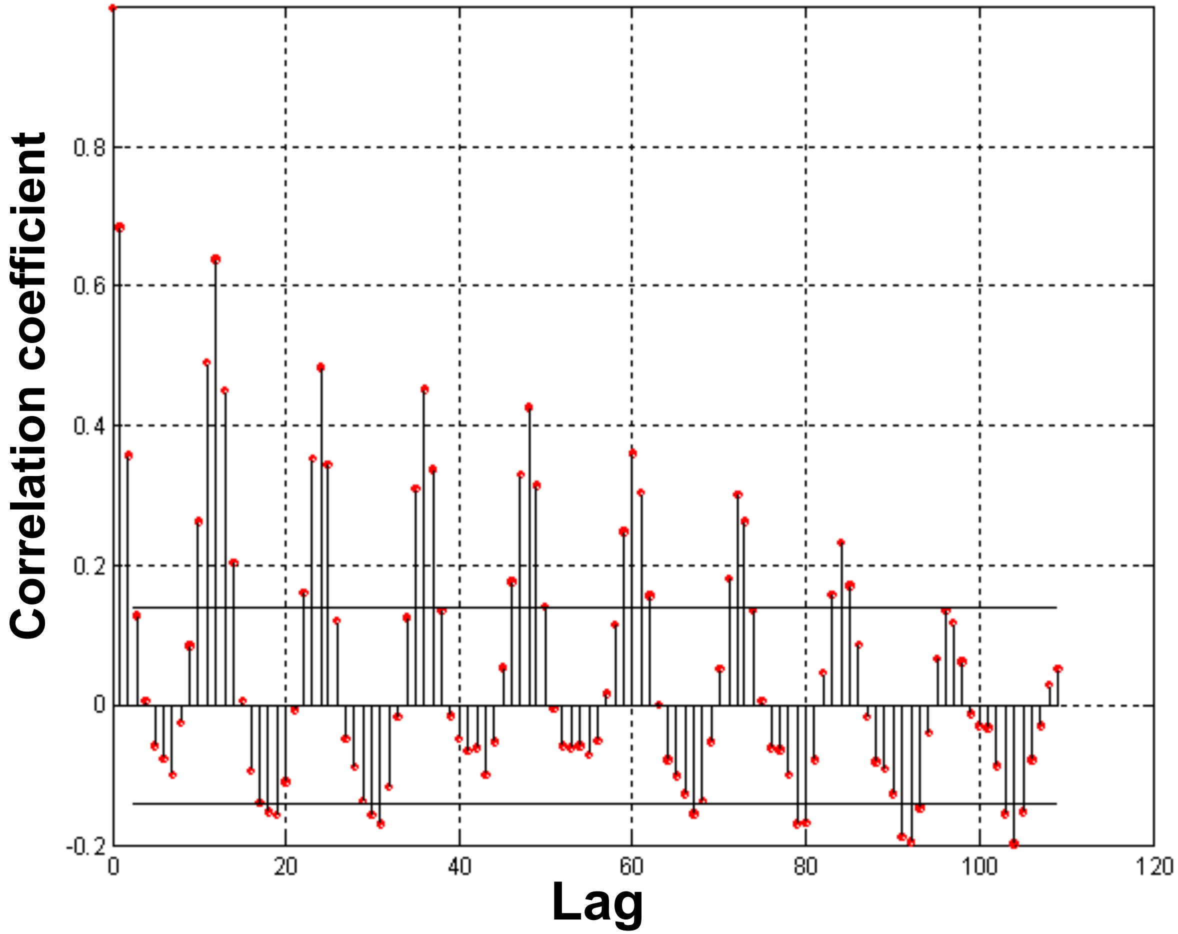
Case study – 4

Monthly Stream flow (1928-1964) of a river;



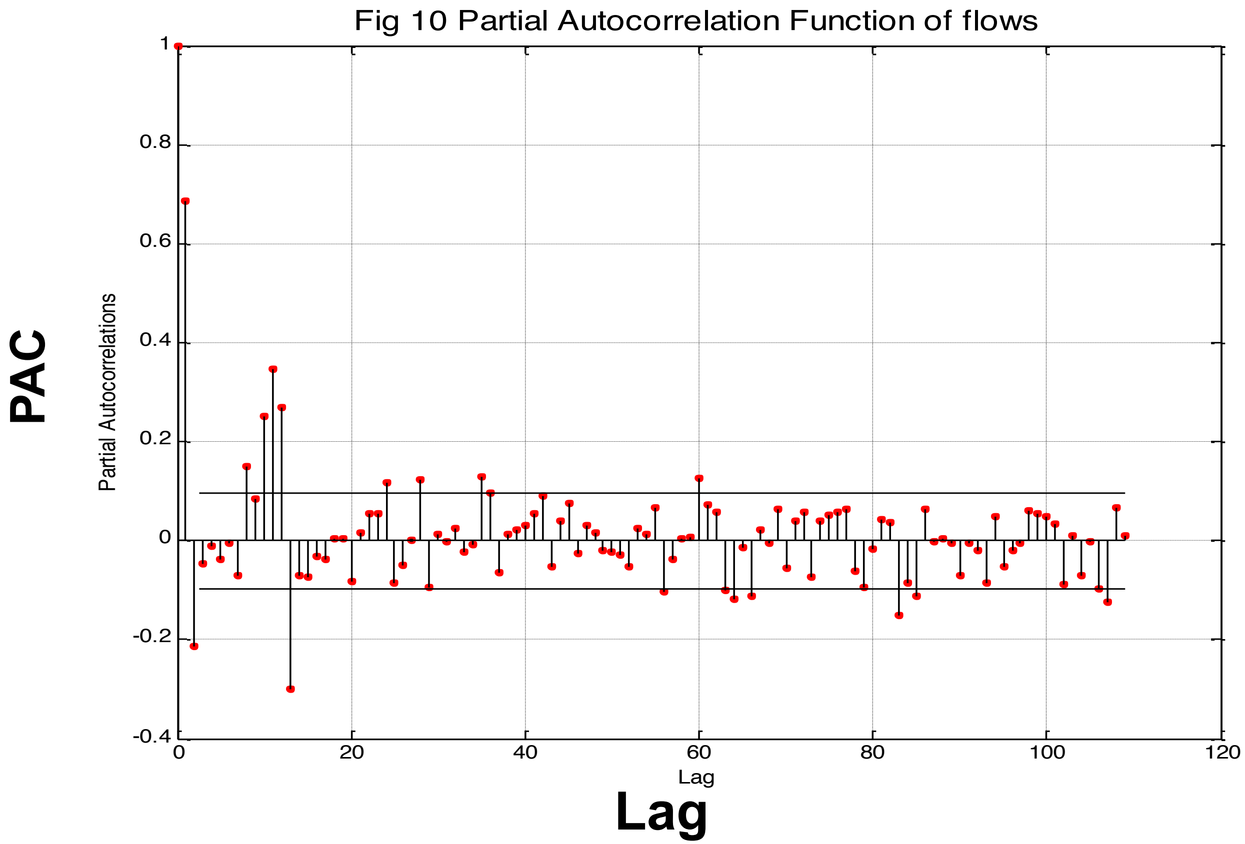
Case study – 4 (Contd.)

- Correlogram



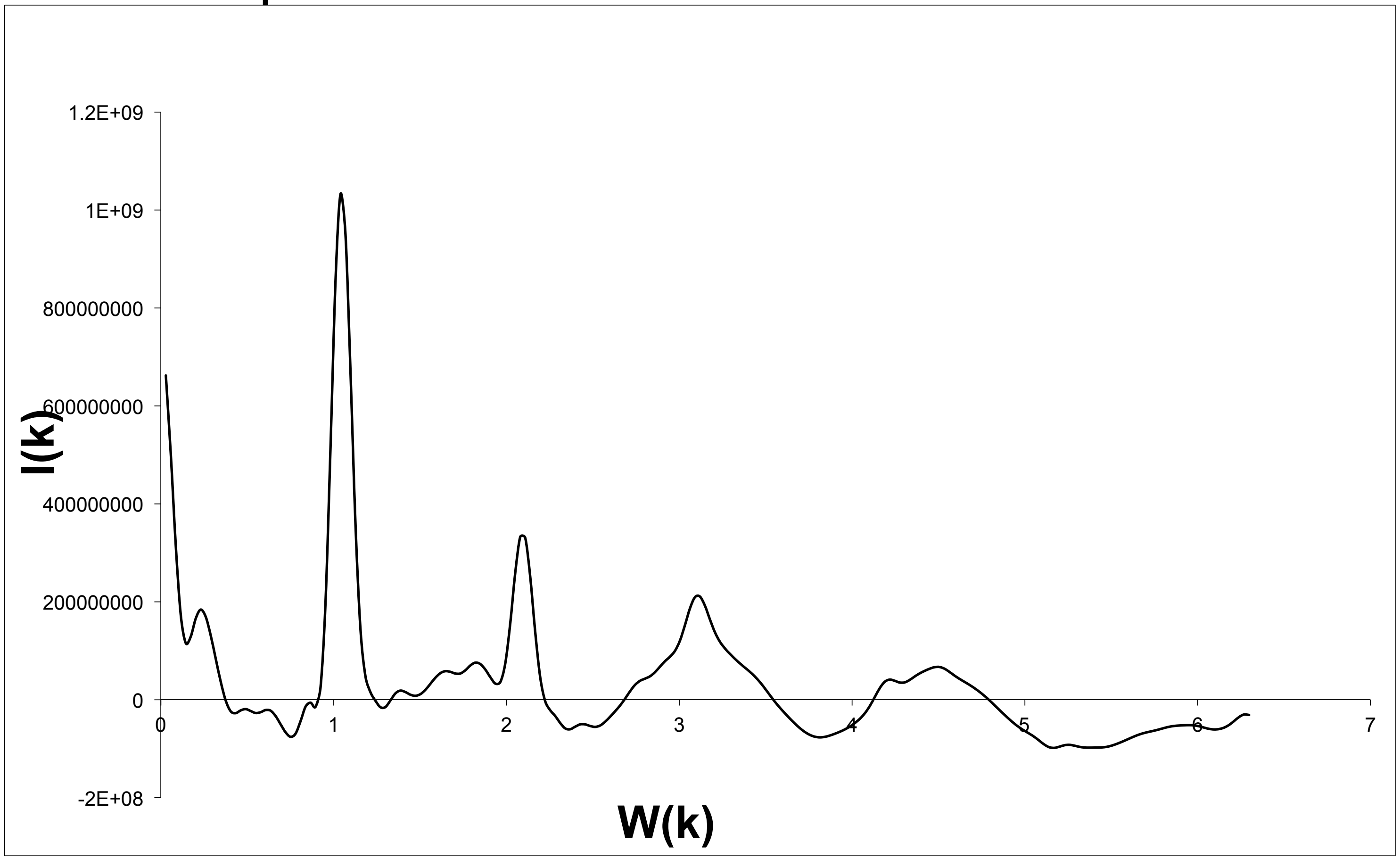
Case study – 4 (Contd.)

- Partial Auto Correlation function



Case study – 4 (Contd.)

- Power Spectrum



Case study – 4 (Contd.)

Model name	constant	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	θ_1	θ_2
ARMA(1,0)	-0.097	0.667									
ARMA(2,0)	0.049	0.042	0.044								
ARMA(3,0)	-0.111	0.767	-0.148	-0.003							
ARMA(4,0)	0.052	0.042	0.058	0.063	0.044						
ARMA(5,0)	0.055	0.042	0.058	0.063	0.048	0.038					
ARMA(6,0)	-0.124	0.764	-0.155	0.034	-0.029	-0.023	-0.021				
ARMA(7,0)	0.056	0.043	0.062	0.065	0.048	0.058	0.075	0.065			
ARMA(8,0)	0.054	0.042	0.060	0.063	0.048	0.058	0.078	0.088	0.061		
ARMA(1,1)	-0.131	0.551								0.216	
ARMA(2,1)	-0.104	0.848	-0.204							-0.083	
ARMA(3,1)	-0.155	0.351	0.165	-0.055						0.418	
ARMA(4,1)	-0.083	1.083	-0.400	0.091	-0.060					-0.318	
ARMA(1,2)	-0.139	0.526								0.241	0.025
ARMA(2,2)	378	1980	1160							1960	461
ARMA(3,2)	577000	30600	38800	32900						7350	6290
ARMA(0,1)	-0.298									0.594	
ARMA(0,2)	-0.297									0.736	0.281

Case study – 4 (Contd.)

Sl. No	Model	Mean Square Error	Likelihood value
1	ARMA(1,0)	0.65	93.33
2	ARMA(2,0)	0.63	97.24
3	ARMA(3,0)	0.63	96.44
4	ARMA(4,0)	0.63	96.14
5	ARMA(5,0)	0.63	95.50
6	ARMA(6,0)	0.63	94.66
7	ARMA(7,0)	0.63	93.80
8	ARMA(8,0)	0.60	101.47
9	ARMA(1,1)	0.63	97.11
10	ARMA(2,1)	0.63	96.25
11	ARMA(3,1)	0.63	95.39
12	ARMA(4,1)	0.63	95.39
13	ARMA(1,2)	0.63	96.16
14	ARMA(2,2)	0.63	95.13
15	ARMA(3,2)	229.86	-777.70
16	ARMA(0,1)	0.73	67.74
17	ARMA(0,2)	0.66	89.28

Case study – 4 (Contd.)

Significance of residual mean:

$$\eta(e) = \frac{N^{1/2}\overline{e}}{\hat{\rho}^{1/2}}$$

$\overline{e}$ is the estimate of the residual mean

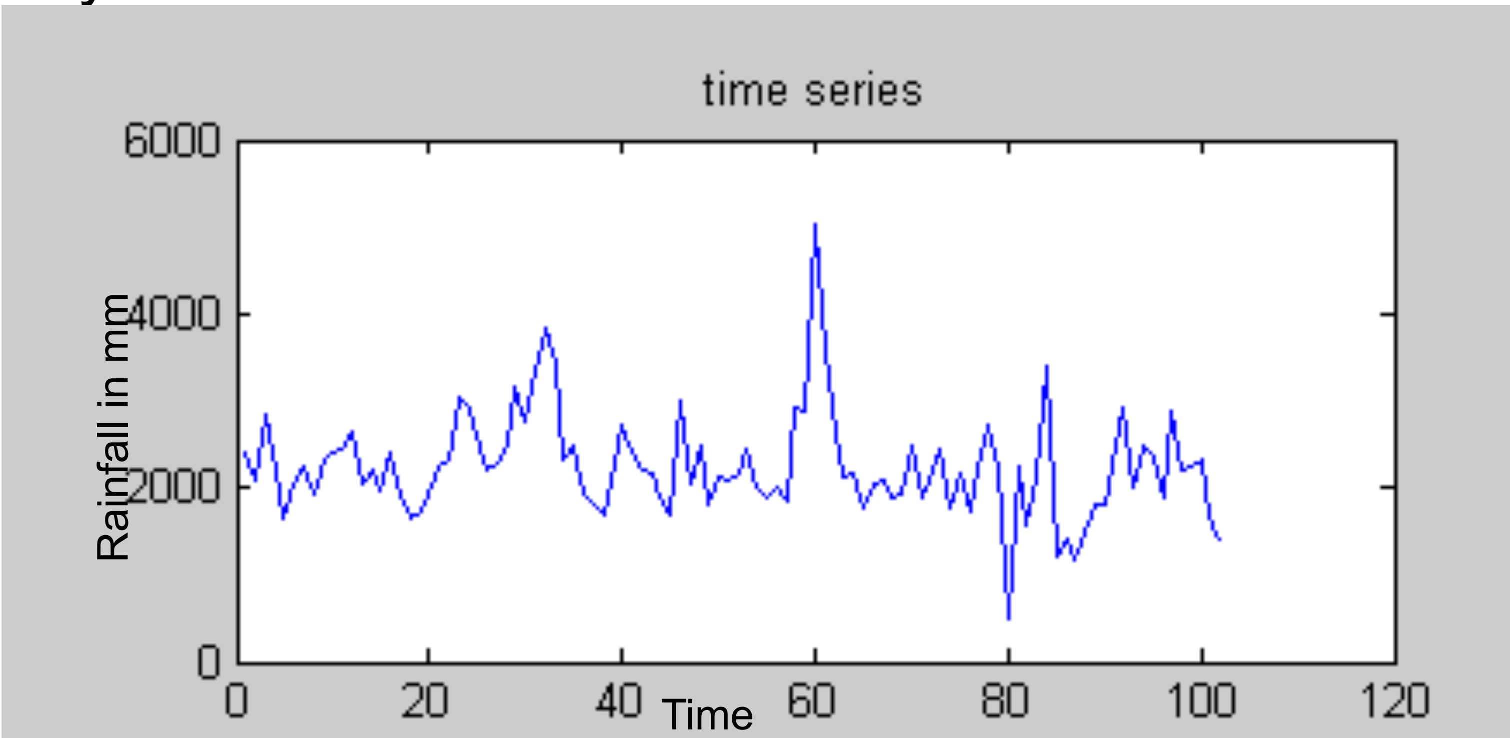
$\hat{\rho}$ is the estimate of the residual variance

- All models pass the test

Model	Test t	
	$\eta(e)$	$t(\alpha, N-1)$
ARMA(1,0)	1.78672E-05	1.645
ARMA(2,0)	6.0233E-06	1.645
ARMA(3,0)	4.82085E-05	1.645
ARMA(4,0)	-3.01791E-05	1.645
ARMA(5,0)	6.84076E-16	1.645
ARMA(6,0)	3.0215E-05	1.645
ARMA(7,0)	-6.04496E-06	1.645
ARMA(8,0)	5.54991E-05	1.645
ARMA(1,1)	-0.001132046	1.645
ARMA(2,1)	-0.002650292	1.645
ARMA(3,1)	-0.022776166	1.645
ARMA(4,1)	0.000410668	1.645
ARMA(1,2)	-0.000837092	1.645
ARMA(2,2)	0.002631505	1.645
ARMA(3,2)	-49.36722756	1.645
ARMA(0,1)	0.022950466	1.645
ARMA(0,2)	0.019847826	1.645

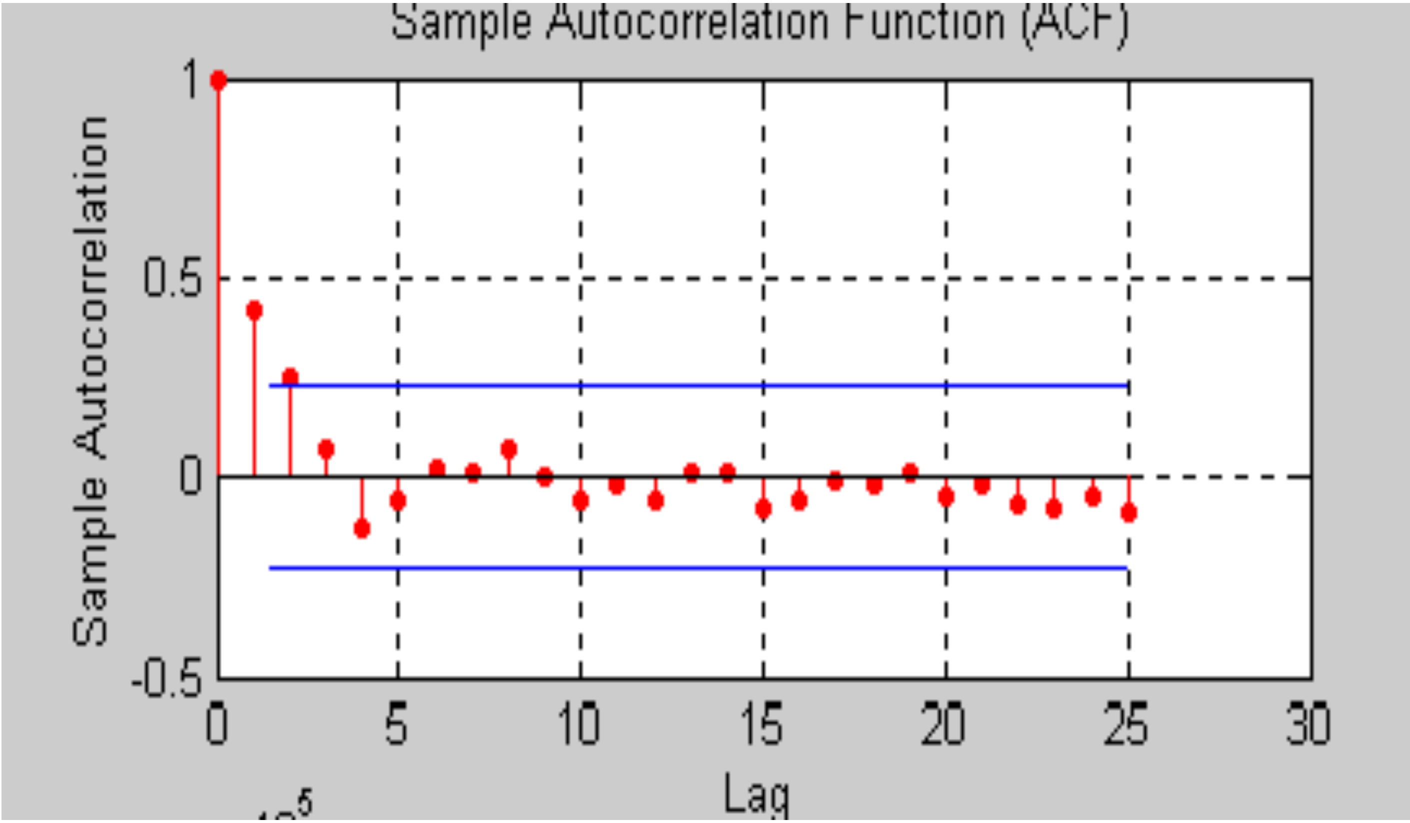
Case study – 5

Sakleshpur Rainfall Data is considered in the case study.



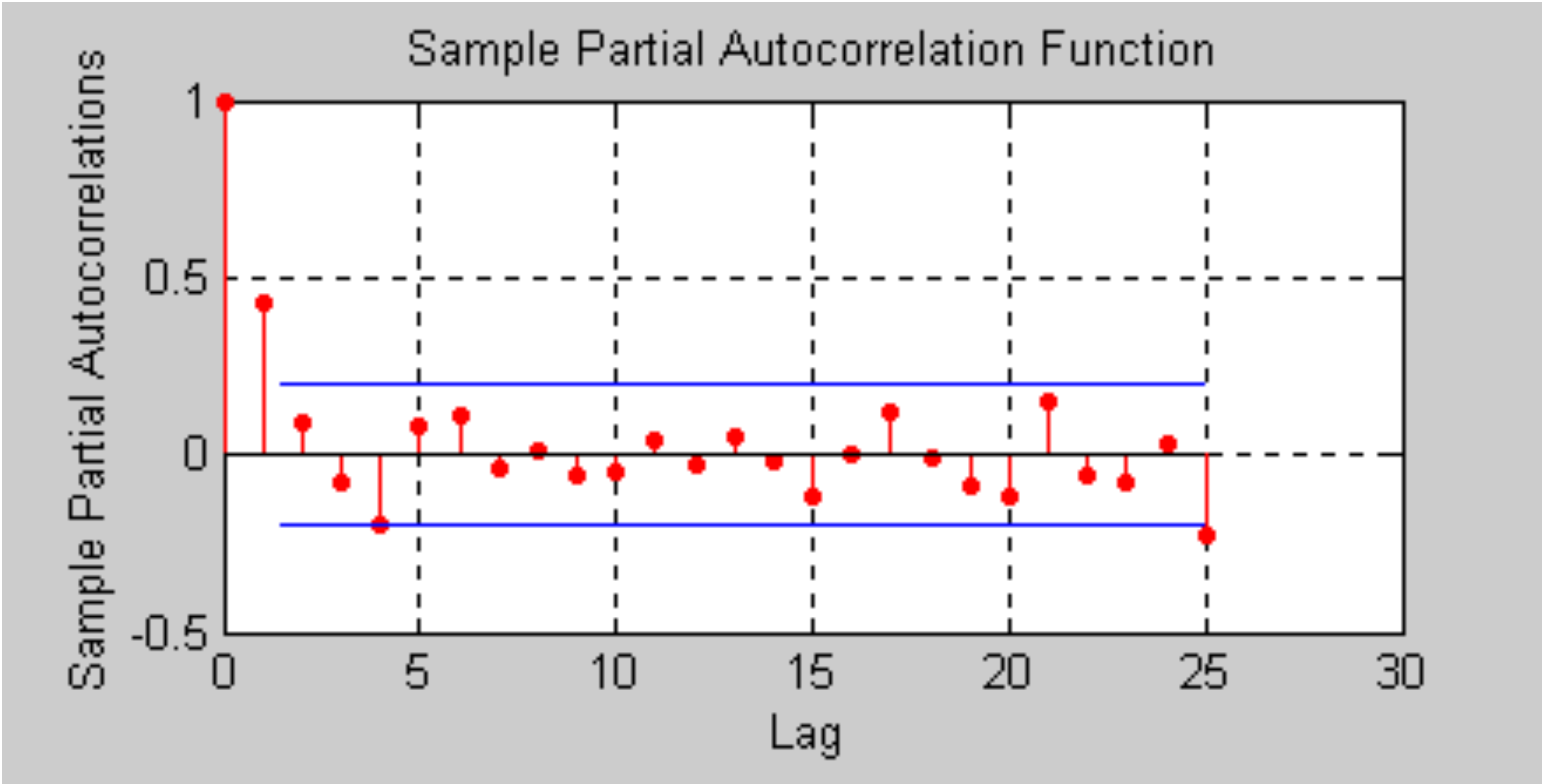
Case study – 5 (Contd.)

Correlogram

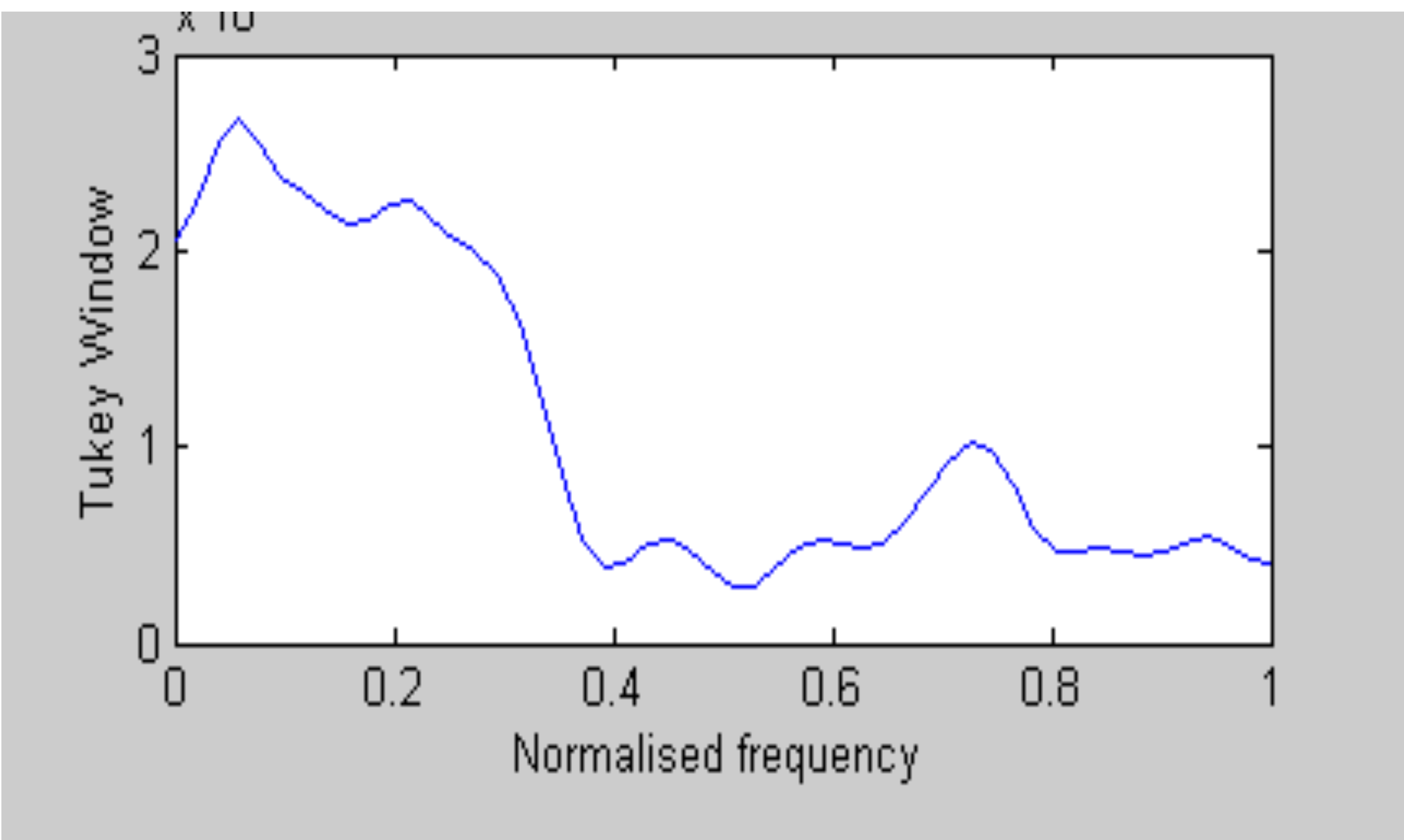


Case study – 5 (Contd.)

PAC function



Case study – 5 (Contd.)



Case study – 5 (Contd.)

Model	Likelihood
AR(1)	9.078037
AR(2)	8.562427
AR(3)	8.646781
AR(4)	9.691461
AR(5)	**9.821681**
AR(6)	9.436822
ARMA(1,1)	8.341717
ARMA(1,2)	8.217627
ARMA(2,1)	7.715415
ARMA(2,2)	5.278434
ARMA(3,1)	6.316174
ARMA(3,2)	6.390390

Case study – 5 (Contd.)

- ARMA(5,0) is selected with highest likelihood value
- The parameters for the selected model are as follows

ϕ_1 = 0.40499

ϕ_2 = 0.15223

ϕ_3 = -0.02427

ϕ_4 = -0.2222

ϕ_5 = 0.083435

Constant = -0.000664

Case study – 5 (Contd.)

- Significance of residual mean

Model	$\eta(e)$	$t_{0.95}(N\ )$
ARMA(5,0)	0.000005	1.6601

Case study – 5 (Contd.)

Significance of periodicities:

Periodicity	η	$F_{0.95}(2,239)$
1st	0.000	3.085
2nd	0.00432	3.085
3rd	0.0168	3.085
4th	0.0698	3.085
5th	0.000006	3.085
6th	0.117	3.085

Case study – 5 (Contd.)

- Whittle's white noise test:

Model	η	$F_{0.95}(2, N-2)$
ARMA(5,0)	0.163	1.783

Case study – 5 (Contd.)

Model	MSE
AR(1)	1.180837
AR(2)	1.169667
AR(3)	1.182210
AR(4)	1.168724
AR(5)	1.254929
AR(6)	1.289385
ARMA(1,1)	1.171668
ARMA(1,2)	**1.156298**
ARMA(2,1)	1.183397
ARMA(2,2)	1.256068
ARMA(3,1)	1.195626
ARMA(3,2)	27.466087

Case study – 5 (Contd.)

- ARMA(1, 2) is selected with least MSE value for one step forecasting
- The parameters for the selected model are as follows

  $\phi_1 = 0.35271$

  $\theta_1 = 0.017124$

  $\theta_2 = -0.216745$

  Constant = -0.009267

Case study – 5 (Contd.)

- Significance of residual mean

Model	$\eta(e)$	$t_{0.95}(N)$
ARMA(1, 2)	-0.0026	1.6601

Case study – 5 (Contd.)

Significance of periodicities:

Periodicity	η	$F_{0.95}(2,239)$
1st	0.000	3.085
2nd	0.0006	3.085
3rd	0.0493	3.085
4th	0.0687	3.085
5th	0.0003	3.085
6th	0.0719	3.085

Case study – 5 (Contd.)

- Whittle's white noise test:

Model	η	$F_{0.95}(2, N\text{-}2)$
ARMA(1, 2)	0.3605	1.783

Markov Chains

Markov Chains:

- Markov chain is a stochastic process with the property that value of process X_t at time t depends on its value at time t-1 and not on the sequence of other values (X_{t-2} , X_{t-3},……. X_0) that the process passed through in arriving at X_{t-1}.

$$P\left[X_t / X_{t-1}, X_{t-2}, \ldots\ldots X_0\right] = P\left[X_t / X_{t-1}\right]$$

Single step Markov chain

Markov Chains

$$P\left[X_t = a_j / X_{t-1} = a_i\right]$$

- The conditional probability gives the probability at time t will be in state 'j', given that the process was in state 'i' at time t-1.
- The conditional probability is independent of the states occupied prior to t-1.
- For example, if X_{t-1} is a dry day, what is the probability that X_t is a dry day or a wet day.
- This probability is commonly called as transitional probability

Markov Chains

$$P\left[X_t = a_j / X_{t-1} = a_i\right] = P_{ij}^t$$

- Usually written as P_{ij}^t indicating the probability of a step from a_i to a_j at time 't'.
- If P_{ij} is independent of time, then the Markov chain is said to be homogeneous.

i.e., $P_{ij}^t = P_{ij}^{t+\tau} \quad \forall \quad t$ and τ

the transitional probabilities remain same across time

Markov Chains

Transition Probability Matrix(TPM):

$$P = \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \\ . \\ . \\ m \end{array} \begin{array}{c} t+1 \rightarrow \\ t \downarrow \\ \left[\begin{array}{cccccc} P_{11} & P_{12} & P_{13} & . & . & P_{1m} \\ P_{21} & P_{22} & P_{23} & . & . & P_{2m} \\ P_{31} & & & & & \\ . & & & & & \\ . & & & & & \\ P_{m1} & P_{m2} & & & & P_{mm} \end{array}\right]_{m \times m} \end{array}$$

Column headings ($t+1$): 1, 2, 3, ., ., m; row headings (t): 1, 2, 3, ., ., m

Markov Chains

$$\sum_{j=1}^{m} P_{ij} = 1 \quad \forall \, j$$

- Elements in any row of TPM sum to unity (stochastic matrix)
- TPM can be estimated from observed data by tabulating the number of times the observed data went from state 'i' to 'j'
- $P_j^{(n)}$ is the probability of being in state 'j' in the time step 'n'.

Markov Chains

- $p_j^{(0)}$ is the probability of being in state 'j' in period t = 0.

$$p^{(0)} = \begin{bmatrix} p_1^{(0)} & p_2^{(0)} & . & . & p_m^{(0)} \end{bmatrix}_{1\times m}$$ …. Probability vector at time 0

$$p^{(n)} = \begin{bmatrix} p_1^{(n)} & p_2^{(n)} & . & . & p_m^{(n)} \end{bmatrix}_{1\times m}$$ …. Probability vector at time 'n'

- Let $p^{(0)}$ is given and TPM is given

$$p^{(1)} = p^{(0)} \times P$$

Markov Chains

$$p^{(1)} = \begin{bmatrix} p_1^{(0)} & p_2^{(0)} & . & . & p_m^{(0)} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} & . & . & P_{1m} \\ P_{21} & P_{22} & P_{23} & . & . & P_{2m} \\ P_{31} & & & & & \\ . & & & & & \\ P_{m1} & P_{m2} & & & & P_{mm} \end{bmatrix}$$

$$= p_1^{(0)} P_{11} + p_2^{(0)} P_{21} + ... + p_m^{(0)} P_{m1}$$ …. Probability of going to state 1

$$= p_1^{(0)} P_{12} + p_2^{(0)} P_{21} + ... + p_m^{(0)} P_{m2}$$ …. Probability of going to state 2

And so on…

Markov Chains

Therefore

$$p^{(1)} = \begin{bmatrix} p_1^{(1)} & p_2^{(1)} & . & . & p_m^{(1)} \end{bmatrix}_{1\times m}$$

$$\begin{aligned} p^{(2)} &= p^{(1)} \times P \\ &= p^{(0)} \times P \times P \\ &= p^{(0)} \times P^2 \end{aligned}$$

In general,

$$p^{(n)} = p^{(0)} \times P^n$$

Markov Chains

- As the process advances in time, $p_j^{(n)}$ becomes less dependent on $p^{(0)}$
- The probability of being in state 'j' after a large number of time steps becomes independent of the initial state of the process.
- The process reaches a steady state ay very large n

$$p = p \times P^n$$

- As the process reach steady state, TPM remains constant

Example – 1

Consider the TPM for a 2-state (state 1 is non-rainfall day and state 2 is rainfall day) first order homogeneous Markov chain as

$$TPM = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Obtain the

1. probability of day 1 is non-rainfall day / day 0 is rainfall day
2. probability of day 2 is rainfall day / day 0 is non-rainfall day
3. probability of day 100 is rainfall day / day 0 is non-rainfall day

Example – 1 (contd.)

1. probability of day 1 is non-rainfall day / day 0 is rainfall day

$$TPM = \begin{array}{c c} & \begin{array}{cc} \text{No rain} & \text{rain} \end{array} \\ \begin{array}{c} \text{No rain} \\ \text{rain} \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

The probability is 0.4

2. probability of day 2 is rainfall day / day 0 is non-rainfall day

$$p^{(2)} = p^{(0)} \times P^2$$

Example – 1 (contd.)

$$p^{(2)} = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.61 & 0.39 \end{bmatrix}$$

The probability is 0.39

3. probability of day 100 is rainfall day / day 0 is non-rainfall day

$$p^{(n)} = p^{(0)} \times P^n$$

Example – 1 (contd.)

$$P^2 = P \times P$$

$$= \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$$P^4 = P^2 \times P^2 = \begin{bmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{bmatrix}$$

$$P^8 = P^4 \times P^4 = \begin{bmatrix} 0.5715 & 0.4285 \\ 0.5714 & 0.4286 \end{bmatrix}$$

$$P^{16} = P^8 \times P^8 = \begin{bmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{bmatrix}$$

Example – 1 (contd.)

Steady state probability

$$p = \begin{bmatrix} 0.5714 & 0.4286 \end{bmatrix}$$

For steady state,

$$p = p \times P^n$$

$$= \begin{bmatrix} 0.5714 & 0.4286 \end{bmatrix} \begin{bmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5714 & 0.4286 \end{bmatrix}$$