Stochastic Structural Dynamics

Lecture-39

Problem solving session-3

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Discussion on properties of processes with Independent increments

Problem 35

Let $X(t)$ be a process with stationary independent increments; assume $t \geq 0$ & $X(0) = 0$. Show that

- $\langle X(t) \rangle = \mu t$;
- $\mathrm{Var}\left[X(t) \right] = \sigma^2 t$
- $\mathrm{Var}\left[X(t) - X(s) \right] = \sigma^2 (t - s)$
- $\mathrm{Cov}\left[X(t) X(s) \right] = \sigma^2 \min(t, s)$

Note: clearly, $\langle X(t=1) \rangle = \mu$ & $\sigma^2 =$ variance of $X(t=1)$.

Let $f(t) = \left\langle X(t) \right\rangle$

$f(t) = \left\langle X(t) - X(0) \right\rangle$

$f(t+s) = \left\langle X(t+s) - X(0) \right\rangle$

$= \left\langle X(t+s) - X(s) + X(s) - X(0) \right\rangle$

$= \left\langle X(t+s) - X(s) \right\rangle + \left\langle X(s) - X(0) \right\rangle$

$= \left\langle X(t) - X(0) \right\rangle + \left\langle X(s) - X(0) \right\rangle$ $\because$ stationary increments

$= f(t) + f(s)$

We get the functional equation $f(t+s) = f(t) + f(s)$

$\Rightarrow f(t) = ct$ is the solution.

Given $f(1) = \left\langle X(1) \right\rangle = \mu \Rightarrow c = \mu$

$\left\langle X(t) \right\rangle = \mu t$

Let $g(t) = \mathrm{Var}\left[X(t)\right]$

$$g(t) = \mathrm{Var}\left[X(t)\right] = \mathrm{Var}\left[X(t) - X(0)\right]$$

$$g(t+s) = \mathrm{Var}\left[X(t+s) - X(0)\right]$$

$$= \mathrm{Var}\left[X(t+s) - X(s) + X(s) - X(0)\right]$$

$$= \mathrm{Var}\left[X(t+s) - X(s)\right] + \mathrm{Var}\left[X(s) - X(0)\right]$$

$$= \mathrm{Var}\left[X(t) - X(0)\right] + \mathrm{Var}\left[X(s) - X(0)\right]$$

$\because$ stationary increments

$$\Rightarrow g(t+s) = g(t) + g(s)$$

$$\Rightarrow g(t) = ct$$

$$g(1) = \mathrm{Var}\left[X(1)\right] = \sigma^2 \Rightarrow c = \sigma^2$$

$$\Rightarrow \mathrm{Var}\left[X(t)\right] = \sigma^2 t$$

Let $t > s$

$$\operatorname{Var}\left[X(t)\right] = \operatorname{Var}\left[X(t) - X(0)\right]$$

$$= \operatorname{Var}\left[X(t) - X(s) + X(s) - X(0)\right]$$

$$= \operatorname{Var}\left[X(t) - X(s)\right] + \operatorname{Var}\left[X(s) - X(0)\right]$$

$$= \operatorname{Var}\left[X(t) - X(s)\right] + \operatorname{Var}\left[X(s)\right]$$

$$\Rightarrow \operatorname{Var}\left[X(t) - X(s)\right] = \operatorname{Var}\left[X(t)\right] - \operatorname{Var}\left[X(s)\right]$$

$$= \sigma^2 (t - s)$$

$$\mathrm{Var}\left[X(t)-X(s)\right]$$

$$=\left\langle\left[X(t)-X(s)-\left\langle X(t)-X(s)\right\rangle\right]^2\right\rangle$$

$$=\left\langle\left[\left\{X(t)-\left\langle X(t)\right\rangle\right\}-\left\{X(s)-\left\langle X(s)\right\rangle\right\}\right]^2\right\rangle$$

$$=\left\langle\left\{X(t)-\left\langle X(t)\right\rangle\right\}^2\right\rangle+\left\langle\left\{X(s)-\left\langle X(s)\right\rangle\right\}^2\right\rangle$$

$$-2\left\langle\left\{X(t)-\left\langle X(t)\right\rangle\right\}\left\{X(s)-\left\langle X(s)\right\rangle\right\}\right\rangle$$

$$=\mathrm{Var}\left[X(t)\right]+\mathrm{Var}\left[X(s)\right]-2\mathrm{COV}\left[X(t),X(s)\right]$$

$$\text{COV}\left[X(t), X(s)\right] =$$

$$\frac{1}{2}\left\{\text{-Var}\left[X(t) - X(s)\right] + \text{Var}\left[X(t)\right] + \text{Var}\left[X(s)\right]\right\}$$

$$= \frac{\sigma^2}{2}\{t + s - (t - s)\} = \sigma^2 s \ \left(\text{assuming that } t > s\right)$$

$$\Rightarrow \text{COV}\left[X(t), X(s)\right] = \sigma^2 \min(t, s)$$

Problem 34

Let $X(t)$ be a stationary Gaussian random process with zero mean and PSD function of the form

$$S_{XX}(\omega)=\frac{\sigma_X^2}{\sqrt{2\pi\alpha}}\exp\left[-\frac{\omega^2}{2\alpha^2}\right];-\infty<\omega<\infty$$

- Determine the autocorrelation and cross correlation functions of the processes $X(t)$ and $\dot{X}(t)$
- Find the average rate of upcrossing of level β
- Find the PDF of time for first crossing of level β
- Find the average rate of peaks above level β
- Find the expected fractional occupation time above level β over a duration 0 to T
- Find the PDF of extreme of $X(t)$ over duration 0 to T

Spectral moments

$$S_{XX}(\omega) = \frac{\sigma_X^2}{\sqrt{2\pi\alpha}} \exp\left[-\frac{\omega^2}{2\alpha^2}\right]; -\infty < \omega < \infty$$

$$\lambda_0 = \int_{-\infty}^{\infty} \frac{\sigma_X^2}{\sqrt{2\pi\alpha}} \exp\left[-\frac{\omega^2}{2\alpha^2}\right] d\omega = \sigma_X^2$$

$$\lambda_2 = \int_{-\infty}^{\infty} \omega^2 \frac{\sigma_X^2}{\sqrt{2\pi\alpha}} \exp\left[-\frac{\omega^2}{2\alpha^2}\right] d\omega = \sigma_X^2 \alpha^2$$

$$\lambda_4 = \int_{-\infty}^{\infty} \omega^4 \frac{\sigma_X^2}{\sqrt{2\pi\alpha}} \exp\left[-\frac{\omega^2}{2\alpha^2}\right] d\omega = 3\sigma_X^2 \alpha^4$$

Autocorrelation function

$$S_{XX}(\omega) = \frac{\sigma_X^2}{\sqrt{2\pi\alpha}} \exp\left[-\frac{\omega^2}{2\alpha^2} \right]; -\infty < \omega < \infty$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma_X^2}{\sqrt{2\pi\alpha}} \exp\left[-\frac{\omega^2}{2\alpha^2} \right] \exp(i\omega\tau)\, d\omega$$

$$= \sigma_X^2 \exp\left(-\frac{\alpha^2\tau^2}{2} \right)$$

Recall: $\left\langle X^m(t) X^n(t+\tau) \right\rangle = (-1)^n \frac{d^{m+n} R_{XX}(\tau)}{d\tau^{m+n}}$

$$\left\langle X\left(t\right)\dot{X}\left(t+\tau\right)\right\rangle = -\frac{dR_{XX}\left(\tau\right)}{d\tau};\left\langle X\left(t\right)\ddot{X}\left(t+\tau\right)\right\rangle = \frac{d^2R_{XX}\left(\tau\right)}{d\tau^2}$$

$$\left\langle \dot{X}\left(t\right)X\left(t+\tau\right)\right\rangle = \frac{dR_{XX}\left(\tau\right)}{d\tau};\left\langle \dot{X}\left(t\right)\dot{X}\left(t+\tau\right)\right\rangle = -\frac{d^2R_{XX}\left(\tau\right)}{d\tau^2};$$

$$\left\langle \dot{X}\left(t\right)\ddot{X}\left(t+\tau\right)\right\rangle = \frac{d^3R_{XX}\left(\tau\right)}{d\tau^3};\left\langle \ddot{X}\left(t\right)X\left(t+\tau\right)\right\rangle = \frac{d^2R_{XX}\left(\tau\right)}{d\tau^2}$$

$$\left\langle \ddot{X}\left(t\right)\dot{X}\left(t+\tau\right)\right\rangle = -\frac{d^3R_{XX}\left(\tau\right)}{d\tau^3};\left\langle \ddot{X}\left(t\right)\ddot{X}\left(t+\tau\right)\right\rangle = \frac{d^4R_{XX}\left(\tau\right)}{d\tau^4}$$

$$R_{XX}(\tau) = \sigma_X^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)$$

$$\frac{d}{d\tau} R_{XX}(\tau) = \sigma_X^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(-\alpha^2\tau\right) //$$

$$\frac{d^2}{d\tau^2} R_{XX}(\tau) = \sigma_X^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(-\alpha^2\tau\right)^2$$

$$+\sigma_X^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(-\alpha^2\right)$$

$$= -\sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(1-\alpha^2\tau^2\right) //$$

$$\frac{d^2}{d\tau^2} R_{XX}(\tau) = -\sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(1-\alpha^2\tau^2\right)$$

$$\frac{d^3}{d\tau^3} R_{XX}(\tau) = -\sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left\{1-\alpha^2\tau^2\right\}\left(-\alpha^2\tau\right)$$

$$-\sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(-2\alpha^2\tau\right)$$

$$= \sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(3\alpha^2\tau - \alpha^4\tau^3\right)$$

$$\frac{d^3}{d\tau^3} R_{XX}(\tau) = \sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(3\alpha^2\tau - 4\alpha^4\tau^3\right)$$

$$\frac{d^4}{d\tau^4} R_{XX}(\tau) = \sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(3\alpha^2\tau - 4\alpha^4\tau^3\right)\left(-\alpha^2\tau\right)$$

$$+\sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(3\alpha^2 - 12\alpha^4\tau^2\right)$$

$$= \sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(-3\alpha^4\tau^2 + 4\alpha^6\tau^4 + 3\alpha^2 - 12\alpha^4\tau^2\right)$$

$$= \sigma_X^2 \alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(3\alpha^2 - 15\alpha^4\tau^2 + 4\alpha^6\tau^4\right) \quad //$$

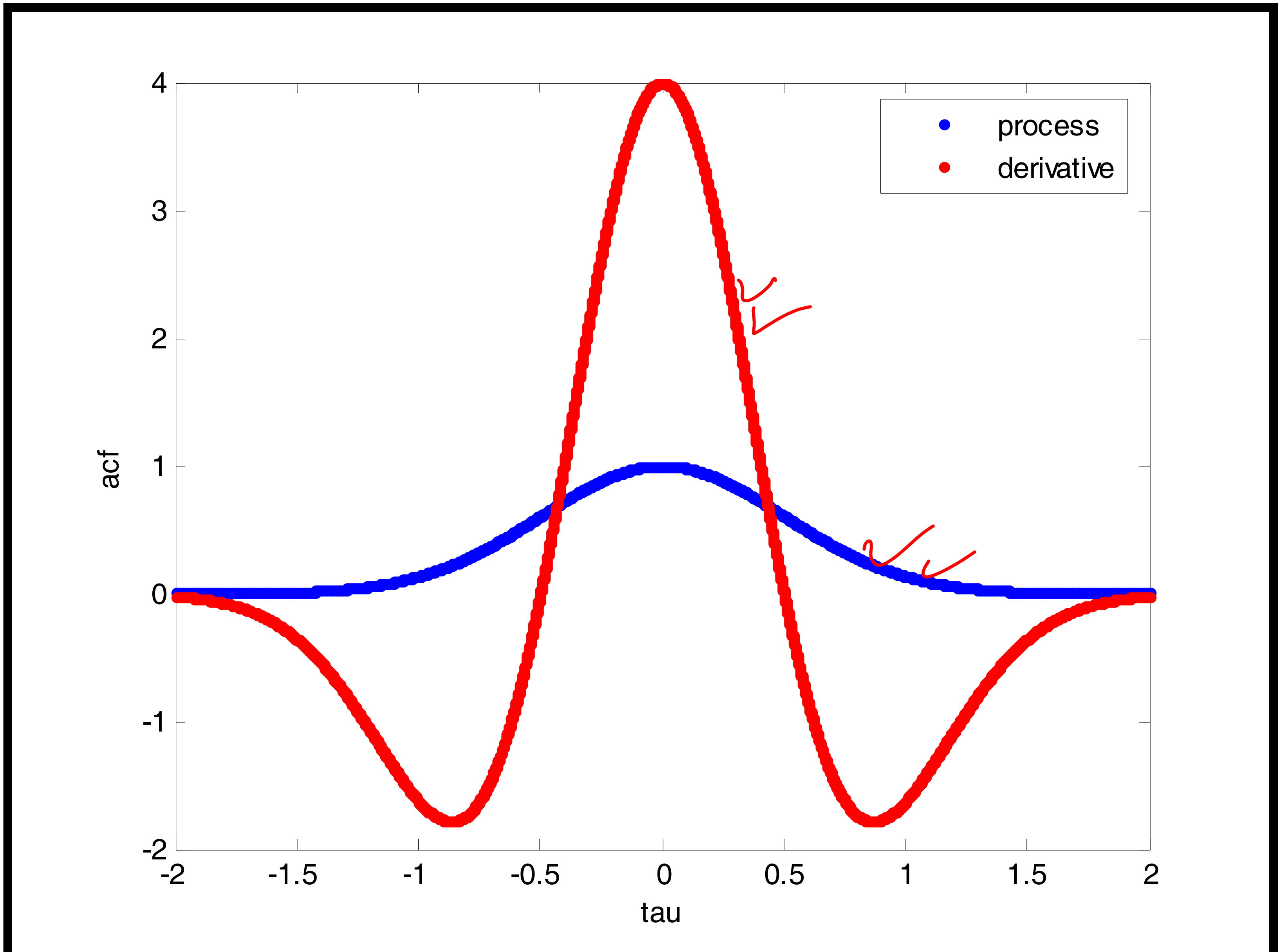
$$R_{XX}(\tau) = \sigma_X^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)$$

$$\frac{d}{d\tau}R_{XX}(\tau) = \sigma_X^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(-\alpha^2\tau\right) \quad \langle X(t)\dot{X}(t+\tau)\rangle$$

$$\frac{d^2}{d\tau^2}R_{XX}(\tau) = -\sigma_X^2\alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(1-\alpha^2\tau^2\right)$$

$$\frac{d^3}{d\tau^3}R_{XX}(\tau) = \sigma_X^2\alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(3\alpha^2\tau - 4\alpha^4\tau^3\right)$$

$$\frac{d^4}{d\tau^4}R_{XX}(\tau) = \sigma_X^2\alpha^2 \exp\left(-\frac{\alpha^2\tau^2}{2}\right)\left(3\alpha^2 - 15\alpha^4\tau^2 + 4\alpha^6\tau^4\right)$$


4
3
2
1
0
-1
-2
acf
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
tau
process
derivative

Level crossing statistics

$$N(\beta,0,T)=\int_0^T \left|\dot{X}(t)\right|\delta\left[X(t)-\beta\right]dt$$

Average rate of upcrossing of level β

$$n^+(\beta,t)=\frac{1}{2}\left\langle\left|\dot{X}(t)\right|\delta\left[X(t)-\beta\right]\right\rangle=\frac{\sigma_{\dot{X}}}{2\pi\sigma_X}\exp\left(-\frac{1}{2}\frac{\beta^2}{\sigma_{\dot{X}}^2}\right)$$

$$\frac{\sigma_{\dot{X}}}{2\pi\sigma_X}=\frac{\sigma_X\alpha}{2\pi\sigma_X}=\frac{\alpha}{2\pi}$$

$$n^+(\beta,t)=\frac{\alpha}{2\pi}\exp\left(-\frac{1}{2}\frac{\beta^2}{\sigma_X^2\alpha^2}\right)=\frac{\alpha}{2\pi}\exp\left(-\frac{1}{2}\frac{a^2}{\alpha^2}\right);a=\frac{\beta}{\sigma_X}$$

PDF of time for first crossing of level β

For high levels of crossings we can approximate the number of times the level is crossed as a Poisson random variable.

$$P\left[N\left(\beta,0,T\right)=k\right]=\exp\left[-\lambda T\right]\frac{\left(\lambda T\right)^{k}}{k!};k=0,1,2,\cdots$$

$$\lambda=n\left(\beta,t\right)=\frac{\alpha}{\pi}\exp\left(-\frac{1}{2}\frac{a^{2}}{\alpha^{2}}\right);a=\frac{\beta}{\sigma_{X}}$$

T_{f} = First passage time

$$\mathrm{P}\left[T_{f}>T\right]=P\left[\text{No points in 0 to T}\right]=P\left[N\left(\beta,0,T\right)=0\right]$$

$$P_{T_{f}}\left(t\right)=1-\exp\left[-\lambda T\right]=1-\exp\left[-\frac{\alpha T}{\pi}\exp\left(-\frac{1}{2}\frac{a^{2}}{\alpha^{2}}\right)\right];T\geq 0$$

Average rate of peaks above level β

$$\langle m(\beta,t)\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_1^2\sigma_2^2}\int_{\beta}^{\infty}[|S|^{\frac{1}{2}}\exp\left(-\frac{\sigma_2^2\sigma_3^2x^2}{2|S|}\right)+\frac{\sigma_2^3}{\sigma_1}x\sqrt{\frac{\pi}{2}}$$

$$\exp\left(-\frac{x^2}{2\sigma_1^2}\right)\left\{1+erf\left(\frac{\sigma_3^2x}{\sigma_1\sqrt{2|S|}}\right)\right\}]dx$$

σ_1^2 var $x(t)$
σ_2^2 var $\dot{x}(t)$
σ_3^2 var $\ddot{x}(t)$

$$S=\begin{bmatrix} \langle X^2(t)\rangle & \langle X(t)\dot{X}(t)\rangle & \langle X(t)\ddot{X}(t)\rangle \\ \langle \dot{X}(t)X(t)\rangle & \langle \dot{X}^2(t)\rangle & \langle \dot{X}(t)\ddot{X}(t)\rangle \\ \langle \ddot{X}(t)X(t)\rangle & \langle \ddot{X}(t)\dot{X}(t)\rangle & \langle \ddot{X}^2(t)\rangle \end{bmatrix}$$

$$=\sigma_X^2\begin{bmatrix} 1 & 0 & -\alpha^2 \\ 0 & \alpha^2 & 0 \\ -\alpha^2 & 0 & 3\alpha^4 \end{bmatrix} \Rightarrow |S| = 3\sigma_X^6\alpha^6\left(1+\alpha^2\right)$$

$$\sigma_1^2=\sigma_X^2$$

$$\sigma_2^2=\alpha^2\sigma_X^2$$

$$\sigma_3^2=3\sigma_X^2\alpha^4$$

Expected fractional occupation time above level β over a duration 0 to T

$$y(\beta,T)=\frac{1}{T}\int_0^T U\left[X(t)-\beta\right]dt$$

$$\left\langle U\left[X(t)-\beta\right]\right\rangle = 1-\int_{-\infty}^{\beta} p_X(x;t)\,dx$$

$$=1-\int_{-\infty}^{\beta}\frac{1}{\sqrt{2\pi}\sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)dx$$

$$=\left[\frac{1}{2}-\operatorname{erf}\left(\frac{\beta}{\sigma_x}\right)\right]=\left\langle y(\beta,T)\right\rangle$$

PDF of extreme of $X(t)$ over duration 0 to T

$$P_{T_f}(t) = 1 - \exp[-\lambda t]$$

$$p_{T_f}(t) = \lambda \exp[-\lambda t] \quad 0 < t < \infty$$

$$\lambda = \frac{\alpha}{2\pi} \exp\left(-\frac{1}{2} \frac{a^2}{\alpha^2} \right); a = \frac{\beta}{\sigma_X}$$

$$P_{X_m}(\beta) = 1 - P\left[T_f(\beta) \le T \right]$$

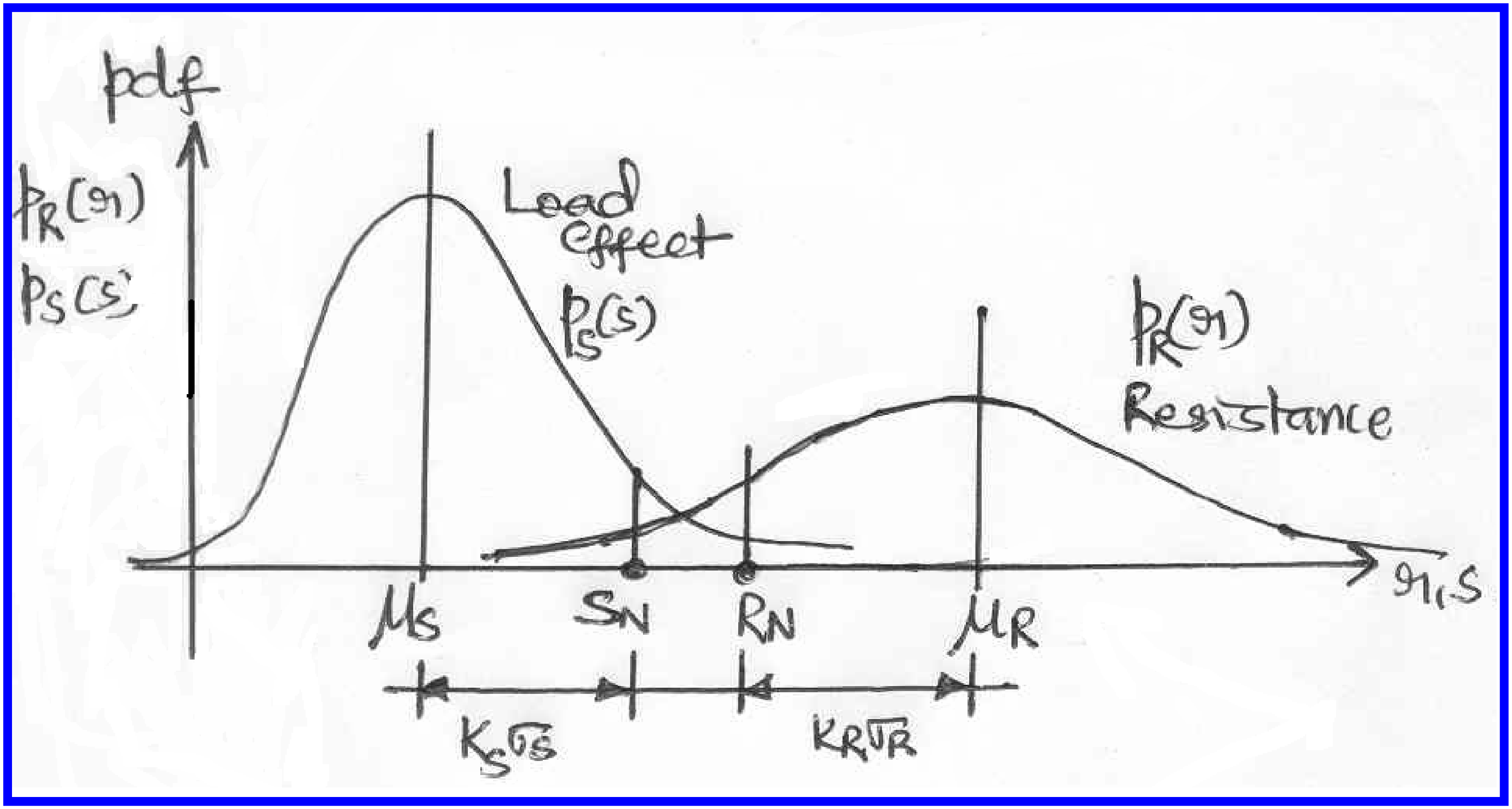
FACTORS OF SAFETY
&
PROBABILITY OF FAILURE

Problem 35

In traditional engineering practice, uncertainties in specifying loads and structural resistance are accounted for by overestimating the loads and underestimating the strucutral resistance. The factors by which these estimates are obtained are calibrated against past experience with existing stock of strucutres. It is of interest to relate this broad principle with the probabilistic modeling of uncertainties. To illustrate this let us consider an idealized situation in which demands on the structure and supply of structural capacity are modeled as a pair of mutually independent Gaussian random variables.

The failure event is defined by exceedance of load effect over the available capacity. If the tolerable level of probability of failure is specified to be P_F, determine the factors by which the expected load and capacity are to be mutiplied.

Extend the discussion to the case when several loads act on the structure (like, for example, dead load, live load, thermal loads, etc.)


pdf
pR(r)
pS(s)
Load effect
pS(s)
pR(r)
Resistance
r, S
μS
SN
RN
μR
KSσS
KRσR

$$R \to N(\mu_R, \sigma_R); \quad S \to N(\mu_S, \sigma_S); \quad R \perp S$$

$$\delta_R = \frac{\mu_R}{\sigma_R}; \quad \delta_S = \frac{\mu_S}{\sigma_S}$$

$$Z = R - S \Rightarrow Z \to N(\mu_R - \mu_S, \sqrt{\sigma_R^2 + \sigma_S^2})$$

$$P_F = 1 - \Phi\left[\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right]$$

$$\Rightarrow \Phi\left[\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right] = 1 - P_F$$

$$\Rightarrow \mu_R = \mu_S + \sqrt{\sigma_R^2 + \sigma_S^2}\,\Phi^{-1}[1 - P_F]$$

$$\mu_R \geq \mu_S + \beta\sqrt{\sigma_R^2 + \sigma_S^2}$$

If β is large, risk is small

$$\varepsilon = \frac{\sqrt{\sigma_R^2 + \sigma_S^2}}{\sigma_R + \sigma_S}; \quad \varepsilon \cong 0.75$$

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} = \frac{\mu_R - \mu_S}{\varepsilon\left(\sigma_R + \sigma_S\right)}$$

$$\Rightarrow \beta\varepsilon\left(\sigma_R + \sigma_S\right) = \mu_R - \mu_S$$

$$\Rightarrow \mu_R - \beta\varepsilon\sigma_R = \mu_S + \beta\varepsilon\sigma_S$$

$$\Rightarrow \mu_R - \beta\varepsilon\delta_R\mu_R = \mu_S + \beta\varepsilon\delta_S\mu_S$$

$$\Rightarrow \mu_R[1 - \beta\varepsilon\delta_R] = \mu_S[1 + \beta\varepsilon\delta_S]$$

$$\bar{\xi} = \frac{\mu_R}{\mu_S} = \frac{[1 + \beta\varepsilon\delta_S]}{[1 - \beta\varepsilon\delta_R]} = \textbf{Central Safety factor}$$

Capacity reduction factor	$\bar{\phi} = 1 - \varepsilon\beta\delta_R$
Load factor	$\bar{\gamma} = 1 + \varepsilon\beta\mu_S$

$$\bar{\phi}\mu_R = \bar{\gamma}\mu_S$$

$$\left[1 - \varepsilon\Phi^{-1}\left(1 - P_F\right)\delta_R\right]\mu_R = \left[1 + \varepsilon\Phi^{-1}\left(1 - P_F\right)\delta_S\right]\mu_S$$

Nominal factor of safety

$$\xi = \frac{R_N}{S_N} = \frac{\mu_R}{\mu_S}\frac{(1-K_R\delta_R)}{(1+K_S\delta_S)} = \left(\frac{1+\varepsilon\beta\delta_S}{1-\varepsilon\beta\delta_R}\right)\frac{(1-K_R\delta_R)}{(1+K_S\delta_S)}$$

$$\Rightarrow$$

$$\varphi R_N \geq \gamma S_N$$

$$\varphi = \frac{1-\varepsilon\beta\delta_R}{1-K_R\delta_R}; \quad \gamma = \frac{1+\varepsilon\beta\delta_S}{1+K_S\delta_S}$$

Case of multiple loads

(Ref: A Haldar and S Mahadevan, 2000, Probability, reliability, and statistical methods in engineering design, John Wiley, NY)

- **Structures need to be designed for more than one loads**
- **It is unlikely that all the loads would act simultaneously**
- **Load combination needs to be considered**
  - **Dead load + live load**
  - **Dead load + live load + wind**
  - **Dead load + live load + earthquake**
  - **Dead load + wind, etc.**

$$S = \sum_{i=1}^{n} S_i; \quad S_i \to N(\mu_i, \sigma_i); \quad i = 1,2,\cdots,n$$

$$\Rightarrow S \to N(\mu_S, \sigma_S)$$

Use formulation already developed.
This leads to a single load factor for S.
Not very useful.
We need different load factors $\boldsymbol{\gamma_i}$ for $i=1,2,..,n$.

$$\mu_R = \mu_S + \beta\sqrt{\sigma_R^2 + \sigma_S^2}$$

Let $\varepsilon = \dfrac{\sqrt{\sigma_R^2 + \sigma_S^2}}{\sigma_R + \sigma_S}$

$$\Rightarrow \mu_R = \mu_S + \varepsilon\beta\left(\sigma_R + \sigma_S\right)$$

$$= \mu_S + \varepsilon\beta\left(\sigma_R + \sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2 + \cdots + \sigma_{S_n}^2}\right)$$

Let $\varepsilon_{nn} = \dfrac{\sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2 + \cdots + \sigma_{S_n}^2}}{\sigma_{S_1} + \sigma_{S_2} + \cdots + \sigma_{S_n}}$

$$\Rightarrow \mu_R = \mu_S + \varepsilon\beta\left[\sigma_R + \varepsilon_{nn}\left(\sigma_{S_1} + \sigma_{S_2} + \cdots + \sigma_{S_n}\right)\right]$$

$$\Rightarrow \mu_R = \mu_{S_1} + \mu_{S_2} + \cdots + \mu_{S_n}$$

$$+\varepsilon\beta\left[\sigma_R + \varepsilon_{nn}\left(\sigma_{S_1} + \sigma_{S_2} + \cdots + \sigma_{S_n}\right)\right]$$

$$\Rightarrow (1 - \varepsilon\beta\delta_R)\mu_R = \mu_{S_1}\left(1 + \varepsilon\beta\varepsilon_{nn}\delta_{S_1}\right) + \mu_{S_2}\left(1 + \varepsilon\beta\varepsilon_{nn}\delta_{S_2}\right) + \cdots$$

$$+\mu_{S_n}\left(1 + \varepsilon\beta\varepsilon_{nn}\delta_{S_n}\right)$$

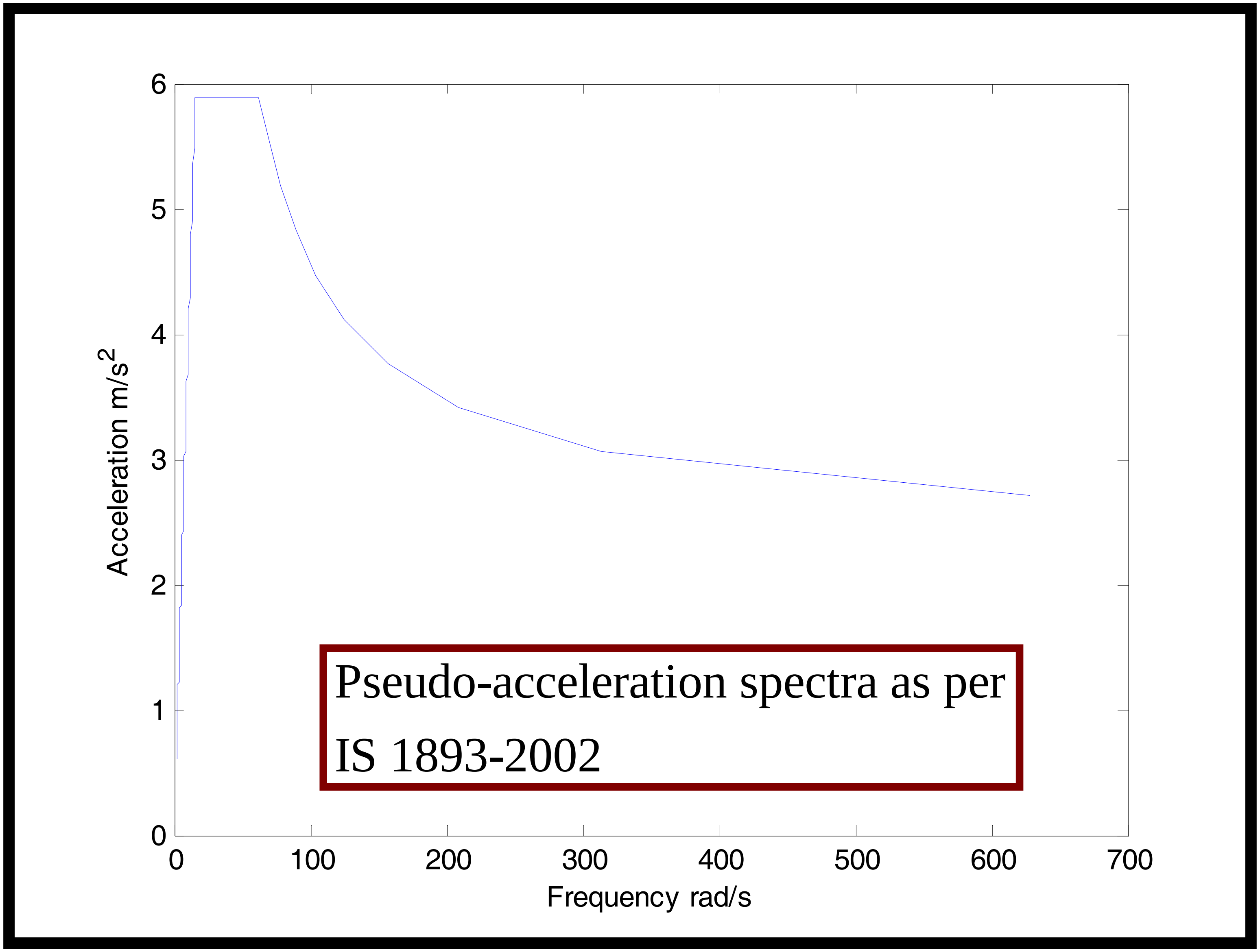
$$\Rightarrow \bar{\phi} = (1 - \varepsilon\beta\delta_R) \quad \& \quad \bar{\gamma}_i = 1 + \varepsilon\beta\varepsilon_{nn}\delta_{S_i}$$

Knowing P_F and variability measures we can find the load and Resistance factors.

Generalization:
Methods of structural reliability analysis

Problem 36

Figure (next slide) shows the pseudo-acceleration spectra for a rocky site according to the IS 1893 (Part 1) : 2002 document. The PGA is taken to be 0.24g. It is of interest to develop a random process model for the ground acceleration that is compatible with this response spectrum. It may be assumed that the ground acceleration can be modeled as a zero mean, stationary Gaussian random process. The duration of the acceleration can be taken to be 30s and the given response spectra may be interpreted as locus of the 84% percentile point and damping may be taken to be 5%.



6
5
4
3
2
1
0
Acceleration m/s2
0
100
200
300
400
500
600
700
Frequency rad/s
Pseudo-acceleration spectra as per
IS 1893-2002

How to generate a response spectrum compatible with a given PSD?

$$\ddot{x} + 2\eta_n \omega_n \dot{x} + \omega_n^2 x = -\ddot{x}_g$$

$\ddot{x}_g(t)$ = zero mean, stationary, Gaussian random process;

$$\ddot{x}_g(t) \sim N\left[0, S_{gg}(\omega)\right]$$

$$X_m = \max_{0<t<T} \left|x(t)\right|$$

$$P_{X_m}(\alpha) = \exp\left[-\nu^+(\alpha)T\right]$$

$$\nu^+(\alpha) = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)$$

with $\sigma_x^2 = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{gg}(\omega) d\omega$ &

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 \omega^2 S_{gg}(\omega) d\omega$$

For a given probability p, the corresponding α is given by

$$p = \exp\left[-\frac{\sigma_{\dot{x}}}{2\pi\sigma_x}\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)T\right]$$

$$\Rightarrow \alpha = \left\{-2\sigma_x^2 \ln\left[-\frac{2\pi\sigma_x}{\sigma_{\dot{x}}T}\ln(p)\right]\right\}^{\frac{1}{2}}$$

Let $R(\omega_n,\eta_n)$ be the given pseudo-acceleration response spectrum.

We interpret $R(\omega_n,\eta_n)$ as the p-th percentile point.

$$\Rightarrow R(\omega_n,\eta_n) = \omega_n^2\left\{-2\sigma_x^2 \ln\left[-\frac{2\pi\sigma_x}{\sigma_{\dot{x}}T}\ln(p)\right]\right\}^{\frac{1}{2}} \quad \text{(typically } p\text{=84\%)}$$

How to generate a PSD compatible with a given response spectrum?

$$\ddot{x} + 2\eta_n\omega_n\dot{x} + \omega_n^2 x = -\ddot{x}_g$$

$\ddot{x}_g(t) =$ zero mean, stationary, Gaussian random process;

$$\ddot{x}_g(t) \sim N\left[0, S_{gg}(\omega)\right]$$

To a first approximation we assume

$$\sigma_x^2 = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{gg}(\omega) d\omega \approx$$

$$(2\eta_n\omega_n)\left|H(\omega_n)\right|^2 S_{gg}(\omega_n) = (2\eta_n\omega_n)\frac{1}{\left(2\eta_n\omega_n^2\right)^2} S_{gg}(\omega_n)$$

$$\& \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \approx \omega_n$$

$$\Rightarrow R^2(\omega_n,\eta_n)=\omega_n^4\left\{-2\frac{S_{gg}(\omega_n)}{2\eta_n\omega_n^3}\ln\left[-\frac{2\pi}{\omega_n T}\ln(p)\right]\right\}$$

To a first approximation we thus get

$$S_{gg}(\omega_n)=\frac{\eta_n R^2(\omega_n,\eta_n)}{\omega_n\left\{-\ln\left[-\frac{1}{\omega_n T}\ln(p)\right]\right\}}.$$

Steps

(1) Set iteration N=1

(2) Start with the initial guess on the PSD given by

$$S^N(\omega_n) = \frac{\eta_n R^2(\omega_n, \eta_n)}{\omega_n \left\{ -\ln\left[-\frac{1}{\omega_n T} \ln(p) \right] \right\}}$$

(3)Evaluate σ_x^2 and $\sigma_{\dot{x}}^2$ using

$$\sigma_x^2 = \int_{-\infty}^{\infty} \left| H(\omega) \right|^2 S_{gg}^N(\omega) d\omega \;\&$$

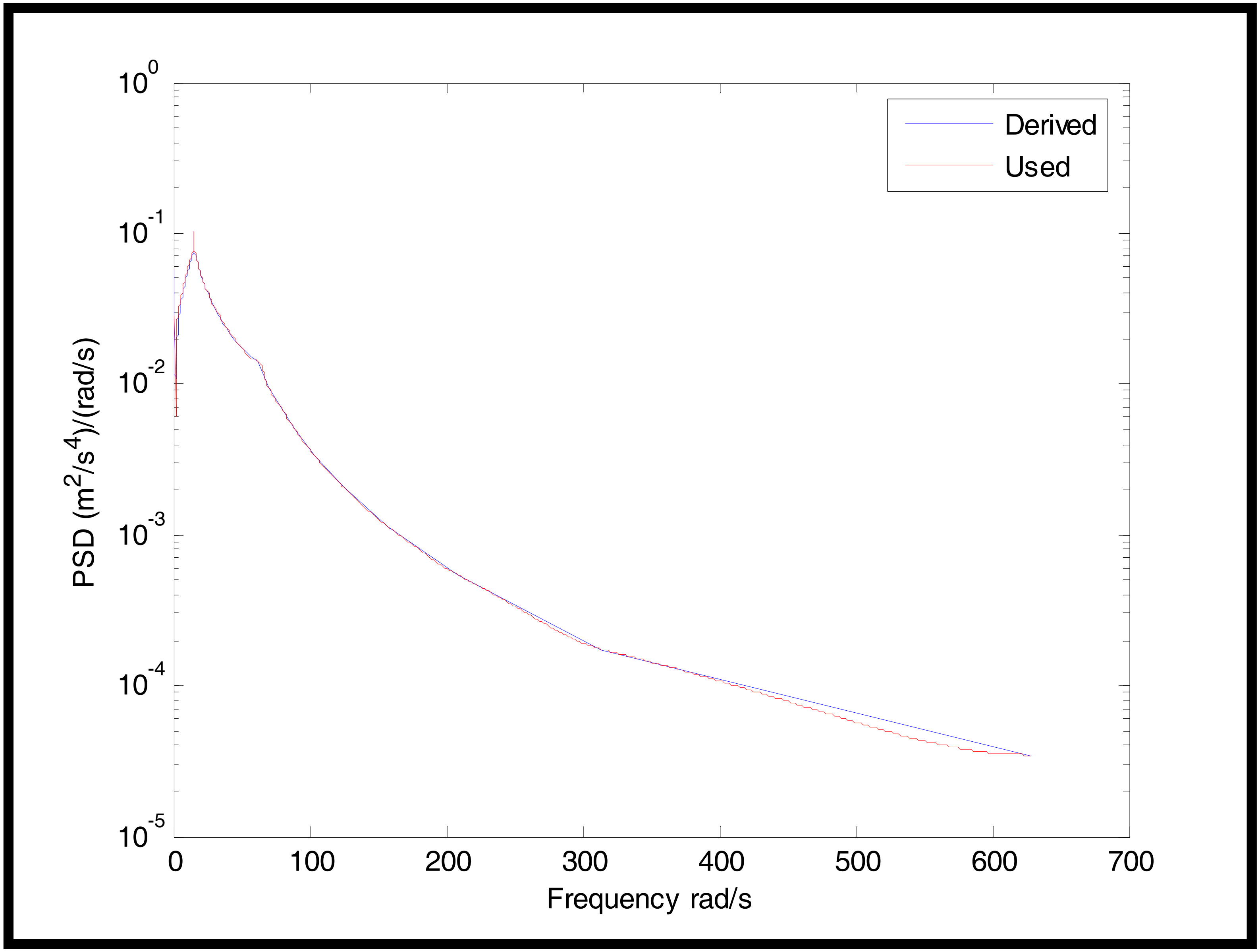
$$\sigma_x^2 = \int_{-\infty}^{\infty} \left| H(\omega) \right|^2 \omega^2 S_{gg}^N(\omega) d\omega$$

(4)Evaluate $R^N\left(\omega_n,\eta_n\right)=\omega_n^2\left\{-2\sigma_x^2\ln\left[-\frac{2\pi\sigma_x}{\sigma_{\dot{x}}T}\ln\left(p\right)\right]\right\}^{\frac{1}{2}}$.

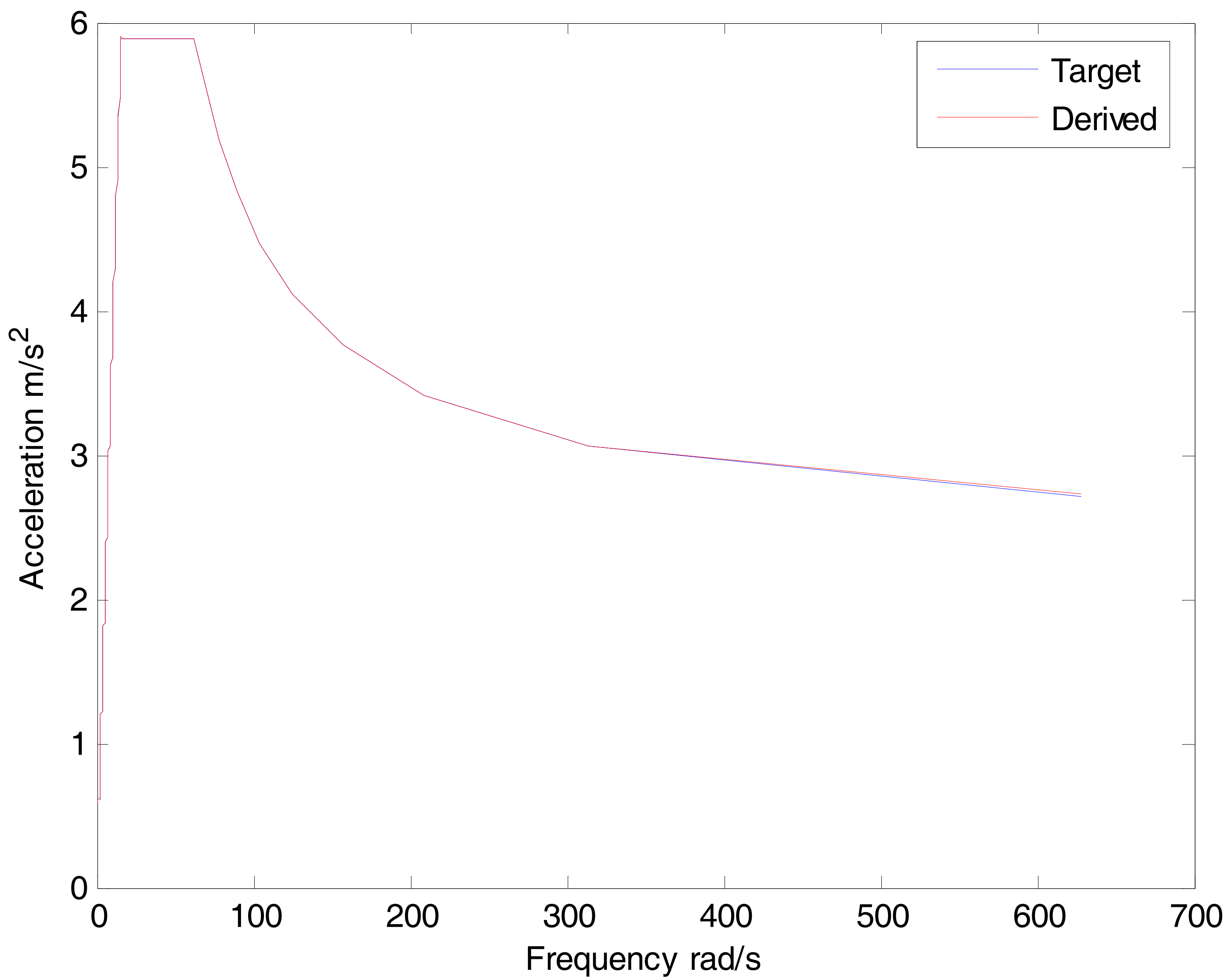
(5)Obtain an improved estimate of PSD using

$$S^{N+1}(\omega)=S^N(\omega)\left[\frac{R(\omega)}{R^N(\omega)}\right]^2$$

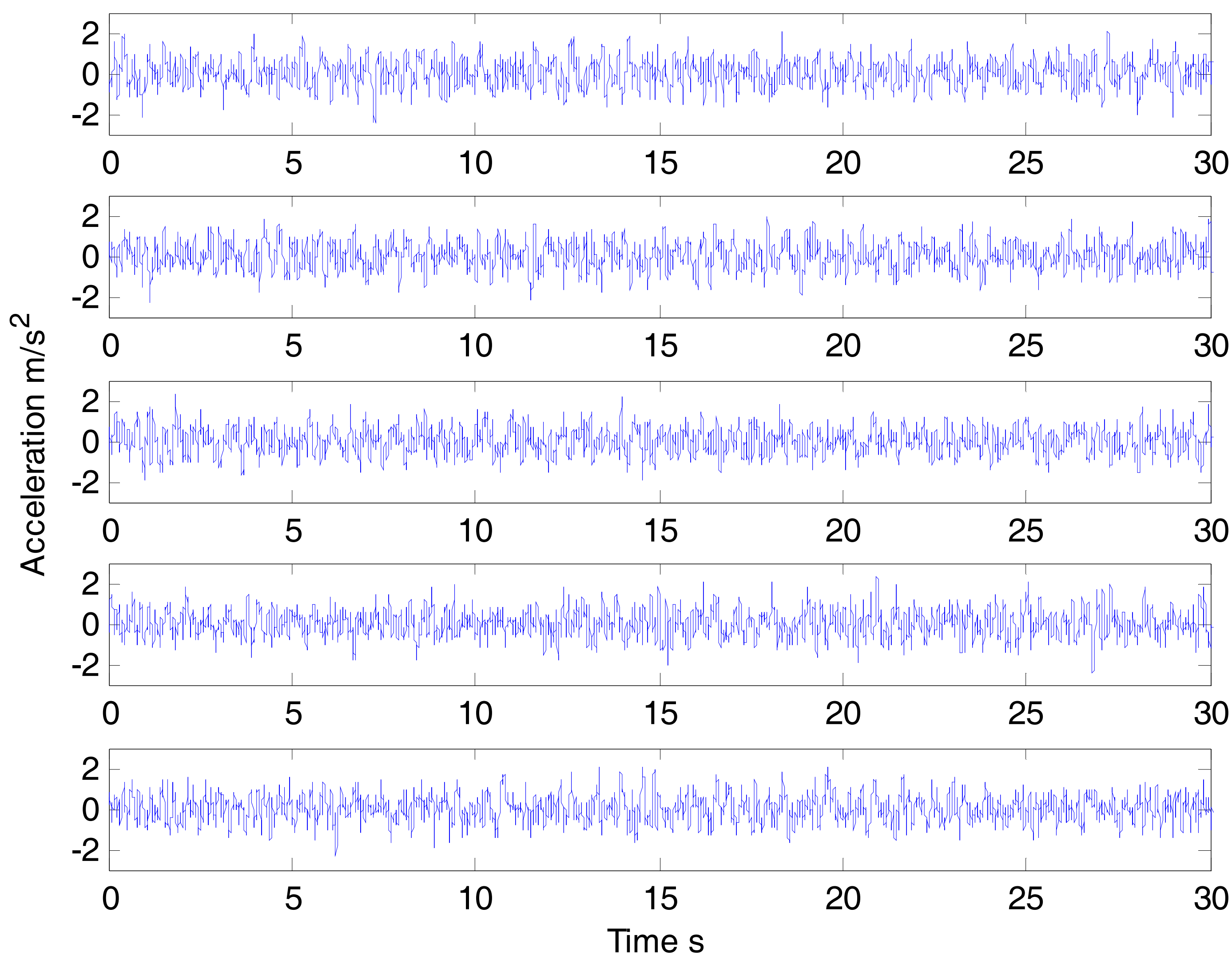
(6)Stop iterations if the PSD function has converged;
if not go to step 3.

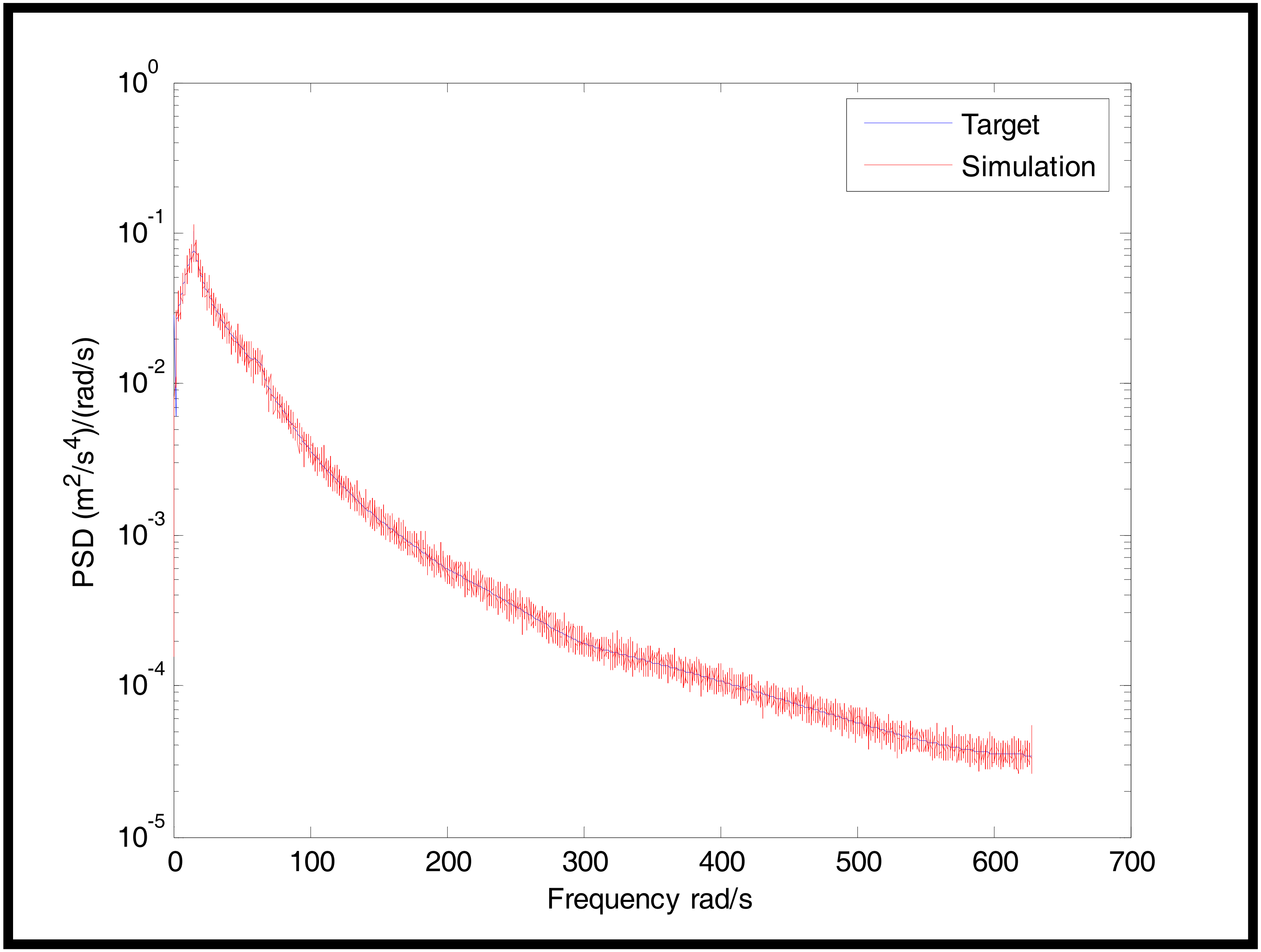


Derived
Used
PSD (m2/s4)/(rad/s)
Frequency rad/s
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
0
100
200
300
400
500
600
700

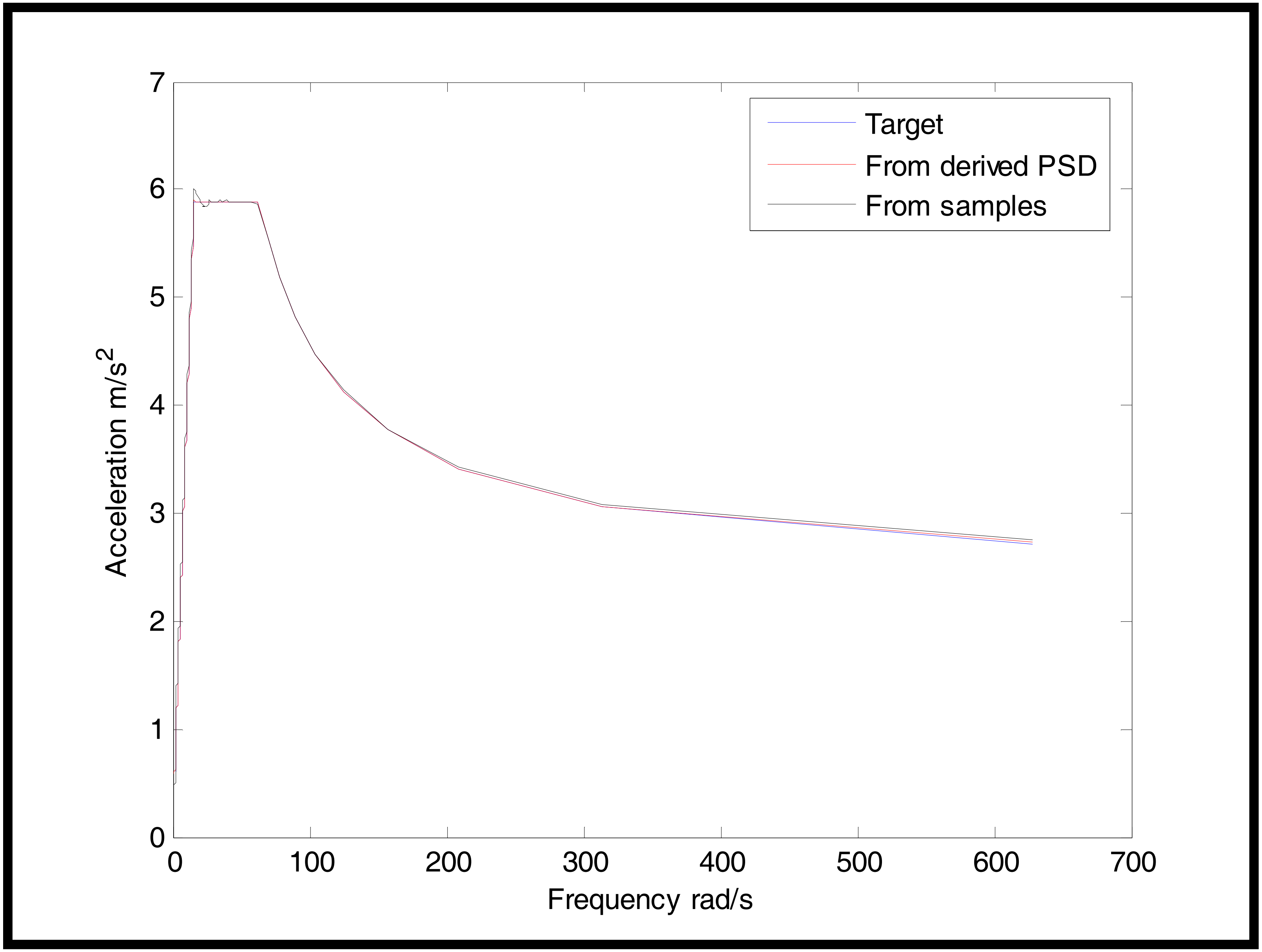


6
5
4
3
2
1
0
Acceleration m/s^2
0
100
200
300
400
500
600
700
Frequency rad/s
Target
Derived


Acceleration m/s^2
Time s
2
0
-2
0
5
10
15
20
25
30



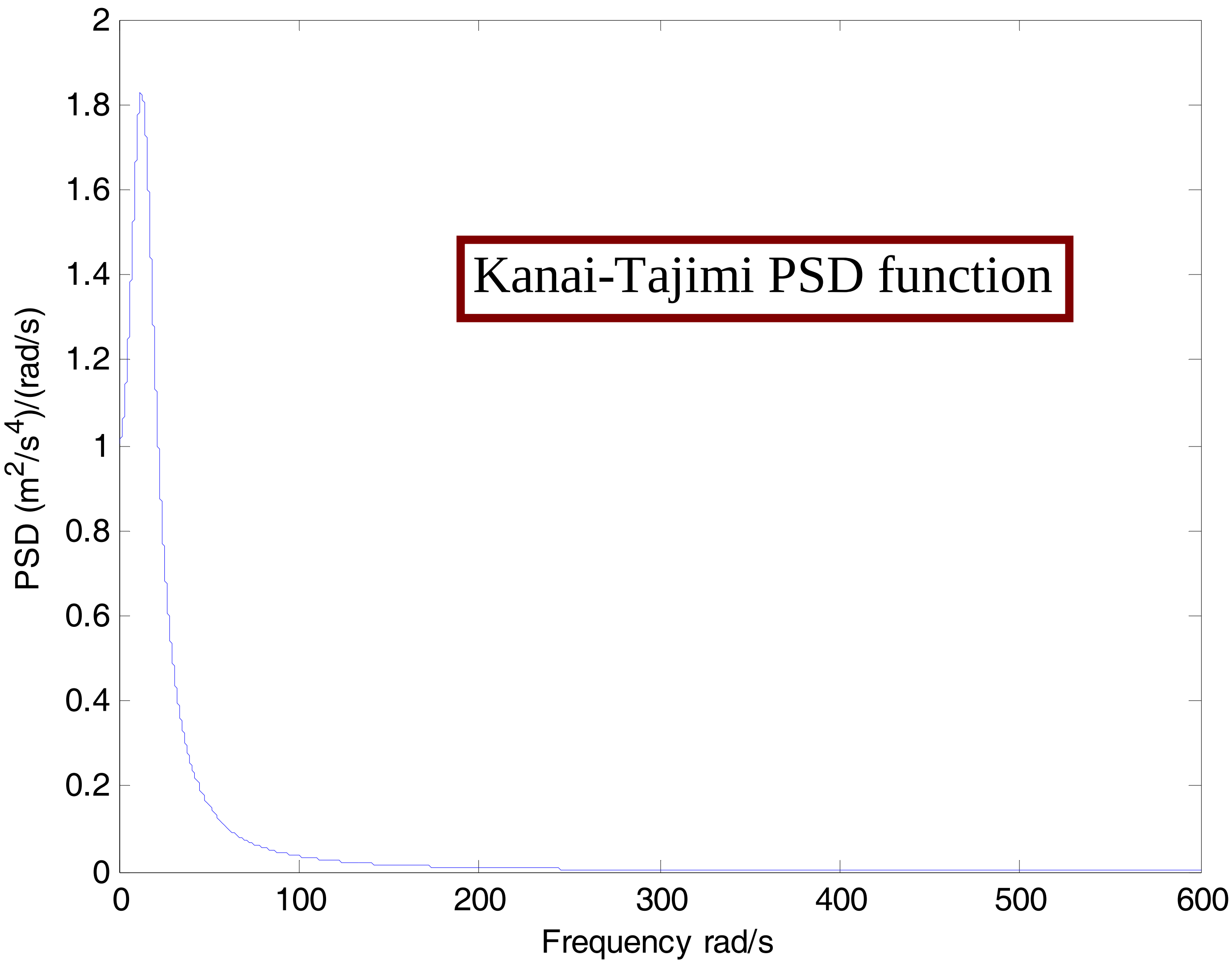
Target
Simulation
PSD (m²/s⁴)/(rad/s)
Frequency rad/s
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
0
100
200
300
400
500
600
700


Target
From derived PSD
From samples
Acceleration m/s²
Frequency rad/s
0
1
2
3
4
5
6
7
0
100
200
300
400
500
600
700

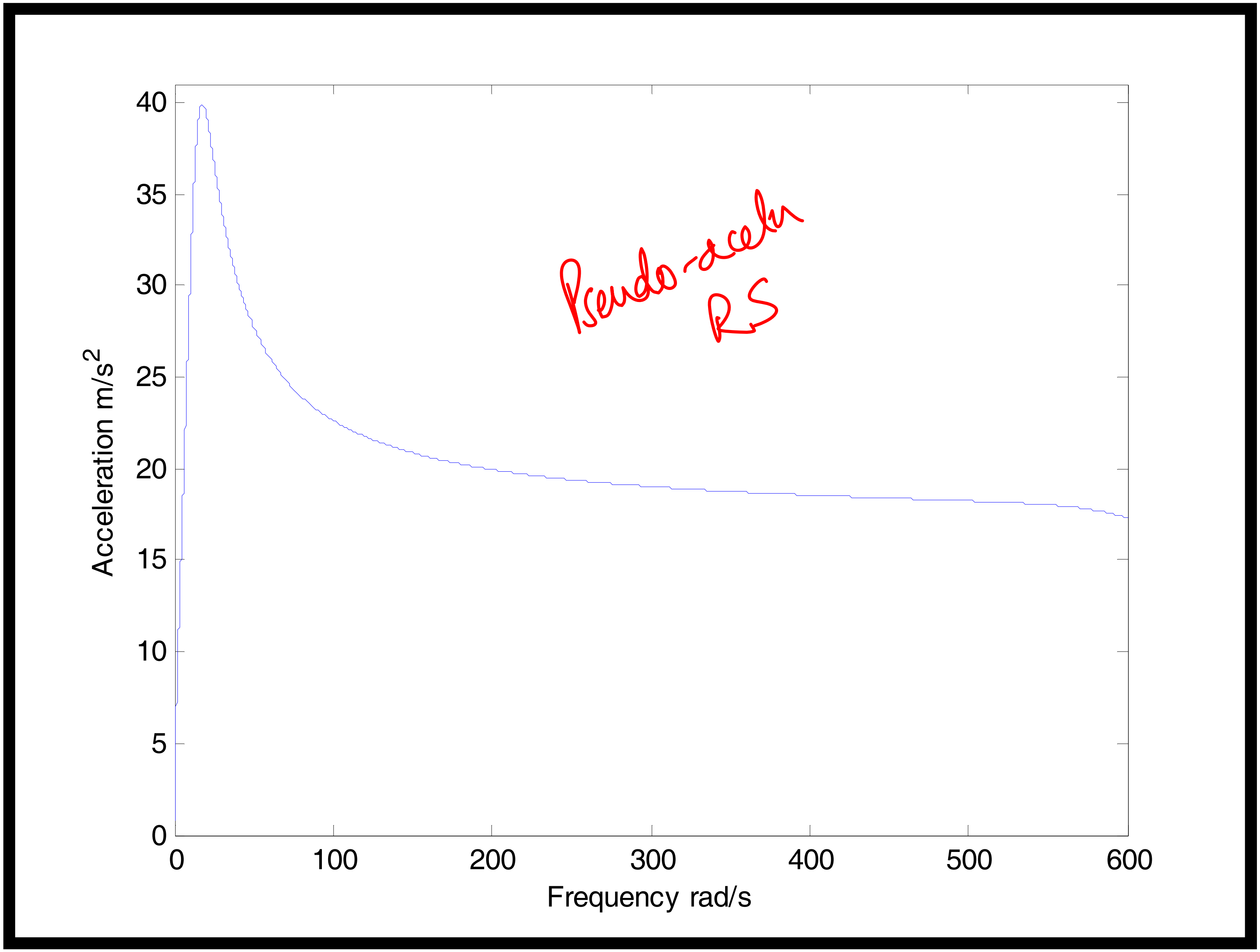
Problem 37

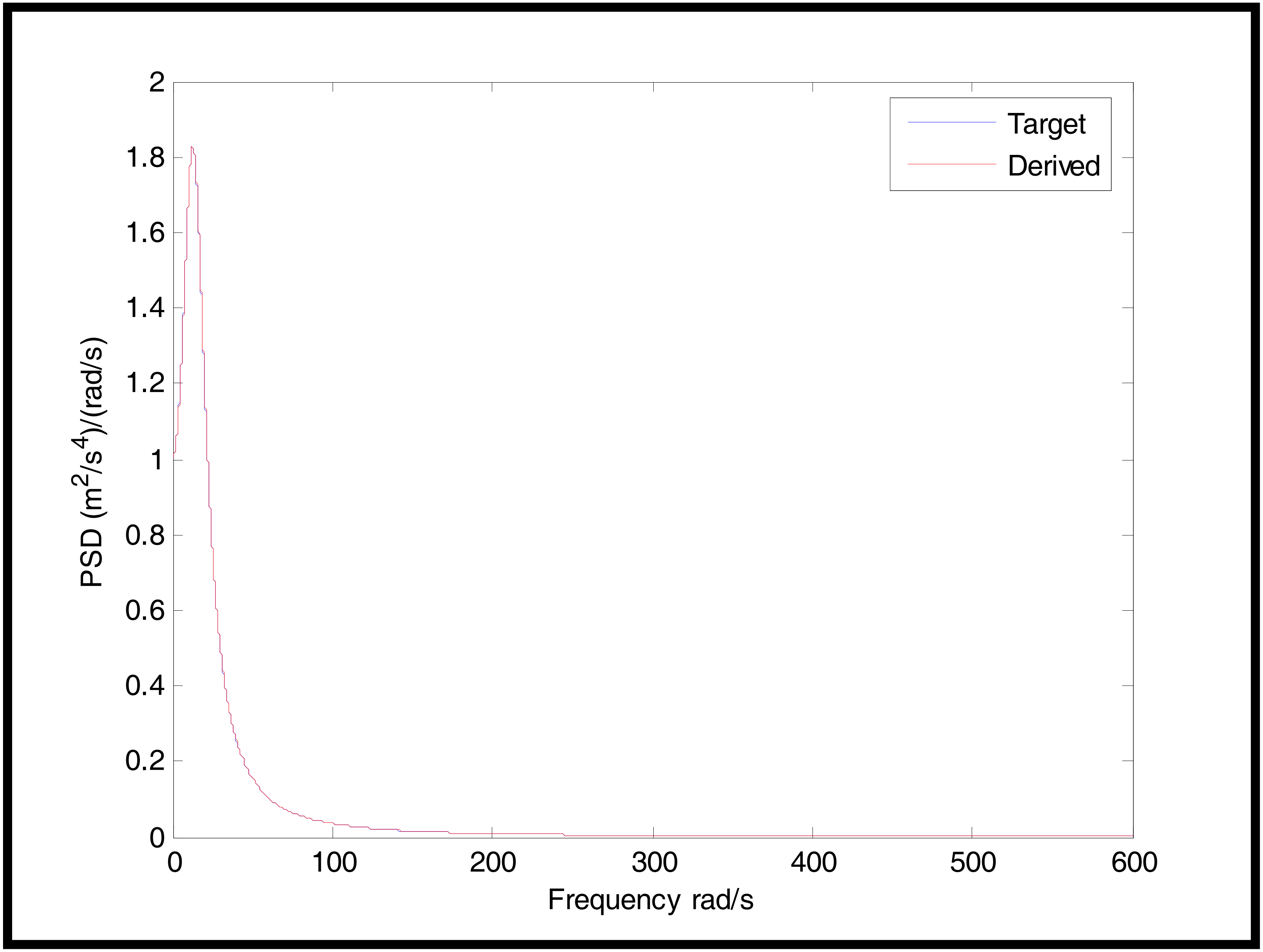
Figure (next slide) shows the psd function of ground acceleration which is modeled using Kanai-Tajimi's approach $\left(\text{with } \omega_g = 15\text{rad/s}, \ \eta_g = 0.6\right)$.

Determine the pseudo-acceleration spectra compatible with this psd function. It may be assumed that the ground acceleration is a zero mean, stationary Gaussian random process. The duration of the acceleration can be taken to be 30s and the target response spectra may be interpreted as the locus of the 84% percentile point and damping may be taken to be 5%.



Kanai-Tajimi PSD function
PSD (m2/s4)/(rad/s)
Frequency rad/s
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
100
200
300
400
500
600


Pseudo-acceln
RS
Acceleration m/s^2
Frequency rad/s
0
5
10
15
20
25
30
35
40
0
100
200
300
400
500
600



Target
Derived
PSD (m^2/s^4)/(rad/s)
Frequency rad/s
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
100
200
300
400
500
600

Discussion on outcrossing theory of random Processes and applications to problems of Load combination

Problem 38 : Load combination

- Let $\text{Q}\left(t\right)$ be a quasi-static load on a structure (e.g., sustained live load). If we are interested in designing the structure for this load, we can estimate $Q_m = \max\limits_t Q\left(t\right)$ and use that is the design.
- What happens if more than one loads act simultaneously?

$$\text{Q}\left(t\right) = Q_1\left(t\right) + Q_2\left(t\right)$$

$$\max_t \text{Q}\left(t\right) = \max_t \left[Q_1\left(t\right) + Q_2\left(t\right)\right] \neq \max_t Q_1\left(t\right) + \max_t Q_2\left(t\right)$$

$\because$ Maximum of $Q_1\left(t\right) \& Q_2\left(t\right)$ do not reach simultaneously.

Consider the failure event of $Q(t)$ crossing a critical barrier $\xi\left(t\right)$ and let $N_\xi\left(T\right) =$ number of times the level $\xi\left(t\right)$ is crossed during the interval 0 to T. Show that if $Q_1\left(t\right)$ and $Q_2\left(t\right)$ are independent, $\text{P}_F \leq P_0 + \text{E}\left[N_\xi\left(T\right)\right]$. Obtain an expression for $\text{E}\left[N_\xi\left(T\right)\right]$.

$$P_F = P\left[\text{Failure at } t = 0 \cup N_\xi(T) \geq 1\right]$$

$$= P\left[\text{Failure at } t = 0\right] + P\left[N_\xi(T) \geq 1\right] -$$

$$P\left[\text{Failure at } t = 0 \bigcap N_\xi(T) \geq 1\right]$$

$$\leq P_0 + P\left[N_\xi(T) \geq 1\right]$$

$$= P_0 + \sum_{n=1}^{\infty} P\left[N_\xi(T) = n\right]$$

$$\leq P_0 + \sum_{n=1}^{\infty} nP\left[N_\xi(T) = n\right]$$

$$= P_0 + \mathrm{E}\left[N_\xi(T)\right]$$

$$\Rightarrow \mathrm{P}_F \leq P_0 + \mathrm{E}\left[N_\xi(T)\right]$$

Recall that in order to characterize the average rate of crossing of a critical barrier by a random process, we need the jpdf of the process and its derivative at the same time instant.

Consider

$$Q(t) = Q_1(t) + Q_2(t)$$

$$\dot{Q}(t) = \dot{Q}_1(t) + \dot{Q}_2(t)$$

$$U = Q_2(t)$$

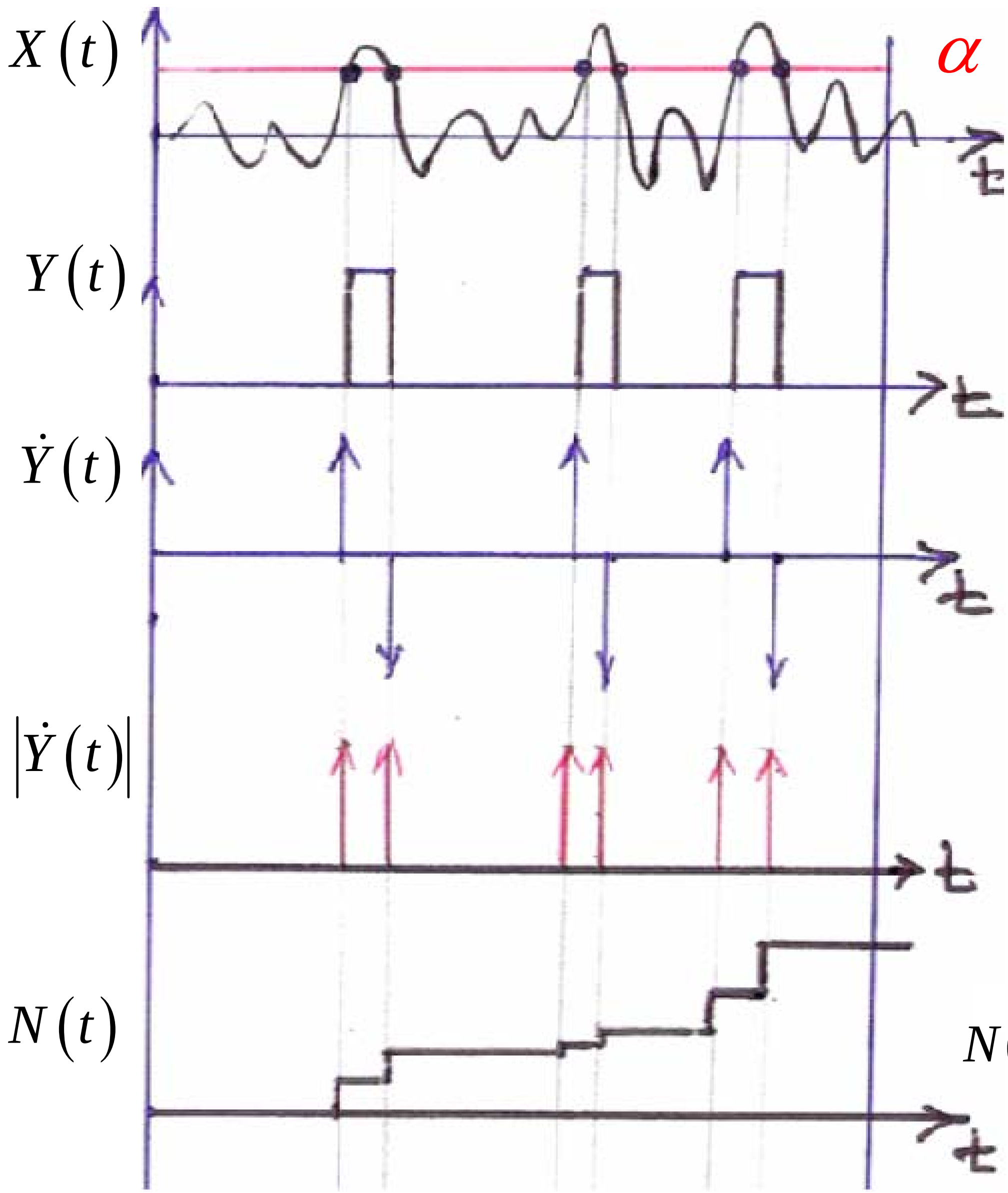
$$V = \dot{Q}_2(t)$$

$$p_{Q\dot{Q}UV}(q,\dot{q},u,v) = p_{Q_1\dot{Q}_1Q_2\dot{Q}}(q-u,\dot{q}-v,u,v)$$

$$= p_{Q_1\dot{Q}_1}(q-u,\dot{q}-v)\,p_{Q_2\dot{Q}}(u,v)$$

$$p_{Q\dot{Q}}(q,\dot{q},u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{Q_1\dot{Q}_1}(q-u,\dot{q}-v)\,p_{Q_2\dot{Q}}(u,v)\,du\,dv$$

RECALL



$$Y(t) = U\left[X(t) - \alpha\right]$$

$$\dot{Y}(t) = \dot{X}(t)\,\delta\left[X(t) - \alpha\right]$$

$$\left|\dot{Y}(t)\right| = \left|\dot{X}(t)\right|\delta\left[X(t) - \alpha\right]$$

$$N(T) = \int_0^T \left|\dot{X}(t)\right|\delta\left[X(t) - \alpha\right]dt$$

$N_{\xi}(T)$ = number of times the level $\xi(t)$ is crossed in $[0,T]$ with positive slope

$$Y(t) = U\left[Q(t) - \xi(t)\right]$$

$$\dot{Y}(t) = \left[\dot{Q}(t) - \dot{\xi}(t)\right]\delta\left[Q(t) - \xi(t)\right]$$

$$\dot{Z}(t) = \left[\dot{Q}(t) - \dot{\xi}(t)\right]\delta\left[Q(t) - \xi(t)\right]U\left[\dot{Q}(t) - \dot{\xi}(t)\right]$$

$$N_{\xi}(T) = \int_0^T \left[\dot{Q}(t) - \dot{\xi}(t)\right]\delta\left[Q(t) - \xi(t)\right]U\left[\dot{Q}(t) - \dot{\xi}(t)\right]dt$$

$$\nu_{\xi}^{+}(0,t) = \mathrm{E}\left\{\left[\dot{Q}(t) - \dot{\xi}(t)\right]\delta\left[Q(t) - \xi(t)\right]U\left[\dot{Q}(t) - \dot{\xi}(t)\right]\right\}$$

$$\nu_{\xi}^{+}(t) = \mathrm{E}\left\{\left[\dot{Q}(t)-\dot{\xi}(t)\right]\delta\left[Q(t)-\xi(t)\right]U\left[\dot{Q}(t)-\dot{\xi}(t)\right]\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left[\dot{q}-\dot{\xi}(t)\right]\delta\left[q-\xi(t)\right]U\left[\dot{q}-\dot{\xi}(t)\right]\right\}p_{Q\dot{Q}}(q,\dot{q};t)\,dq\,d\dot{q}$$

$$= \int_{\xi(t)}^{\infty}\left[\dot{q}-\dot{\xi}(t)\right]p_{Q\dot{Q}}\left(\xi(t),\dot{q};t\right)d\dot{q}$$

$$= \int_{\xi(t)}^{\infty}\left[\dot{q}-\dot{\xi}(t)\right]\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p_{Q_1\dot{Q}_1}\left(\xi(t)-u,\dot{q}-v\right)p_{Q_2\dot{Q}_2}(u,v)\,du\,dv\right\}d\dot{q}$$

$$E\left[N_{\xi}(T)\right] = \int_{0}^{T}\nu_{\xi}^{+}(t)\,dt$$

Remark

The evaluation of

$$v_{\xi}^{+}(t)$$

$$= \int_{\xi(t)}^{\infty} \left[\dot{q} - \dot{\xi}(t)\right] \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{Q_1\dot{Q}_1}\left(\xi(t) - u, \dot{q} - v\right) p_{Q_2\dot{Q}_2}(u, v)\, du\, dv \right\} d\dot{q}$$

is possible for Gaussian models for loads. A general solution is difficult to obtain.

Problem 39

Discussion of fatigue crack growth modeling under random loads using fracture mechanics concepts

Fracture mechanics based approaches

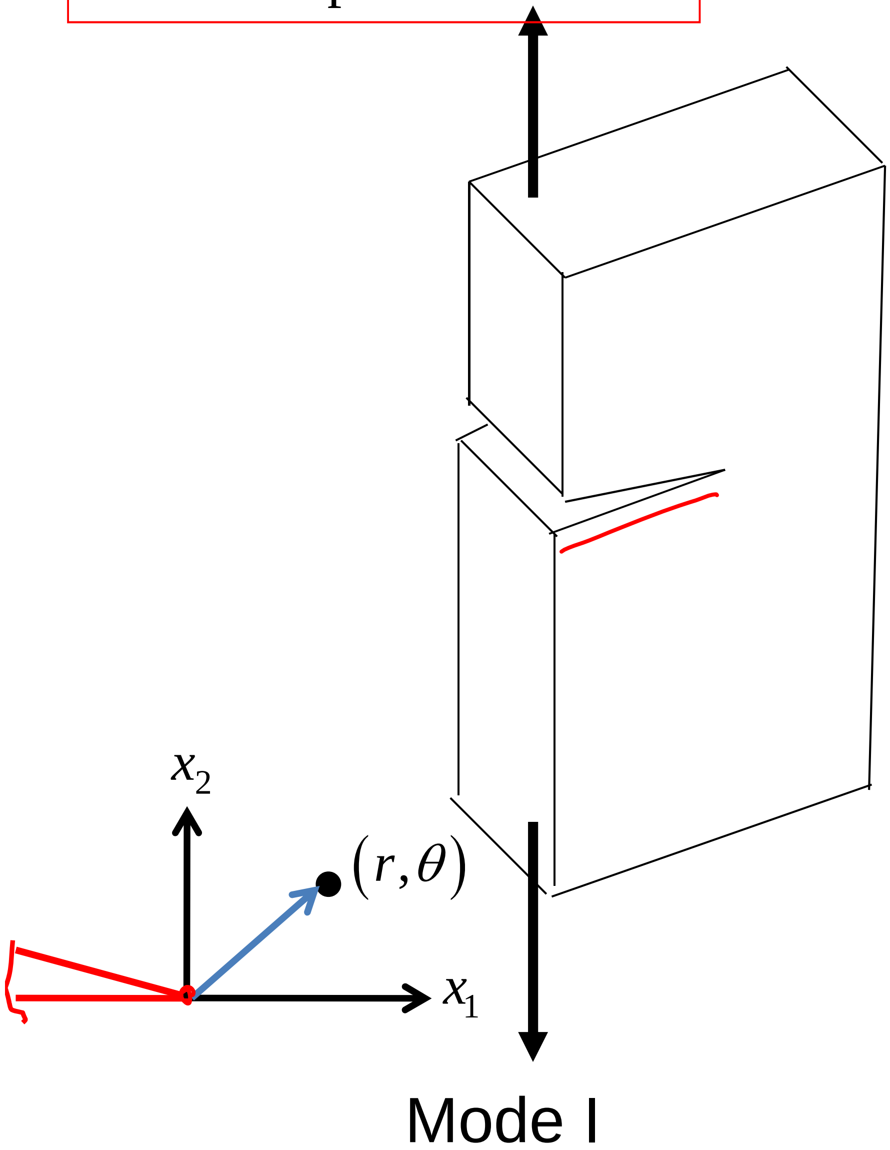
Basic assumption:
there exists a crack in the structural component.

Question:
Given the geometry of the crack, loads, boundary conditions, can we say if the crack is likely to grow?

Parameters for measuring the potency of the crack

- **Stress intensity factor** //
- Energy release rate
- J-integral
- Crack tip opening displacement

Stress near the cracktip in an infinite plate



Mode I (plane strain)

$$\sigma_{11} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{22} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{33} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \cdots$$

$$u_1 = \frac{\sigma\sqrt{\pi a}}{\mu}\sqrt{\left(\frac{r}{2\pi}\right)} \cos\frac{\theta}{2}\left[1 - 2\nu + \sin^2\frac{\theta}{2}\right]$$

$$u_2 = \frac{\sigma\sqrt{\pi a}}{\mu}\sqrt{\left(\frac{r}{2\pi}\right)} \sin\frac{\theta}{2}\left[2 - 2\nu + \cos^2\frac{\theta}{2}\right]$$

$$u_3 = 0$$

Stress Intensity Factor (SIF) and Critical SIF

In the expressions for stress and displacement components the quantities σ and $\sqrt{\pi a}$ appear together.

Can we give a name to the quantitiy $\sigma\sqrt{\pi a}$?

Recall: $EI,\ mv,\ 0.5mv^2, \xi = x - ct, \cdots$ Reynold's number, Froude's number$\cdots$

Definition

$$K_I = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{22}\left(r,\theta=0\right)$$

K_I = Mode I stress intensity factor $=\sigma\sqrt{\pi a}$.

Definition

Crack propagates if $\mathrm{K_I} > K_{Ic}$

K_{Ic} = critical stress intensity factor

Critical SIF is a material property.

Analogy

Stress	Yield stress
SIF	Critical SIF

Mode I (plane strain)

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

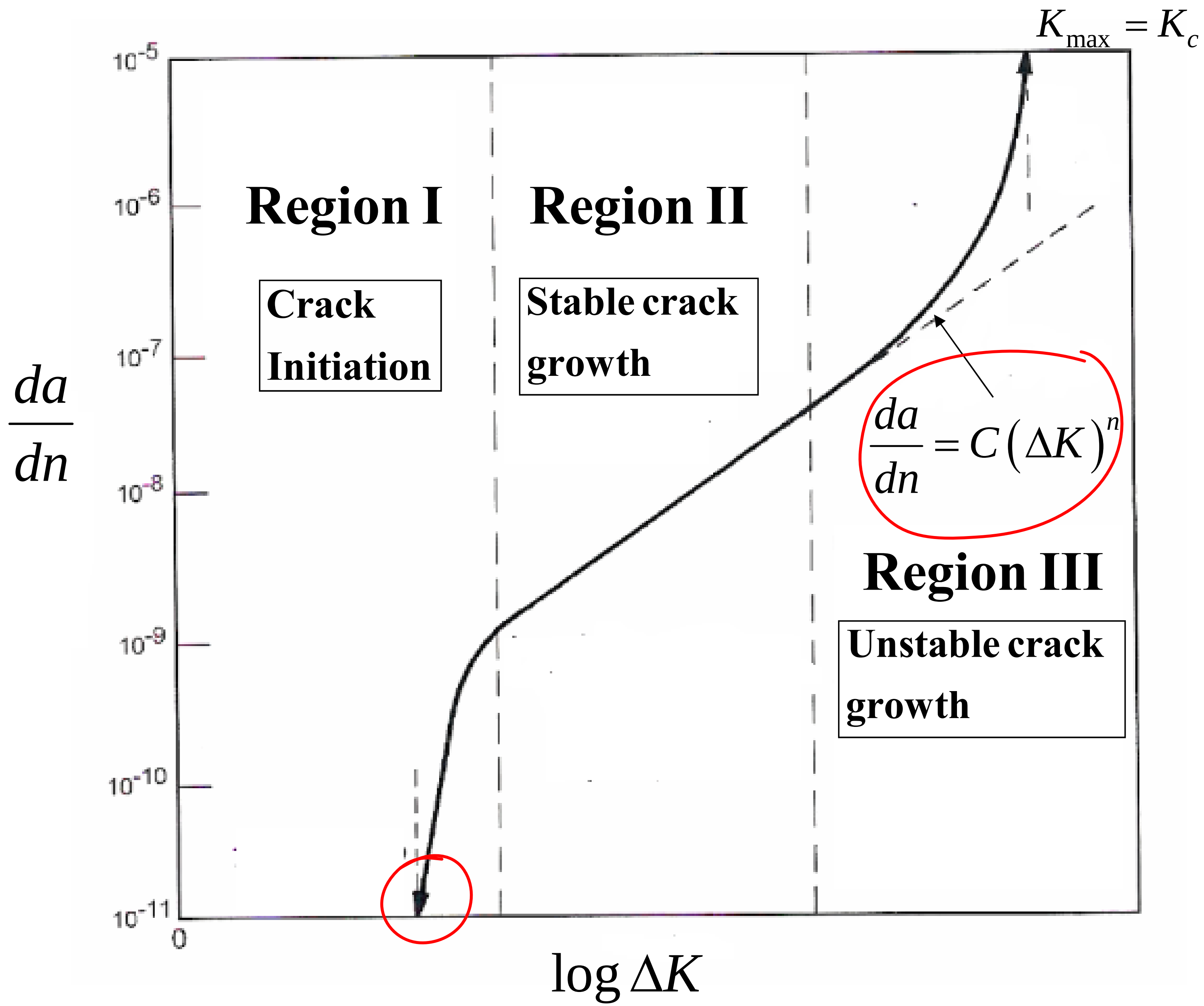
$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{33} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$u_1 = \frac{K_I}{\mu}\sqrt{\left(\frac{r}{2\pi}\right)} \cos\frac{\theta}{2}\left[1 - 2\nu + \sin^2\frac{\theta}{2}\right]$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\left(\frac{r}{2\pi}\right)} \sin\frac{\theta}{2}\left[2 - 2\nu + \cos^2\frac{\theta}{2}\right]$$

$$u_3 = 0$$


$K_{max} = K_c$
$\frac{da}{dn}$
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
0
Region I
Crack Initiation
Region II
Stable crack growth
$\frac{da}{dn} = C(\Delta K)^n$
Region III
Unstable crack growth
$\log \Delta K$

Model for Stage II crack growth

$$\frac{da}{dN} = f\left[\Delta K, K_{\max}, K_{\min}, \Delta K_{th}, E, \nu, \sigma_{ys}, \sigma_{ult}, \varepsilon_i, k_i\right]$$

ε_i = environmental variables

(temperature, humidity, salinity, etc.,)

k_i = other material or mechanics variables

(frequency of excitation, grain size, ...)

Dimensional analysis

$$\frac{da}{dN} = \left(\frac{\Delta K}{E}\right)^2 F\left[R, \frac{K_{Ic}}{\Delta K}, \frac{\Delta K_{th}}{\Delta K}, \frac{\sigma_{ys}}{E}, \frac{\sigma_{ult}}{E}, \alpha_i \varepsilon_i, \beta_i k_i\right]$$

Paris - Erdogan model

$$\frac{da}{dN} = C\left(\Delta K\right)^m ; \Delta K > 0; a\left(0\right) = a_0$$

$$\Rightarrow \log\left(\frac{da}{dN}\right) = \log C + m \log \Delta K$$

Example $\left(a \text{ in m } \& \ \Delta K \text{ in Mpa}\sqrt{\text{m}}\right)$

Ferrite pearlite steel : $\dfrac{da}{dN} = 6.80 \times 10^{-12}\left(\Delta K\right)^{3.00}$

69

Modeling of uncertainties

Sources

- Macro-properties of specimens (geometry, dimensions, and material properties may differ between specimens).
- External loading.
- Inhomgenous microstructure.

Tests on identical specimens

- Behavior of crack length of identical specimens is random
- The crack length behavior is nonlinear in time
- The curves of different specimens intermingle.

Two approaches

- Treat constants appearing in the differential equation for evolution of a as a fucntion of N as random variables.

$$\frac{da}{dN} = C\left(\Delta K\right)^m ; \Delta K > 0; a\left(0\right) = a_0$$

- Introduce random process models

$$\frac{da}{dN} = C\left(\Delta K\right)^m X\left(t\right); \Delta K > 0; a\left(0\right) = a_0$$

$$* \; N = \frac{\lambda t}{2\pi}$$

Cumulative jump models

(Reference: *K Sobczyk and B F Spencer Jr.*, 1992, *Random fatigue from data to theory*, *Academic Press*)

Define $A(t,\gamma)=$ random process: length of the dominant crack at time t.

- $\gamma \in \Omega$ (sample point). To be suppressed in further description.

$$A(t) = A_0 + \sum_{i=1}^{N(t)} Y_i; \quad Y_i = \Delta A_i$$

- $A_0 =$ Initial crack length; sufficiently long to propagate; could be random.
- $N(t)$= a counting process; homogeneous Poisson process; counts the number of crack increments in 0 to t.

$$P\left[N(t)=k\right]=\exp(-\lambda_0 t)\frac{(\lambda_0 t)^k}{k!};k=0,1,2,\cdots,\infty$$

- $\{Y_i\}_{i=1}^{\infty}=$ iid sequence of non-negative rvs with a common pdf $p_Y(y)$
- $N(t)\perp\{Y_i\}_{i=1}^{\infty}$
- $P\left[A(t)\leq a\right]=P_A(a;t)$ [PDF]
- $p_A(a;t)=\dfrac{dP_A(a;t)}{da}$ [pdf]

Let $A(t) = A_0 + A_1(t)$ with $A_1(t) = \sum_{i=1}^{N(t)} Y_i$

Consider the moment generating function of $A_1(t)$.

$$\langle \exp(-sA_1) \rangle = \left\langle \exp\left(-s \sum_{i=1}^{N(t)} Y_i \right) \right\rangle =$$

$$\sum_{k=0}^{\infty} \left\langle \exp\left(-s \sum_{i=1}^{N(t)} Y_i \right) \middle| N(t) = k \right\rangle P\left[N(t) = k \right]$$

$$= \sum_{k=0}^{\infty} \langle \exp(sY_i) \rangle^k \frac{(\lambda_0 t)^k}{k!} \exp(-\lambda_0 t)$$

$$\langle \exp(-sA_1)\rangle == \sum_{k=0}^{\infty} \langle \exp(sY_i)\rangle^k \frac{(\lambda_0 t)^k}{k!} \exp(-\lambda_0 t)$$

$$= \sum_{k=0}^{\infty} [G(s)]^k \frac{(\lambda_0 t)^k}{k!} \exp(-\lambda_0 t)$$

Here $G(s)$ is the moment generating function of Y_i.

That is, $G(s) = \langle \exp(-sY)\rangle$

Let us assume $p_Y(y) = \alpha \exp(-\alpha y); y \geq 0$

$$\Rightarrow G(s) = \frac{\alpha}{\alpha + s}; s > 0.$$

$$\Rightarrow p_{A_1}(a;t) = \exp(-\lambda_0 t - \alpha a) \sum_{k=0}^{\infty} \frac{(\alpha \lambda_0 t)^{k+1} a^k}{k!(k+1)!}; a > 0$$

$$p_{A_1}(a;t) = \exp(-\lambda_0 t - \alpha a) \sum_{k=0}^{\infty} \frac{(\alpha\lambda_0 t)^{k+1} a^k}{k!(k+1)!}; a > 0$$

$$= \sqrt{\frac{(\alpha\lambda_0 t)}{a}} \exp(-\lambda_0 t - \alpha a) I_1\left(2\sqrt{\lambda_0 \alpha a t}\right); a > 0$$

where $\mathrm{I}_1(\cdot)$ = Bessel's function of the first order.

$$A(t) = A_0 + A_1(t) \Rightarrow p_A(a;t) = p_{A_1}(a - A_0, t)$$

Model for life time

Let ξ be the critical crack length

(esimtated from the knowledge of K_{Ic}).

T = time required for $A(t)$ to reach the critical length ξ.

$$P\left(T > t\right) = P\left[A\left(t\right) \le \xi\right]$$

$$\Rightarrow P_T\left(t\right) =$$

$$1 - \int_0^{\xi} \sqrt{\frac{\left(\alpha\lambda_0 t\right)}{a - A_0}} \exp\left(-\lambda_0 t - \alpha\left\{a - A_0\right\}\right) I_1\left(2\sqrt{\lambda_0\alpha\left\langle a - A_0\right\rangle t}\right) da$$

It can be shown that

$$p_T\left(t\right) = \lambda_0 \exp\left(-\lambda_0 t - \alpha\left\{a - A_0\right\}\right) I_0\left[2\sqrt{\lambda_0\alpha\left\langle a - A_0\right\rangle t}\right]; 0 < t < \infty$$

Estimation of system parameters

Model parameters: λ_0 associated with the process $N(t)$;

α : associated with $p_Y(y)$.

Idea: derive these model parameters from laws such as the Paris law. An approximate method to achieve this would be to modify the Paris law to allow for randomness in applied stress and system parameters.

$$\frac{da_p}{dN} = C(\Delta K)^m ; a_p(0) = a_0$$

Let $S(t)$ be the stress field that is modeled as a Gaussian, stationary random process.

- **What is meant by cycle?**

$$N = \omega_s t \Rightarrow \frac{d}{dN} = \frac{d}{dt}\frac{dt}{dN} = \frac{1}{\omega_s}\frac{d}{dt}$$

$$\Rightarrow \frac{da_p}{dt} = \omega_s C\left[\left(S_{\max} - S_{\min}\right)\sqrt{\pi a_p}\,\right]^m ; a_p\left(0\right) = a_0$$

Interpret ω_s as the average rate of peaks in $S(t)$.

$\omega_s =$ average rate of zero crossing of $\dot{S}(t)$.

$$\omega_s = \frac{1}{2\pi}\left[\frac{\int_{-\infty}^{\infty} \omega^4 S_s(\omega)\,d\omega}{\int_{-\infty}^{\infty} \omega^2 S_s(\omega)\,d\omega}\right]^{\frac{1}{2}}$$

•Interpretation of ΔK

Recall: $\Delta K = \Delta\sigma\sqrt{\pi a}$. Interpret $\Delta\sigma = \langle S_{\max} - S_{\min}\rangle =$ mean range.

$$S_{mr} = \langle S_{\max} - S_{\min}\rangle = 2S_{rms}\sqrt{\frac{\pi}{2}\left(1-\varepsilon^2\right)}$$

$$\varepsilon = \left(1 - \frac{\Lambda_2^2}{\Lambda_0\Lambda_4}\right)^{\frac{1}{2}};\Lambda_n = \int_{-\infty}^{\infty} \omega^n S_s(\omega)\,d\omega$$

$$\Rightarrow \frac{da_p}{dt} = \omega_s C\left(\sqrt{\pi}\right)^m a_p^{\frac{m}{2}} \langle S_{\max} - S_{\min}\rangle^m ; a_p(0) = a_0$$

•Interpretation of λ_0

If we take $N(t)$=number of peaks above a level s_0, then λ_0 becomes the average rate of peaks in $S(t)$ above level s_0. Select s_0 = fatigue limit of material (that is the endurance limit).

$$\lambda_0 = \frac{1}{2\pi}\left\{\left(\frac{\Lambda_4}{\Lambda_2}\right)^{\frac{1}{2}}\left[1-\Phi\left(s_0\sqrt{\frac{\Lambda_4}{\Lambda_4-\Lambda_2^2}}\right)\right]\right.$$

$$\left.+\sqrt{(2\pi\Lambda_2)}\phi(s_0)\Phi\left(\frac{s_0\Lambda_2}{\sqrt{\Lambda_4-\Lambda_2^2}}\right)\right\}$$

How to find α?

Select α such that

$$F(\alpha) = \int_0^{t^*} \left\langle \left[A_p(t) - A(t) \right]^2 \right\rangle dt$$

is minimized.

Here t^* = time required by $A_p(t)$ to reach ξ.