Stochastic Structural Dynamics

Lecture-2

Scalar random variables-1

Dr C S Manohar
Department of Civil Engineering
Professor of Structural Engineering
Indian Institute of Science
Bangalore 560 012 India
manohar@civil.iisc.ernet.in

Recall

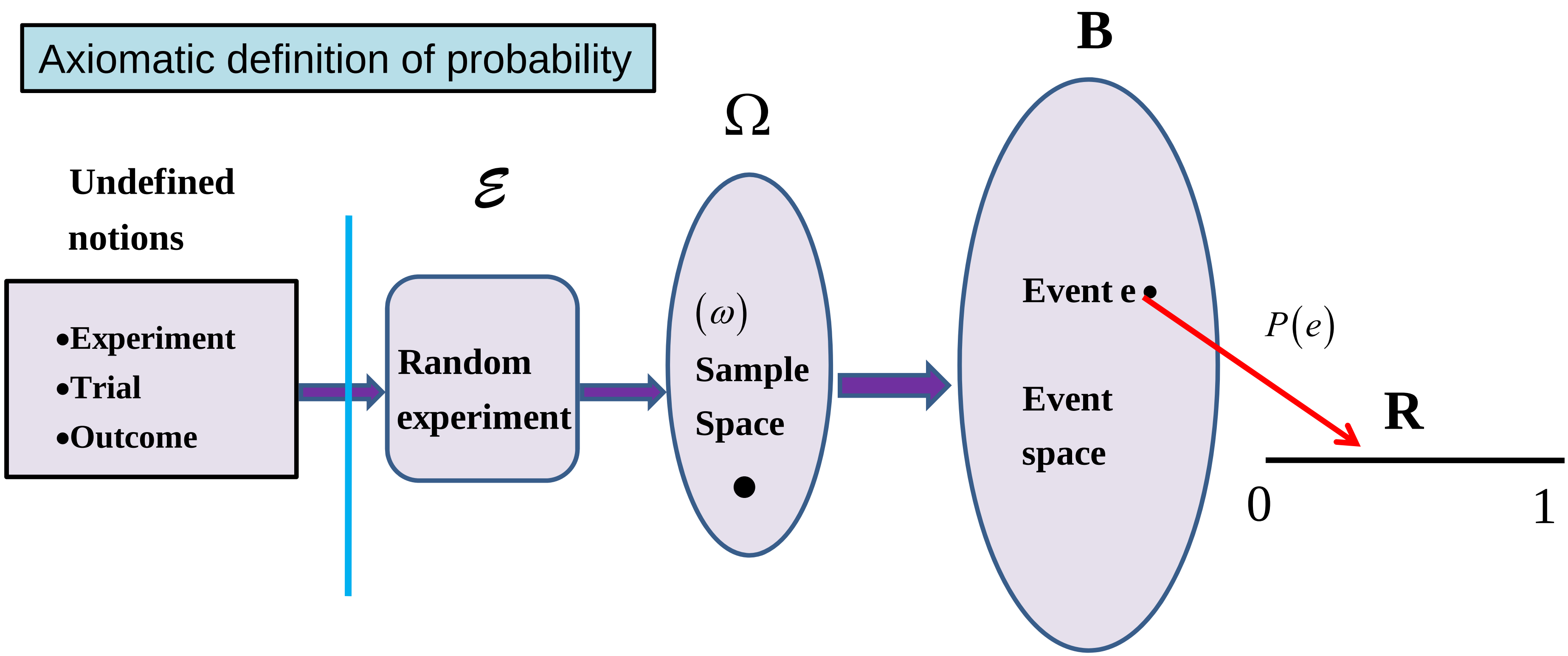
- Uncertainty modeling using theories of probability and random processes
- Definitions of probability
  - Classical definition $P(A)=m/n$
  - Relative frequency definition

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

  - Axiomatic definition

Recall (continued)

Axiomatic definition of probability



- Sample point: element of sample space
- Events are subsets of sample space on which we assign probability
- Axiomatic definition does not prescribe how to assign probability

Axioms

$P(A_i) \geq 0$

$P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B)$

if $A \cap B = \phi$

Recall (continued)

- Conditional Probability

Definition

$$P(A \mid B) = \text{Probability of event A given that B has occurred}$$

$$= \frac{\mathrm{P}(\mathrm{A} \cap \mathrm{B})}{P(B)}; P(B) \neq 0.$$

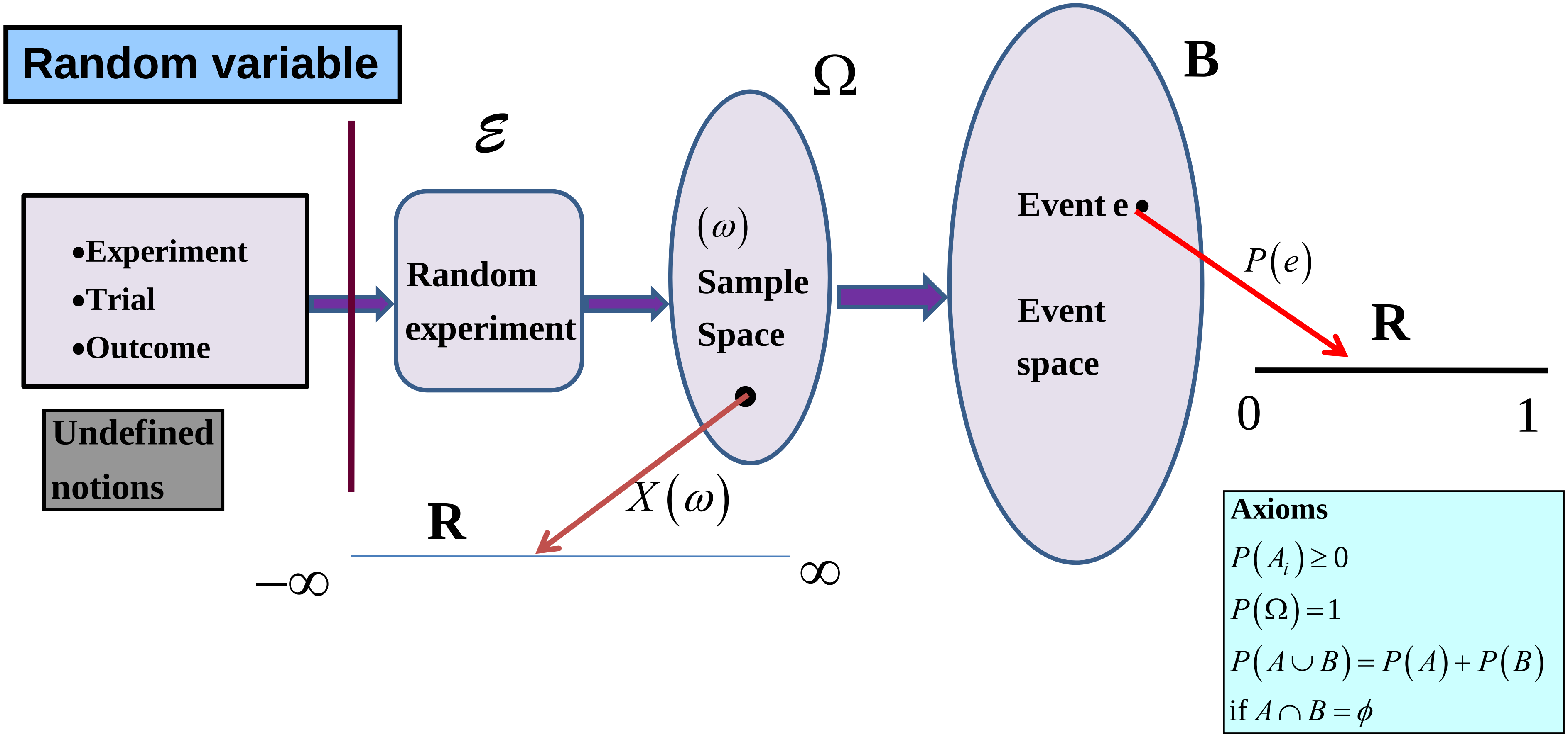
- Stochastic independence

Notation : A and B are independent

$$A \perp B \Rightarrow P(A \cap B) = P(A)P(B)$$

- Total probability theorem
- Bayes theorem

Random variable



Random variable is a function from sample space into real line such that

(1) for every $x \in R$, $\{\omega : X(\omega) \leq x\}$ is an event,

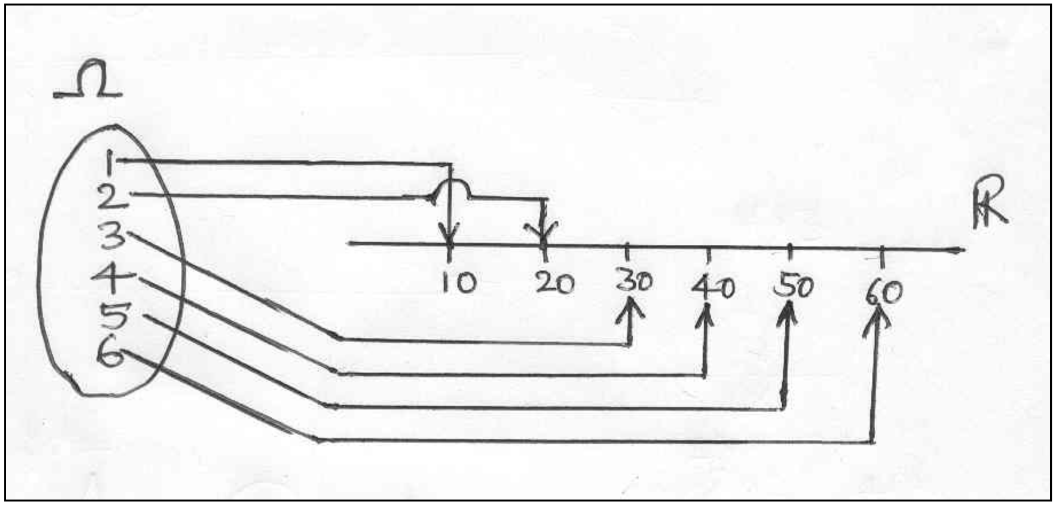
(2) $P(\omega : X(\omega) = \pm\infty) = 0$

Meaning of $\{X(\omega) \le x\}$

Example: Consider the experiment of die tossing.

$\Omega = \{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6\}$

Define $X(\omega_i) = 10i$



$\{X \le 40\} = \{1 \quad 2 \quad 3 \quad 4\}$

$\{X < 5\} = \phi$

$\{X \le 100\} = \Omega$

$\{20 \le X \le 50\} = \{2 \quad 3 \quad 4 \quad 5\}$

Observation

$\{X(\omega) \le x\}$ is a subset of Ω and hence an element of B and hence an event on which we assign probabilities.

We write $\{X(\omega) \le x\} = \{X \le x\}$.

Need for introducing the notion of a random variable

(a) Enables us to deal with Ω which is not numerically valued.

(b) Enables quantification of uncertainties.

(c) Forms the basis on which qunatities such as mean, standard deviation, covariance, etc., are defined.

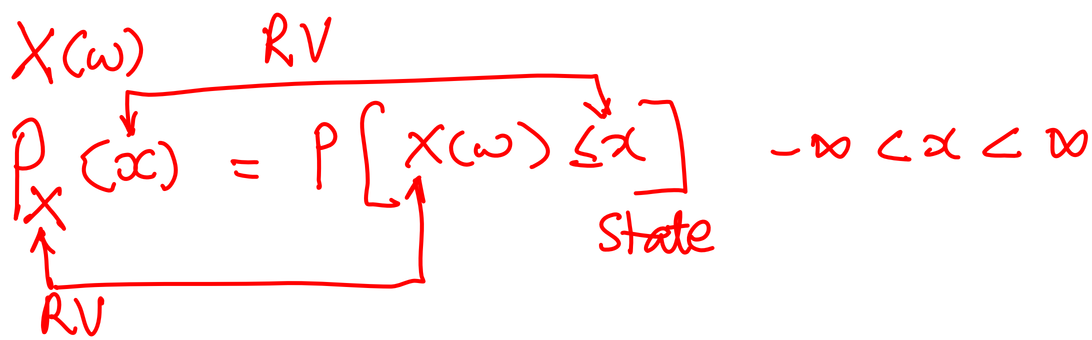
Probability Distribution Function (PDF) $\left[P_X(x)\right]$

Definition

$$P_X(x,\omega) = P\left\{X(\omega) \le x\right\} \; ; \; -\infty < x < \infty$$

Notation

$P_X(x,\omega) = P_X(x)$ (the dependence on ω is not explicitly displayed)



Properties

$(a)\ 0 \le P_X(x) \le 1$

$(b)\ P_X(\infty) = P\{X \le \infty\} = P(\Omega) = 1$

$(c)\ P_X(-\infty) = P\{X \le -\infty\} = P(\varphi) = 0$

$(d)\ x_2 > x_1 \Rightarrow P_X(x_2) \ge P_X(x_1)$

PDF is monotone nondecreasing.

Let $x_2 > x_1$.

$\Rightarrow \{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$

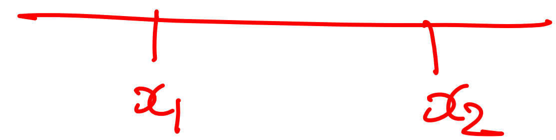
$\Rightarrow \{X \le x_1\} \cap \{x_1 < X \le x_2\} = \phi$

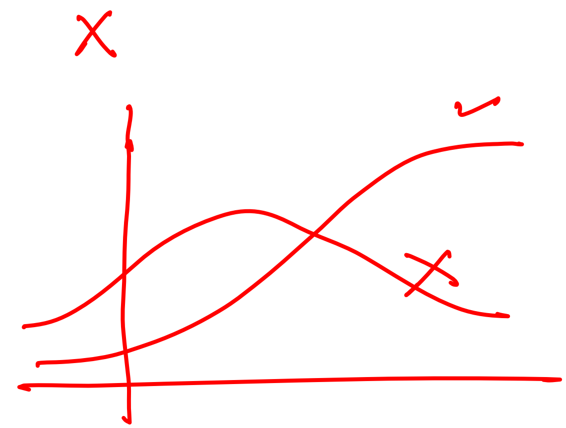
$\Rightarrow P\{X \le x_2\} = P\{X \le x_1\} + P\{x_1 < X \le x_2\}$

$\Rightarrow P\{X \le x_2\} \ge P\{X \le x_1\}$

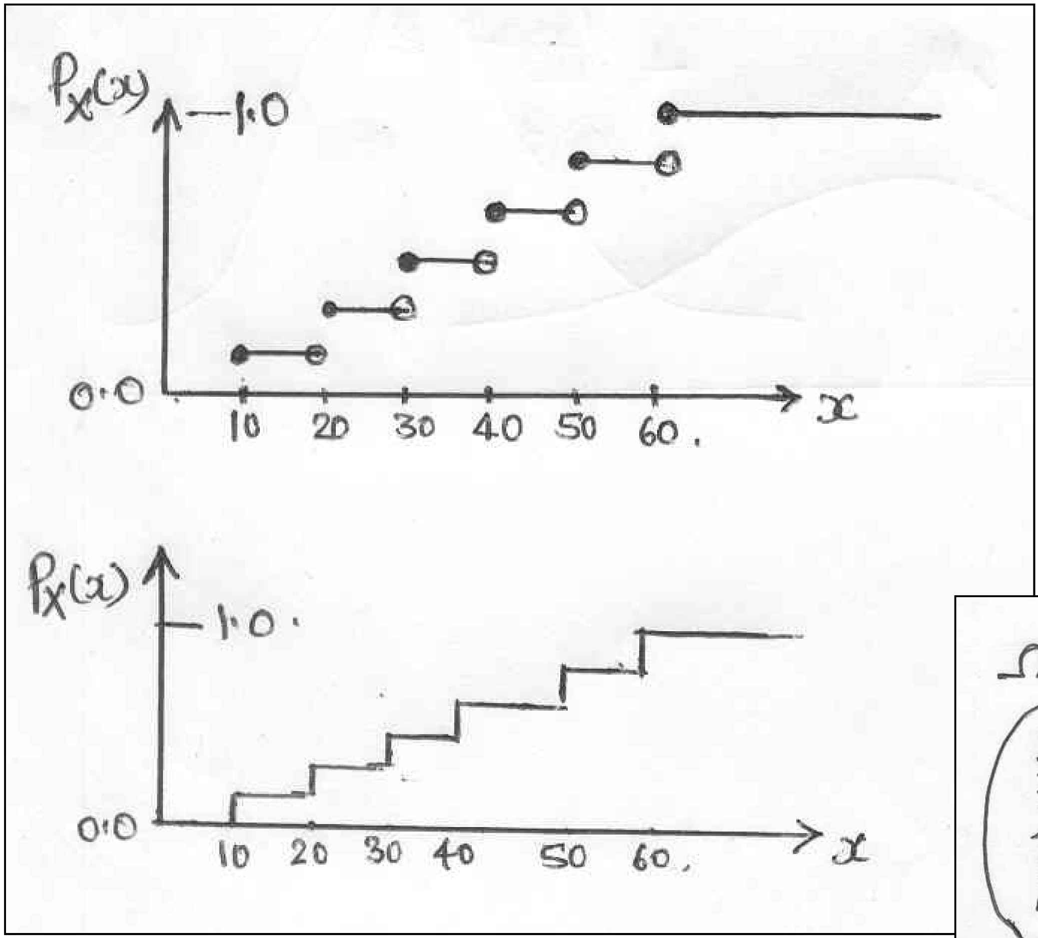
$(e)\ \lim_{0<\varepsilon\to 0} P_X(x+\varepsilon) = P_X(x)$

PDF is right continuous.

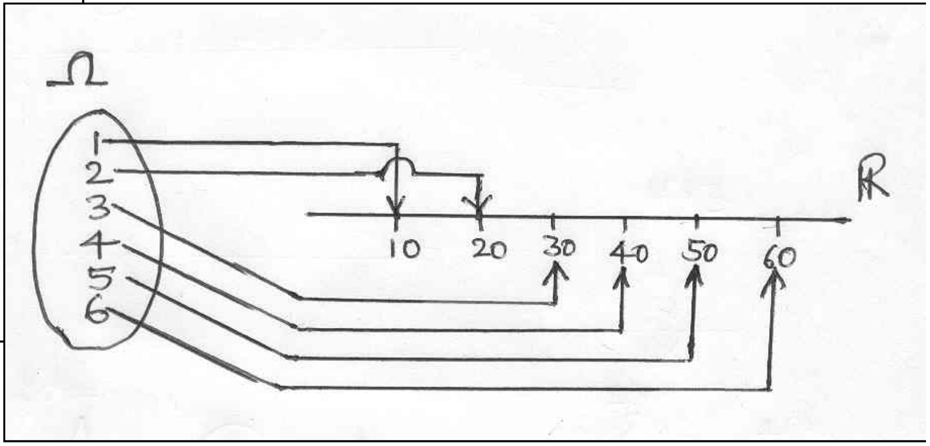




Example: Die tossing: $\Omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}; \quad X(\omega_i) = 10i$

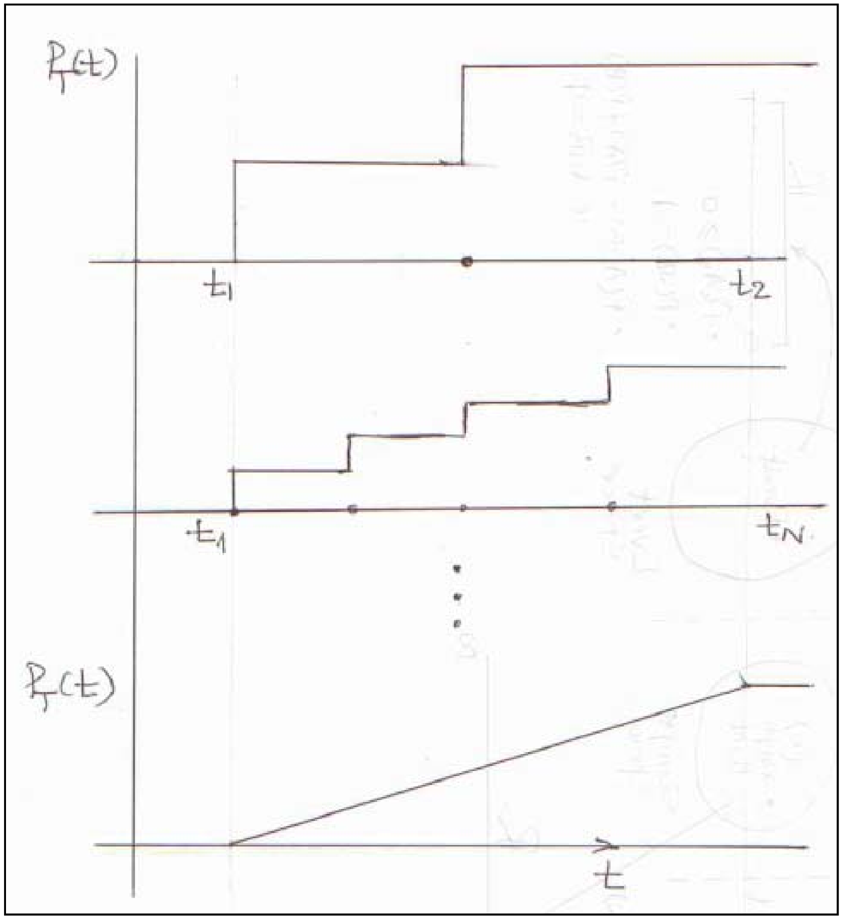


x	*{X≤x}*	P*{X≤x}*
0	*φ*	0
-100	*φ*	0
600	*Ω*	1
5	*φ*	0
10	{1}	1/6
20	{1,2}	2/6
22	{1,2}	2/6
35	{1,2,3}	3/6
53	{1,2,3,4,5}	5/6



Example

- Consider the time of arrival of a train on the platform.
- Let T = arrival time.
- Let any time instant in the interval (t_1, t_2) be equally likely.
- $\Omega = (t_1, t_2)$
- Let T be the random variable which denotes the time of arrival.
- Let us divide the interval (t_1, t_2) into N discrete time segments (t_n, t_{n-1}) with $t_n = t_1 + \frac{n}{N}(t_2 - t_1); n = 0,1,2,\cdots,N.$
- We take $\mathrm{T} = t_n$ if $t_n < t \le t_{n+1}; n = 1,2,\cdots,N-1.$
- For $\mathrm{N} < \infty$, the PDF of T would have discontinuities at t_n, $n = 1,2,\cdots,N-1.$
- As $\mathrm{N} \to \infty$, the PDF of T becomes continuous.

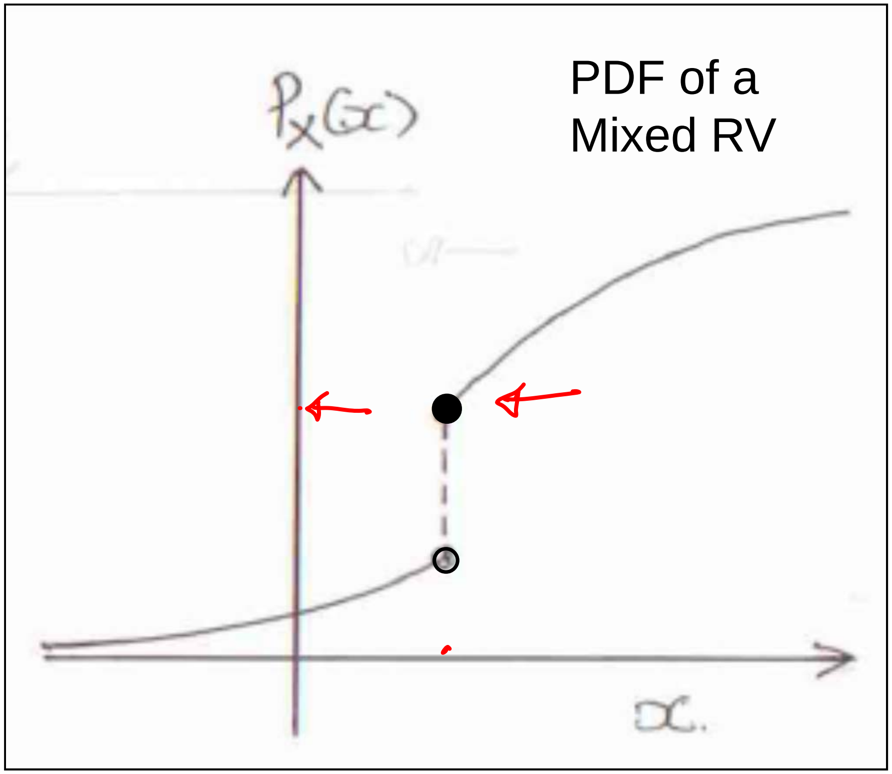


Random variables

Discrete : PDF proceeds only through jumps.

Continuous : PDF proceeds without any jumps.

Mixed : PDF proceeds with both jumps and continuously



$$\lim_{0<\varepsilon\to 0} P_X(x+\varepsilon) \to P_X(x)$$

Probability density function (pdf) $p_X(x)$

Definition

$$p_X(x) = \frac{dP_X(x)}{dx}$$

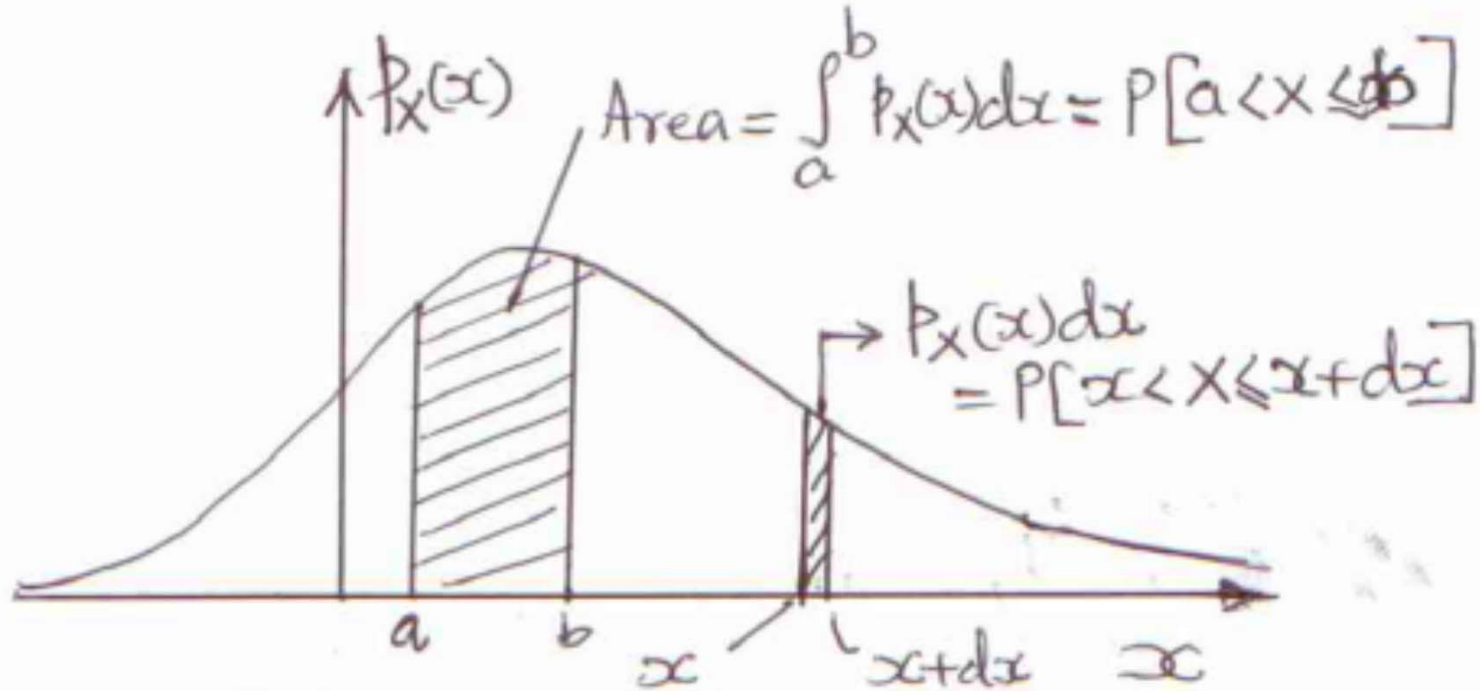
$$\Rightarrow P_X(x) = \int_{-\infty}^{x} p_X(u)du$$

Properties

$$P_X(\infty) = P\{X \le \infty\} = P(\Omega) = 1 = \int_{-\infty}^{\infty} p_X(u)du$$

$$P\{a \le X < b\} = \int_a^b p_X(u)du$$

$$p_X(x)dx = P\{x \le X < x + dx\}$$



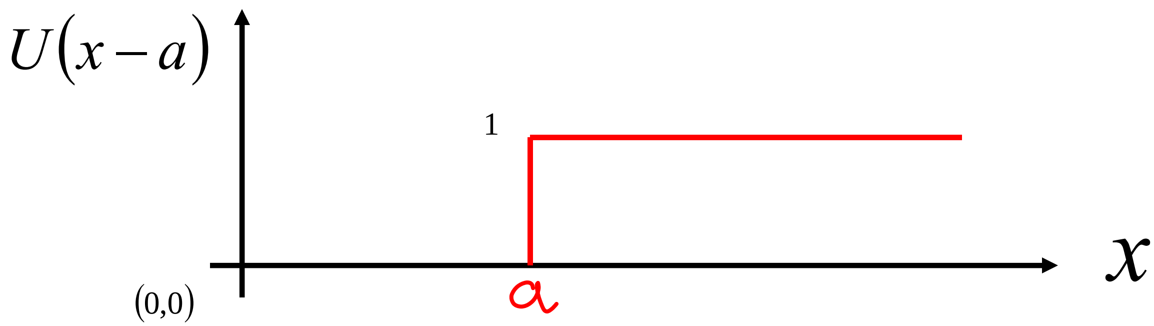
- Total area under the curve = 1
- Non-negative.

Notes :

(a) $P\left(X=k\right)=p_k; k=0,1,2,\cdots$ is also knwon as probability mass function (PMF).

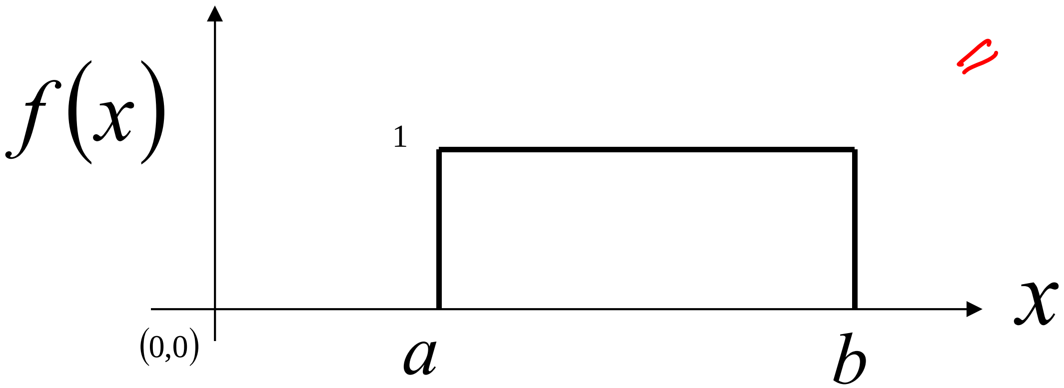
(b) PDF is also known as cumulative probability distribution function (CDF).

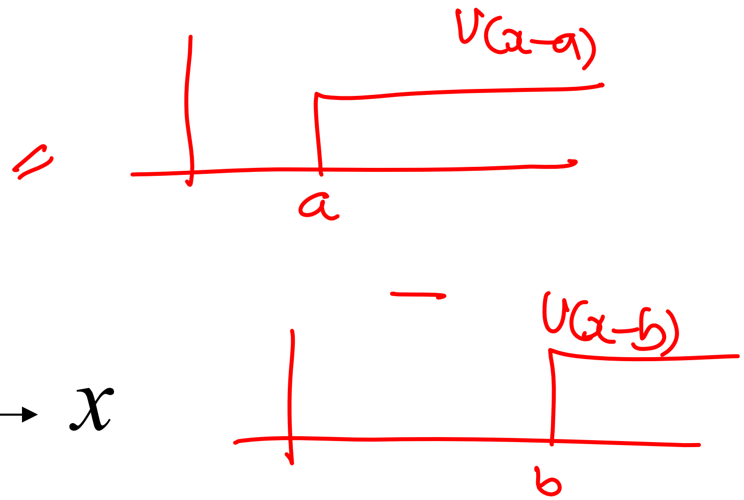
Heaveside's step function



$$
U(x-a) = \begin{cases} 0 & x < a \\ 1 & x > a \\ \dfrac{1}{2} & x = a \end{cases}
$$

Example 1: Box function





$$f(x) = U(x-a) - U(x-b)$$

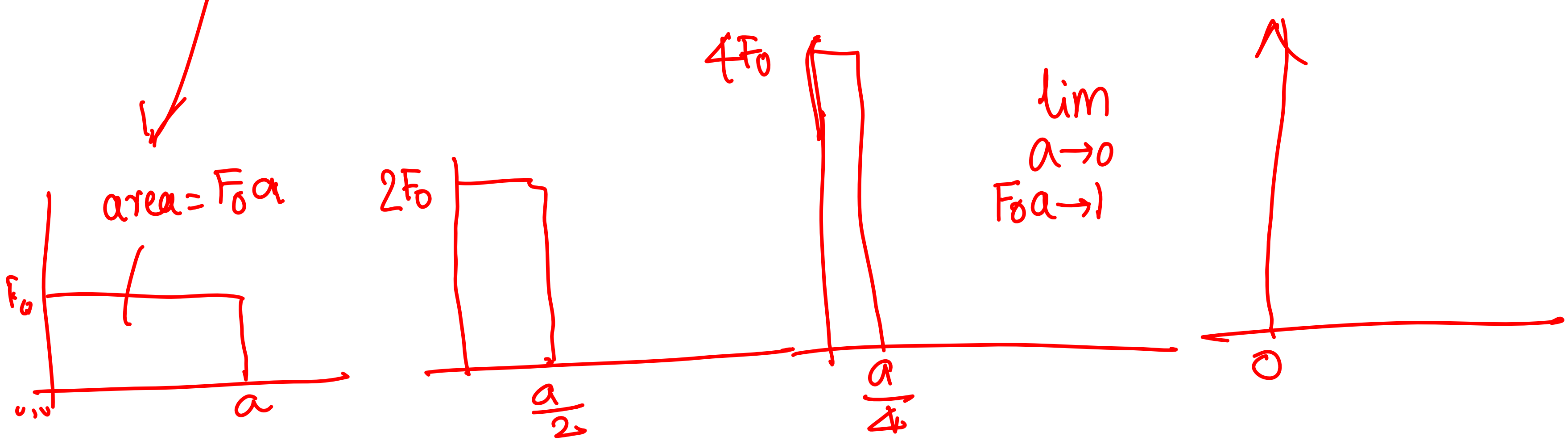
Dirac's delta function

$$f(x) = F_0\left[U(x) - U(x-a)\right]$$

$$\lim_{\substack{a\to 0\\ F_0 a = 1}} f(x) = \lim_{\substack{a\to 0\\ F_0 a = 1}} F_0\left[U(x) - U(x-a)\right]$$

$$= \lim_{\substack{a\to 0\\ F_0 a = 1}} F_0 a \frac{\left[U(x) - U(x-a)\right]}{a}$$

$$= \frac{d}{dx}U(x) = \delta(x)$$



Dirac's delta function

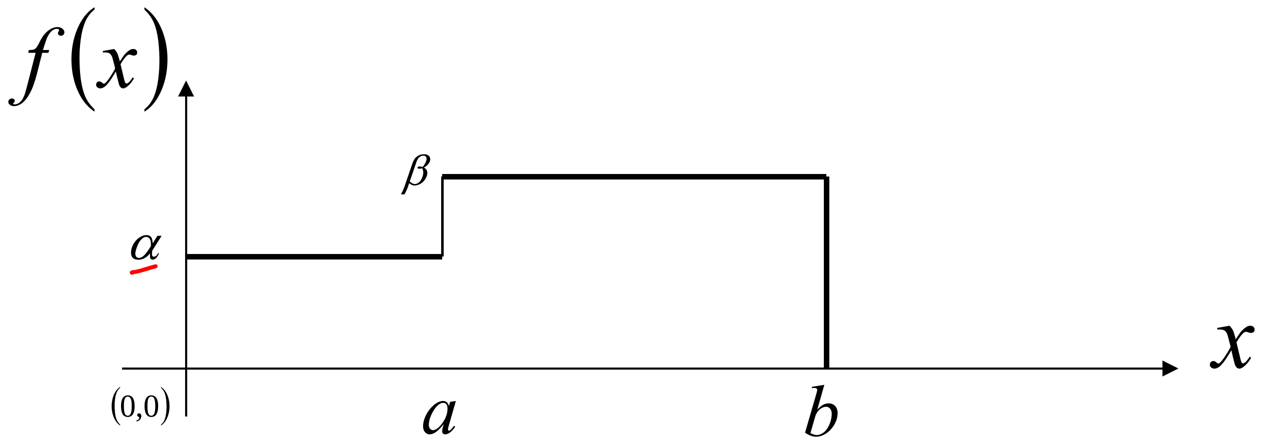
$$\delta(x-a)=0 \quad \text{for } x\neq a$$

$$\int_{-\infty}^{\infty}\delta(x-a)dx=1$$

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$$

$$\frac{d}{dx}U(x-a)=\delta(x-a)$$

Example-2: Stair case function



$$f(x) = \alpha[U(x) - U(x-a)] + \beta[U(x-a) - U(x-b)]$$

$$\frac{df}{dx} = \alpha[\delta(x) - \delta(x-a)] + \beta[\delta(x-a) - \delta(x-b)]$$

Commonly encountered random variables

- Models for rare events
- Models for sums
- Models for products
- Models for extremes
  - Highest
  - Lowest
- Models for waiting times

Limit theorems

Bernoulli random variable

The random experiment has only two outcomes : success and failure.

$$\Omega = \left(\text{S} \quad \text{F} \right)$$

Let

$$P(S) = P(X = 0) = p$$

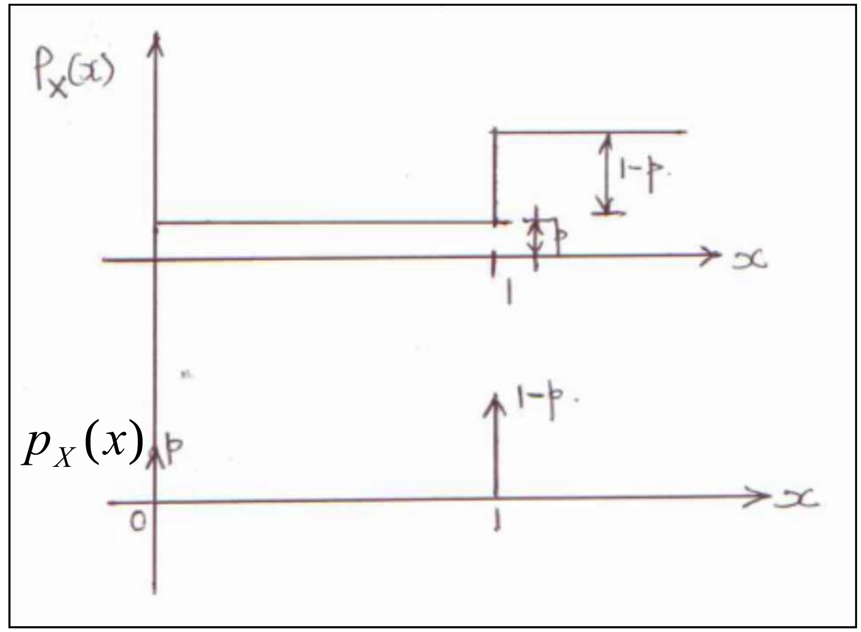
$$P(F) = P(X = 1) = 1\text{-}p$$

$$P_X\left(x\right) = pU(x) + (1-p)U(x-1)$$

$$p_X\left(x\right) = p\delta(x) + (1-p)\delta(x-1).$$

Check :

$$\int_{-\infty}^{\infty} p_X(x)dx = \int_{-\infty}^{\infty} \left[p\delta(x) + (1-p)\delta(x-1) \right] dx = 1.$$



Remarks:

- *p* is the parameter of the Bernoulli random variable.
- Discrete random variable
- Finite sample space
- Basic building block

Repeated Bernoulli trials: **Binomial random variable**.

The random experiment here consists of N repeated Bernoulli trials.

Assumptions

(a) The random experiment consists of N independent trials.

(b) Each trial results in only two outcomes (success/failure)

(c) P(success) remains constant during all trials.

Define X =number of successes in N trials; $X = 0,1,2,3,\cdots,N$.

$$P\left(X = k\right) = {}^{N}C_{k}\, p^{k}(1-p)^{N-k};k = 0,1,2,\cdots,N$$

Notes

Binomial theorem: $\left(p+q\right)^{n} = \sum_{r=0}^{n} {}^{n}C_{r}\, p^{r} q^{n-r}$

$${}^{N}C_{k} = \frac{N!}{(N-k)!k!}$$

Consider a sequence of N trials resulting in k successes.

Occurrence of k successes implies the occurrence of $(N - k)$ failures.

Probability of occurrence of one such sequence=$p^k(1-p)^{N-k}$.

Number of such possible sequences=${}^N C_k$.

These sequences are mutually exclusive.

$$\therefore P[X=k] = {}^N C_k p^k (1-p)^{N-k}$$

$$\Rightarrow P[X \le m] = \sum_{k=0}^{m} {}^N C_k p^k (1-p)^{N-k}$$

$$\Rightarrow P[X \le N] = \sum_{k=0}^{N} {}^N C_k p^k (1-p)^{N-k} = 1$$

(By virtue of binomial theorem.)

Hence the name binomial random variable.

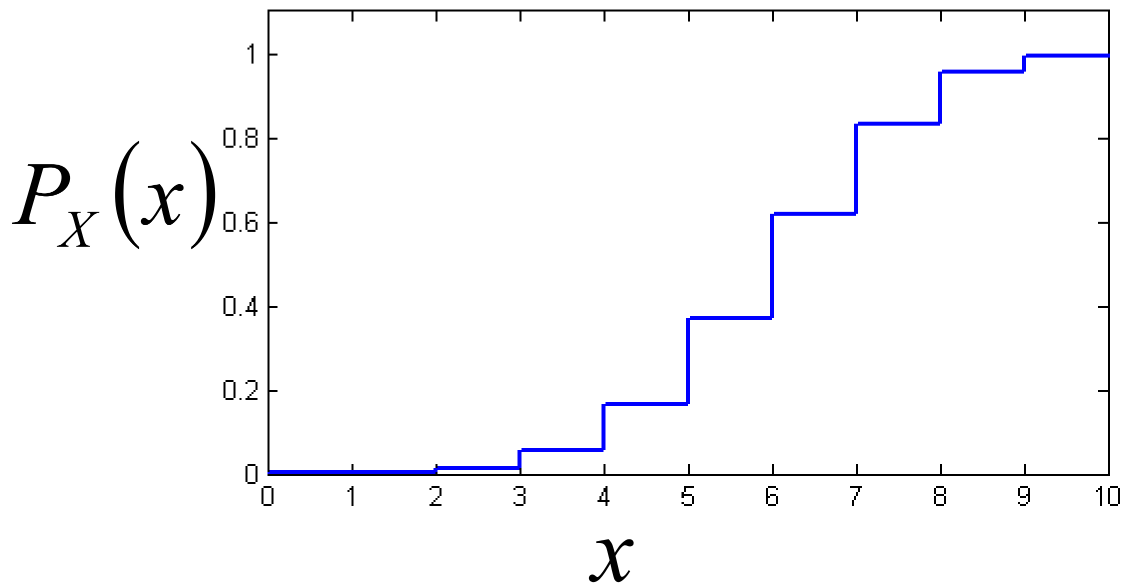
Remarks

A binomial random variable is denoted by $B(N, p)$.

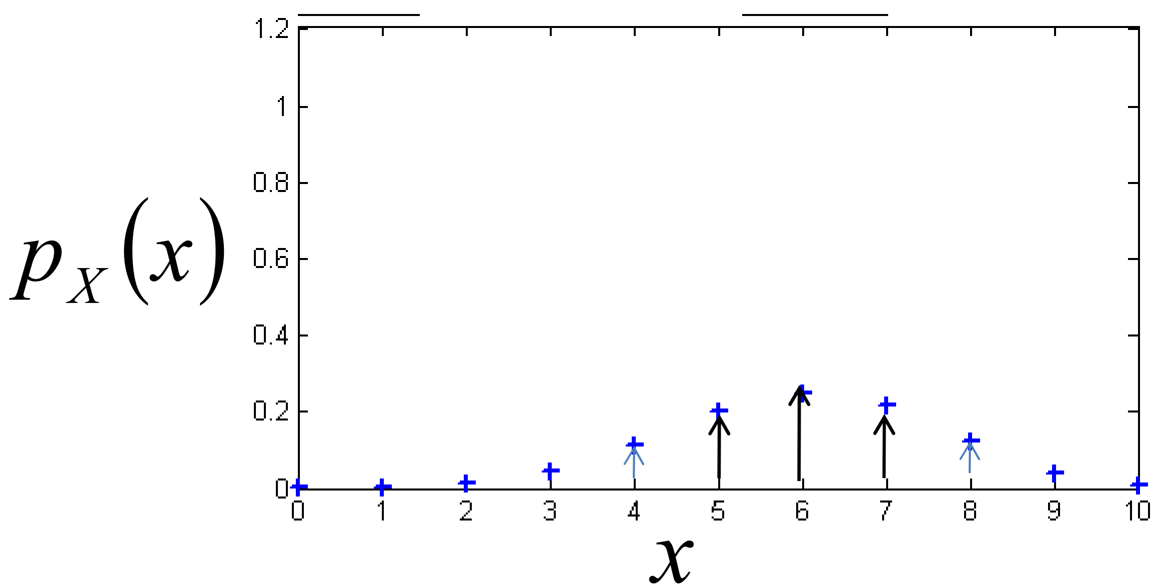
N and p are the paramters of this random variable.

- Discrete random variable
- Sample space is finite

$$B(10, 0.5)$$



$P_X(x)$
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10
x



$p_X(x)$
1.2
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10
x

Geometric random variable

Random experiments: as in binomial random variable.

N=number of trials for the first success; $N = 1, 2, \cdots, \infty$.

$P\left(\text{first success in N-th trial}\right) = P(\text{success on the } N\text{-th trial} \cap \text{failures on the first } (N-1) \text{ trials})$

$\Rightarrow P\left(N = n\right) = (1-p)^{n-1} p; n = 1, 2, \cdots, \infty.$

$\sum_{n=1}^{\infty} P\left(N = n\right) = \sum_{n=1}^{\infty} (1-p)^{n-1} p$ (this must be =1).

Ex: let p=0.6

$$\Rightarrow \sum_{n=1}^{\infty} (1-p)^{n-1} p = \sum_{n=1}^{\infty} 0.4^{n-1} \times 0.6$$

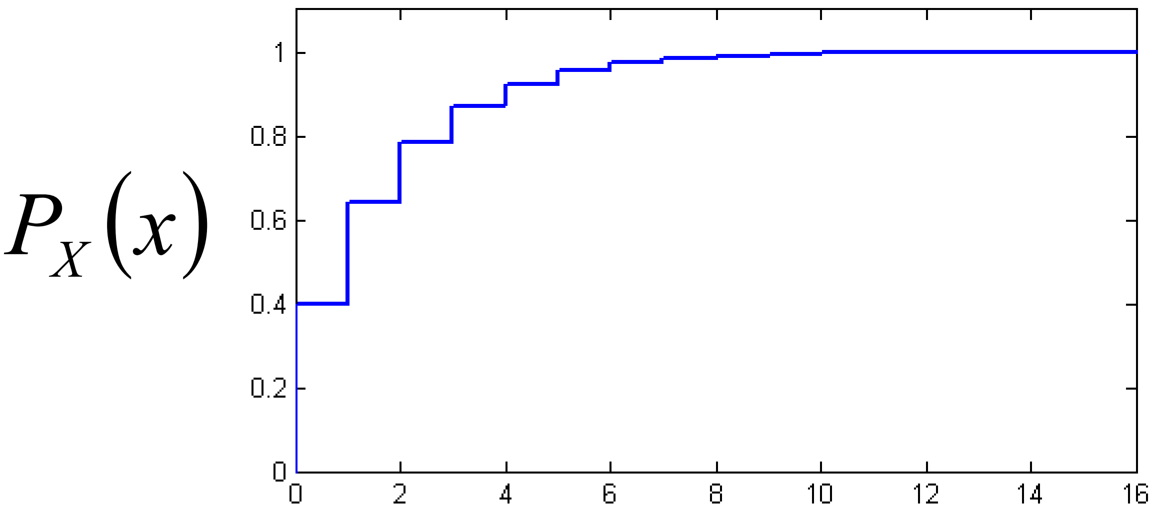
$$= 0.6\left[1 + 0.4 + 0.4^2 + 0.4^3 + \cdots\right]$$

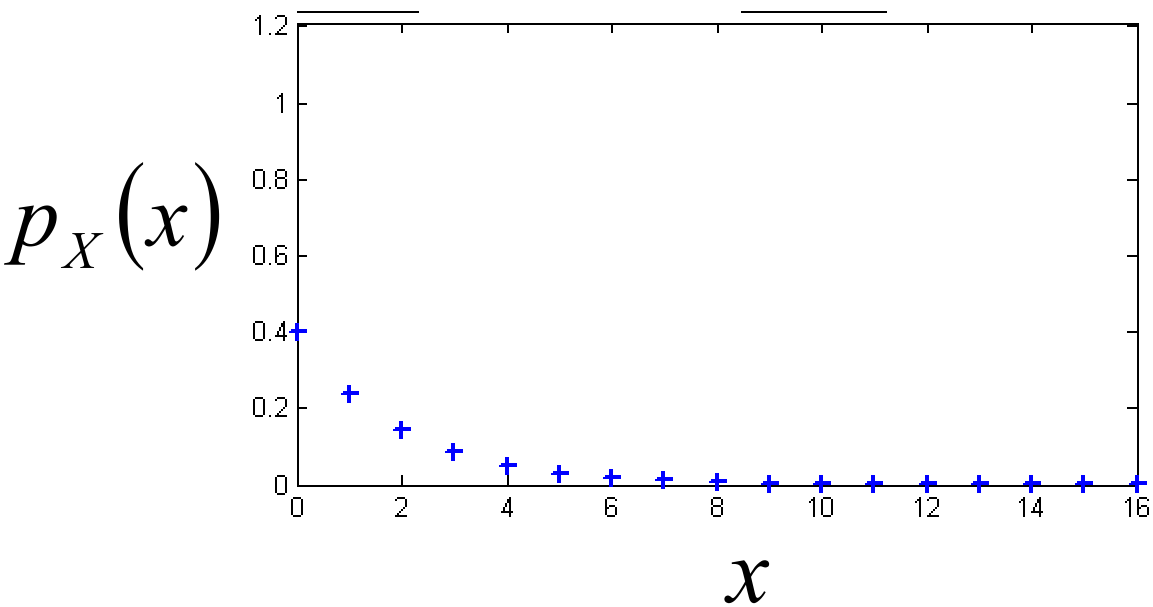
The expression inside the bracket is a geometric progression.

Hence the name geometric random variable.

- Discrete random variable
- Countably infinite sample space
- Useful in modeling life times of engineering systems

Geometric random variable with p=0.4





Pascal or negative binomial distribution

Define $W_k =$ Number of trials to the k-th success.

It can be shown that

$$P\left(W_k = w\right) = {}^{(w-1)}C_{(k-1)}(1-p)^{w-k}\, p^k ; w = k, k+1, k+2, \cdots, \infty$$

Models for rare events : Poisson random variable

(a) We are looking for occurrence of an isolated phenomenon in a time/space continuum.

(b) We cannot put an upper bound on the number of occurrences.

(c) Actual number of occurrences is relatively small.

Examples : goals in football match (time continuum), defect in a yarn (1-d space continuum), typos in a manuscript (2-d continuum), defect in a solid (3-d continuum). **Stress at a point exceeding elastic limit during the life time of a structure.**

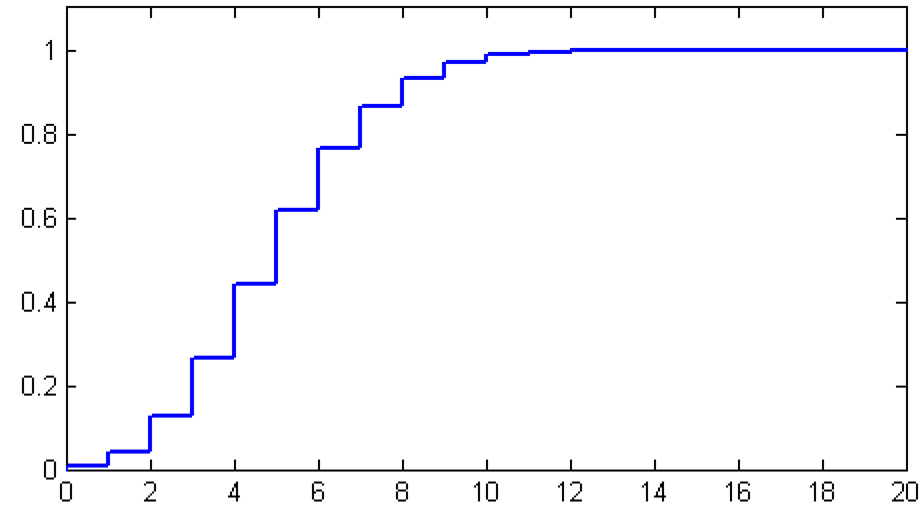
$$P(X=k)=\exp(-a)\frac{a^k}{k!};k=0,1,2,\cdots$$

Check

$$P(X\le\infty)=\sum_{k=0}^{\infty}\exp(-a)\frac{a^k}{k!}=\exp(-a)\sum_{k=0}^{\infty}\frac{a^k}{k!}=\exp(-a)\exp(a)=1.$$

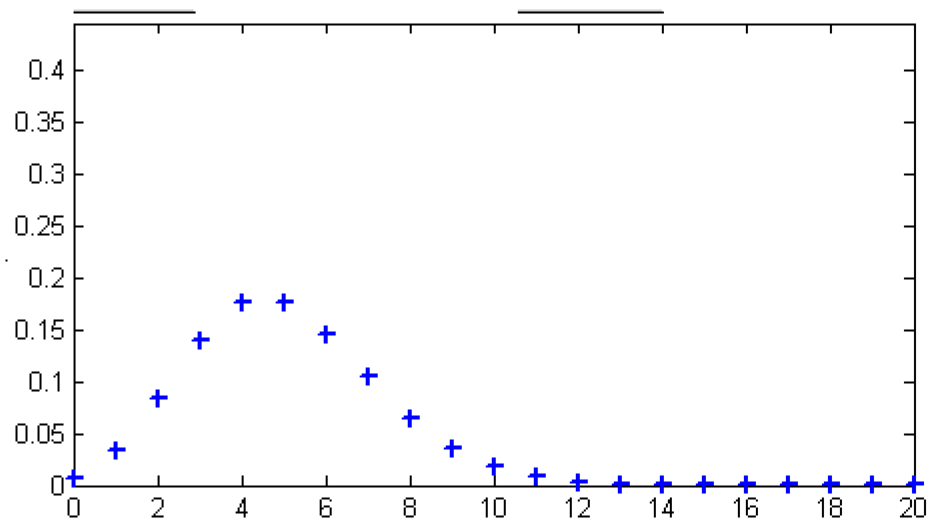
Poisson random variable with a=5

$$P_X(x)$$



$$x$$

$$p_X(x)$$



$$x$$

- Discrete RV
- Countably infinite sample space
- Useful in wide variety of contexts