Stochastic Structural Dynamics

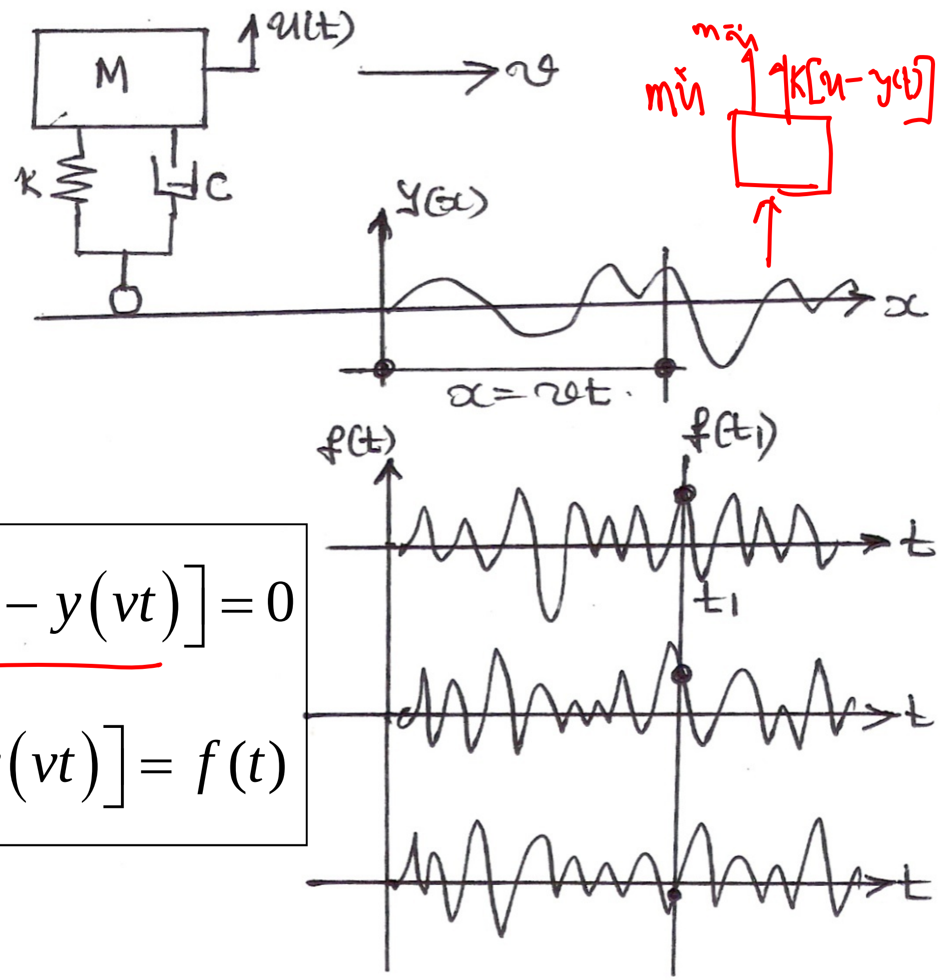
Lecture-6

Random processes-1

Dr C S Manohar
Department of Civil Engineering
Professor of Structural Engineering
Indian Institute of Science
Bangalore 560 012 India
manohar@civil.iisc.ernet.in

Review of theory of Random processes

Guideway unevenness



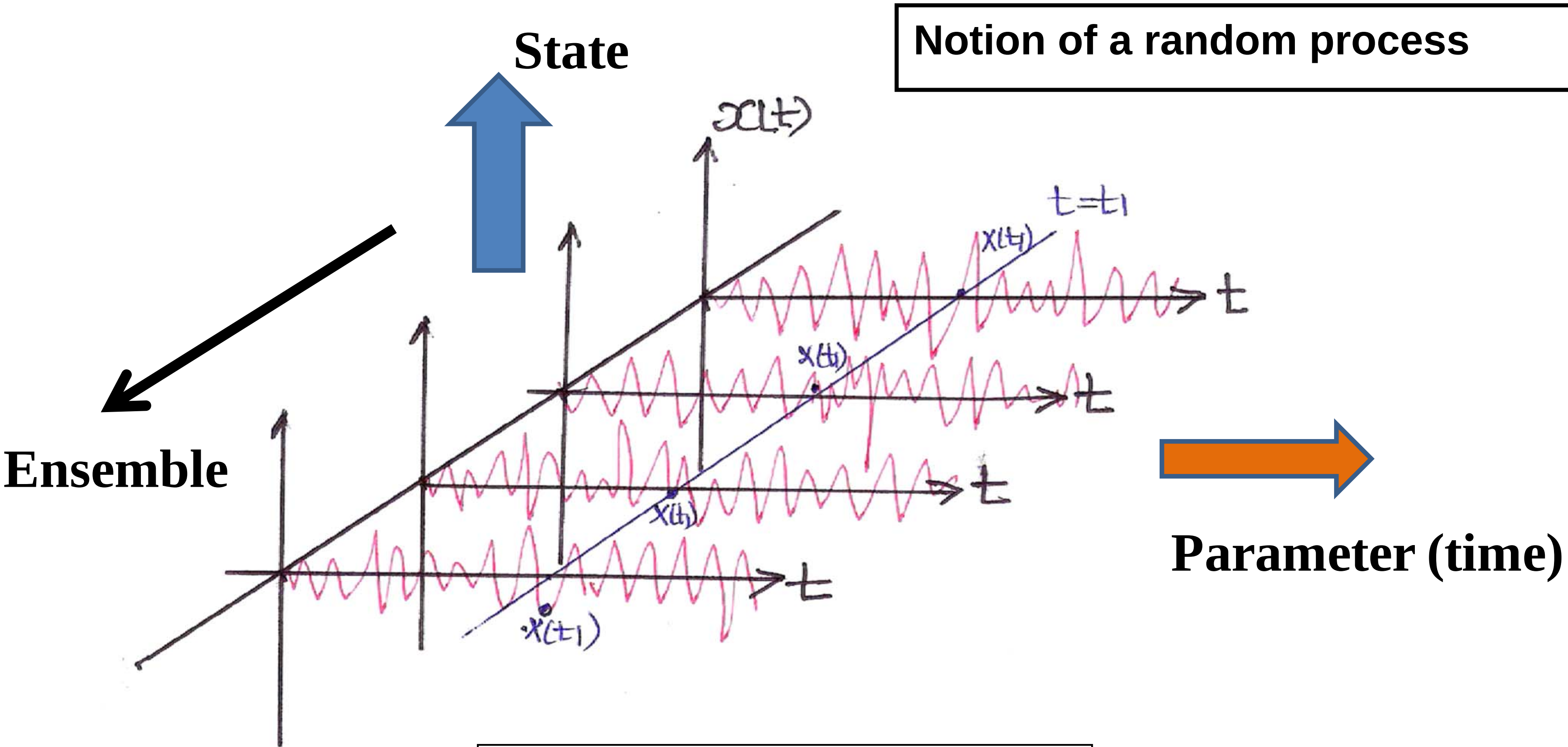
$$m\ddot{u} + c\frac{d}{dt}\left[u(t) - y(vt)\right] + k\left[u(t) - y(vt)\right] = 0$$

$$m\ddot{u} + c\dot{u} + ku = c\frac{d}{dt}\left[y(vt)\right] + k\left[y(vt)\right] = f(t)$$

For example, if $y(x) = \Delta \sin \lambda x,\ f(t) = -c\lambda v \Delta \cos(\lambda vt) + k\Delta \sin(\lambda vt)$.

For more complicated forms of guideway uneveness, $f(t)$ would be more complicated.

Notion of a random process



Working definition:
A random variable that evolves in time.
Or
Parametered family of random variables.

Analogy

Random variable — Statics

Random process — Dynamics

When to model a quantity as random variable and when to model it as a random process?

This is analogous to asking when to model a system as static and when as dynamic.

Recall

Random variable is a function from sample space into real line such that

(1) for every $x \in R$, $\{\omega : X(\omega) \le x\}$ is an event,

(2) $P\left(\omega : X(\omega) = \pm\infty\right) = 0$

A random process is a function: $\Omega \times R \to R$

and is denoted by $X\left(t, \omega\right)$ $\left[\text{and is written as } X(t)\right]$ such that

(a)for a fixed value of t, $X\left(t, \omega\right)$is a random variable,

(b) for a fixed value of ω, $X\left(t, \omega\right)$is a function of time (a realization),

(c) for fixed values of t and ω, $X\left(t, \omega\right)$is a number, and

(d) for varying t and ω, $X\left(t, \omega\right)$ is collection of time histories (ensemble).

Terminology

Evolution in time : Random processes
Evolution in space: Random fields

Mathematically it is not necessary to maintain this distinction

Stochastic processes
Stochastic field
Random functions
Time series

A scheme for classification of random processes

Let $X(t,\omega)$ be a random process.

t=parameter; values taken by $X(t,\omega)$=state.

For fixed value of t

If $X(t,\omega)$ is a discrete random variable, then $X(t,\omega)$ is a random process with a discrete state space.

If $X(t,\omega)$ is a continuous random variable, then $X(t,\omega)$ is a random process with a continuous state space.

If t takes only discrete values, we say that $X(t,\omega)$ is a random process with discrete paramters.

If t takes continuous values, we say that $X(t,\omega)$ is a random process with continuous parameters.

Four categories of random processes

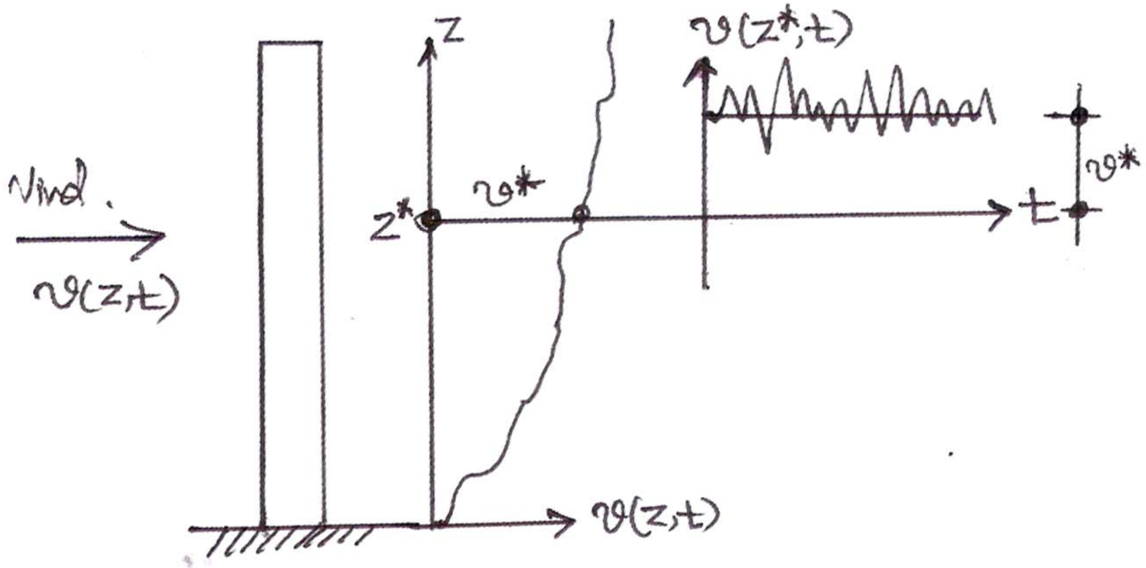
(a) discrete state discrete parameter random processes

(b) discrete state continuous parameter random processes

(a) continuous state discrete parameter random processes

(a) continuous state continuous parameter random processes

Parameter need not always be time…



Evolution of wind velocity in space and time

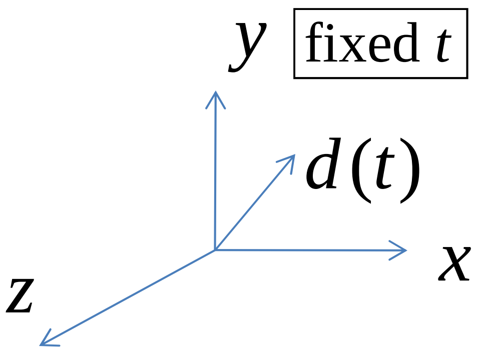
Other examples

(a) Road roughness (evolution in space)

(b) wave heights (evolution in space and time)

(c) Thickness of a cylindrical shell (evolution in an angle)

(d) FRF-s evolution in frequency (and space)

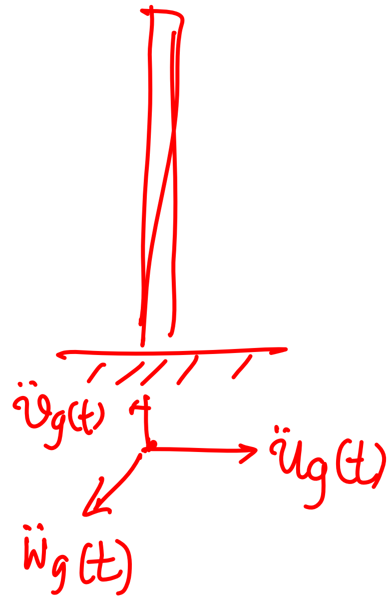


Vector random process

$$d(t)=\begin{Bmatrix} u_g(t) \\ v_g(t) \\ w_g(t) \end{Bmatrix}:\ \text{ground displacement}$$

$$v(t)=\begin{Bmatrix} \dot{u}_g(t) \\ \dot{v}_g(t) \\ \dot{w}_g(t) \end{Bmatrix}:\ \text{ground velocity}$$

$$a(t)=\begin{Bmatrix} \ddot{u}_g(t) \\ v_g(t) \\ \ddot{w}_g(t) \end{Bmatrix}:\ \text{ground acceleartion}$$



Description of a random process

First order Probability Distribution Function

$$P_X(x;t) = P[X(t) \le x]$$

First order probability density function

$$p_X(x;t) = \frac{\partial P_X(x;t)}{\partial t}$$

Random variable

$$P_X(x;t)$$

State variable

Parameter

Second order Probability Distribution Function

$$P_{XX}\left(x_1, x_2; t_1, t_2\right) = P\left[X\left(t_1\right) \leq x_1 \bigcap X\left(t_2\right) \leq x_2\right]$$

Second order probability density function

$$p_{XX}\left(x_1, x_2; t_1, t_2\right) = \frac{\partial^2 P_{XX}\left(x_1, x_2; t_1, t_2\right)}{\partial x_1 \partial x_2}$$

n - th order Probability Distribution Function

$$P_{\tilde{X}}\left(\tilde{x};\tilde{t}\right) = P\left[\bigcap_{i=1}^{n}\left\{X\left(t_i\right) \le x_i\right\}\right]$$

n - th order probability density function

$$p_{\tilde{X}}\left(\tilde{x};\tilde{t}\right) = \frac{\partial^n P_{\tilde{X}}\left(\tilde{x};\tilde{t}\right)}{\partial x_1 \partial x_2 \cdots \partial x_n}$$

Complete description of a random process

Specify $P_{\tilde{X}}\left(\tilde{x};\tilde{t}\right)$ for all n and for any choice of $\tilde{t}$.

OR

Specify $p_{\tilde{X}}\left(\tilde{x};\tilde{t}\right)$ for all n and for any choice of $\tilde{t}$.

$X(t)$: RP

Expectation of a random process

$$m_X(t) = \langle X(t) \rangle = \int_{-\infty}^{\infty} x \, p_X(x;t)\, dx$$

RP $X(t)$

Mean

$$\sigma_X^2(t) = \int_{-\infty}^{\infty} [x - m_X(t)]^2 \, p_X(x;t)\, dx$$

$$= \langle [X(t) - m_X(t)]^2 \rangle$$

Variance

t_1 & t_2 $\quad X(t_1)$ $\quad X(t_2)$

Autocovariance

$$C_{XX}(t_1, t_2) = \langle [X(t_1) - m_X(t_1)][X(t_2) - m_X(t_2)] \rangle$$

$$= \int_{-\infty}^{\infty}\int [x_1 - m_X(t_1)][x_2 - m_X(t_2)] \, p_{XX}(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

Autocorrelation

$$R_{XX}(t_1, t_2) = \langle X(t_1)\, X(t_2) \rangle$$

$$= \int_{-\infty}^{\infty}\int x_1 x_2 \, p_{XX}(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

Autocorrelation coefficient

$$r_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sigma_X(t_1)\, \sigma_X(t_2)}$$

Remarks

(a) $C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2)$ if $m_X(t_1) = m_X(t_2) = 0$

(b) $\sigma_{\mathrm{X}}^2(t) = C_{XX}(t, t)$

(c) $\left| r_{XX}(t_1, t_2) \right| \leq 1$ (prove it)

Gaussian random process

Let $X(t)$ be a random process and consider its 1st and 2nd order pdf-s.

$$p_X\left(x;t\right)=\frac{1}{\sqrt{2\pi}\sigma_X\left(t\right)}\exp\left[-\frac{1}{2}\left\{\frac{x-m_X\left(t\right)}{\sigma_X\left(t\right)}\right\}^2\right];-\infty<x<\infty$$

$$p_{XX}\left(x_1,x_2;t_1,t_2\right)=\frac{1}{\left(2\pi\right)\sigma_1\sigma_2\sqrt{\left[1-r_{12}^2\right]}}$$

$$\exp\left[-\frac{1}{2\left\{1-r_{12}^2\right\}}\left\{\frac{\left(x_1-m_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-m_2\right)^2}{\sigma_2^2}-2r_{12}\frac{\left(x_1-m_1\right)\left(x_2-m_2\right)}{\sigma_1\sigma_2}\right\}\right]$$

$$-\infty<x_1,x_2<\infty$$

$$m_1=m_X\left(t_1\right);m_2=m_X\left(t_2\right);\sigma_1=\sigma_X\left(t_1\right);\sigma_2=\sigma_X\left(t_2\right);r_{12}=r_{XX}\left(t_1,t_2\right)$$

Continuing further, consider n time instants $\{t_i\}_{i=1}^{n}$ and associated random variables $\{X(t_i)\}_{i=1}^{n}$.

Let the jpdf of $\{X(t_i)\}_{i=1}^{n}$ be given by

$$p_{XX\cdots X}(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n) =$$

$$\frac{1}{(2\pi)^{\frac{n}{2}} |S|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\eta)^t S^{-1}(x-\eta)\right]; -\infty < x_i < \infty \forall i \in [1, n]$$

$$S_{ij} = \left\langle \left[X(t_i) - m_X(t_i)\right]\left[X(t_j) - m_X(t_j)\right]\right\rangle$$

Note: $S^t = S$ & S is positive definite.

$$\eta = \left[m_X(t_1) \quad m_X(t_2) \quad \ldots \quad m_X(t_n)\right]^t$$

$$x = \left[x_1 \quad x_2 \quad \cdots \quad x_n\right]^t$$

Definition

$X(t)$ is said to be a Gaussian random process if the above form of pdf is true for any n and for any choice of $\{t_i\}_{i=1}^{n}$.

Remarks

(a) A Gaussian random process is completely specified through its mean $m_X(t)$ and covariance $C_{XX}(t_1, t_2)$.

(b) $X(t)$ is stationary $\Rightarrow m_X(t) = m_X \;\&\; C_{XX}(t_1, t_2) = C_{XX}(t_1 - t_2)$

$\Rightarrow p_{XX}(x_1, x_2; t_1, t_2) = p_{XX}(x_1, x_2; t_1 - t_2)$

$\Rightarrow X(t)$ is 2nd order SSS $\Rightarrow X(t)$ is SSS.

(c) A stationary Gaussian random process with zero mean is completely desribed bv its autocovariance function

(d) Linear transformation of Gaussian random processes preserve the Gaussian nature. Gaussian distributed loads on linear systems produce Gaussian distributed responses.

Stationarity of a random process

Analogous to concept of steady state in vibration problems

One or more of the properties of random process becomes independent of time

Strong sense stationarity (SSS)
: defined with respect to pdf-s

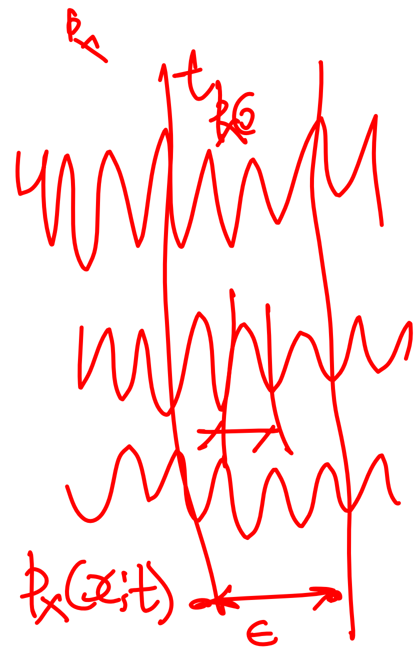
Wide sense stationarity (WSS)
: defined with respect to moments

1st order, 2nd order, *n*-th order SSS

$$p_X(x;t) = p_X(x;t+\epsilon) \text{ for any } \epsilon$$

$X(t)$: 1st order SSS

$$= p_X(x)$$



$$p_{XX}(x_1, x_2; t_1, t_2) = p_{XX}(x_1, x_2; t_1+\epsilon, t_2+\epsilon) \ \forall \epsilon$$

$X(t)$ is 2nd order SSS

$$= p_{XX}(x_1, x_2; t_2 - t_1)$$

$$p_{XX\cdots X}(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = p_{XX\cdots X}(x_1, x_2, \dots, x_n; t_1+\epsilon, t_2+\epsilon, \dots, t_n+\epsilon)$$

$X(t)$ is nth order SSS

$X(t)$ is said to be SSS

$$p_{XX\cdots X}\left(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n\right)$$

$$= p_{XX\cdots X}\left(x_1, x_2, \cdots, x_n; t_1+\varepsilon, t_2+\varepsilon, \cdots, t_n+\varepsilon\right) \forall\, \varepsilon, n \,\&\, \{t_i\}_{i=1}^{n}$$

If the above result is true only for $m \le n$, and not for all values of n, then we say that $X(t)$ is m-th order SSS.

Remark (a)

What happen to mean and variance of a 1st order SSS process?

$$m_X(t) = \int_{-\infty}^{\infty} x\, f_X(x;t)\, dx$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = m_X \text{ independent of } t.$$

$$\sigma_X^2(t) = \int_{-\infty}^{\infty} [x - m_X(t)]^2 f_X(x;t)\, dx$$

$$= \int_{-\infty}^{\infty} [x - m_X]^2 f_X(x)\, dx = \sigma_X^2 \text{ independent of } t.$$

Remark (b)

Exercise

Show that 2^{nd} order SSS implies 1^{st} order SSS

Remark (c)

What happens to covariance of a 2nd order SSS process?

$$C_{XX}(t_1,t_2) = \int_{-\infty}^{\infty}\int [x_1 - m_X(t_1)][x_2 - m_X(t_2)]\, f_{XX}(x_1,x_2;t_1,t_2)\,dx_1\,dx_2$$

$$= \int_{-\infty}^{\infty}\int [x_1 - m_X][x_2 - m_X]\, f_{XX}(x_1,x_2;t_2-t_1)\,dx_1\,dx_2$$

$$= C_{XX}(t_2-t_1).$$

Remark (d)

$X(t)$ is said to be 2nd order WSS if

$m_X(t)$ is independent of time and

$C_{XX}(t_1, t_2) = C_{XX}(t_2 - t_1)$

Remarks (Continued)

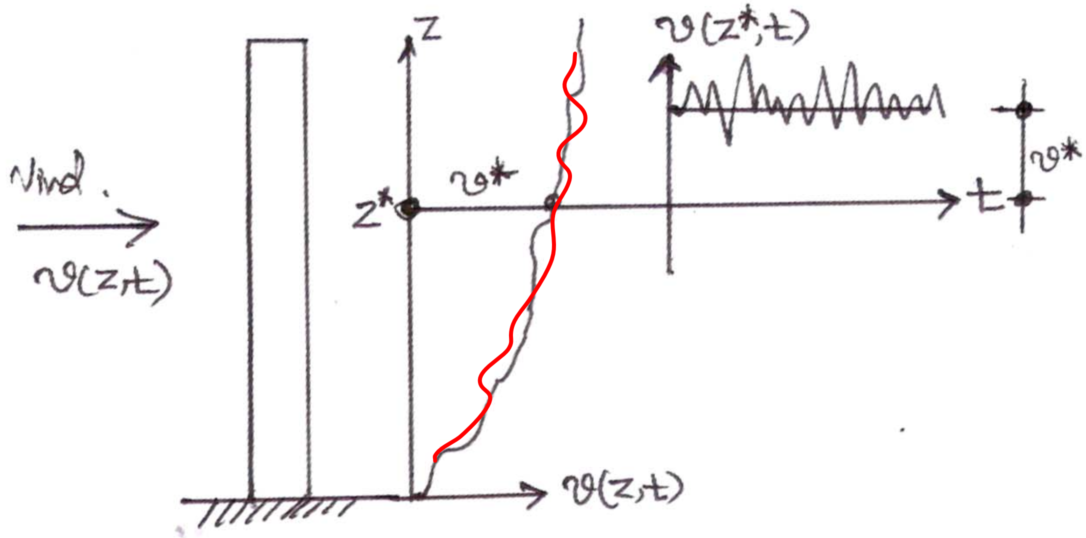
(e) The default notion of stationarity is 2nd order WSS.

(f) For a process that is evolving in space the term homogeneity is used to denote stationarity.

(g) A process that is not stationary is called nonstationary.

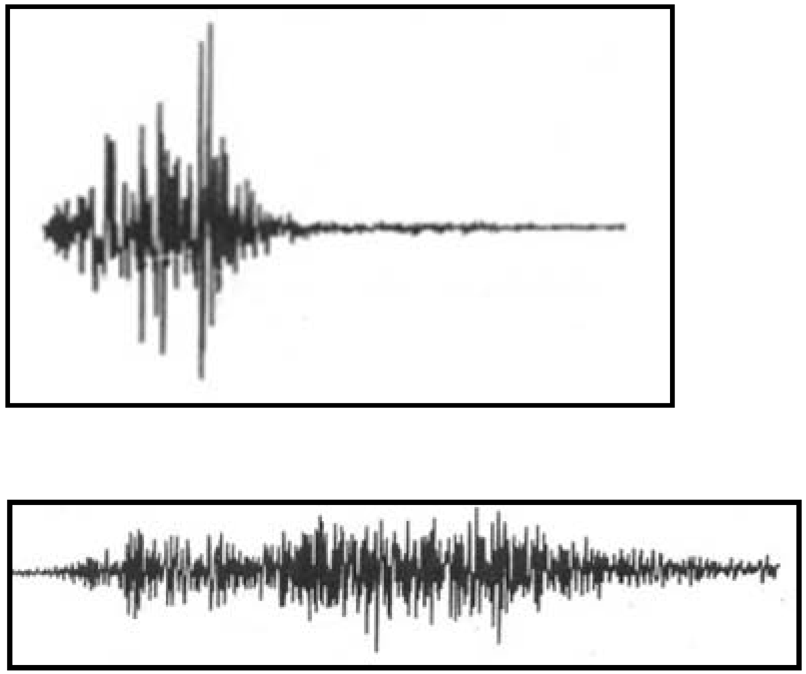
(h) Notion of joint stationarity of two or more random processes can also be defined.

Wind velocity:
Stationary in time
Nonstationary in space



Earthquake ground acceleration

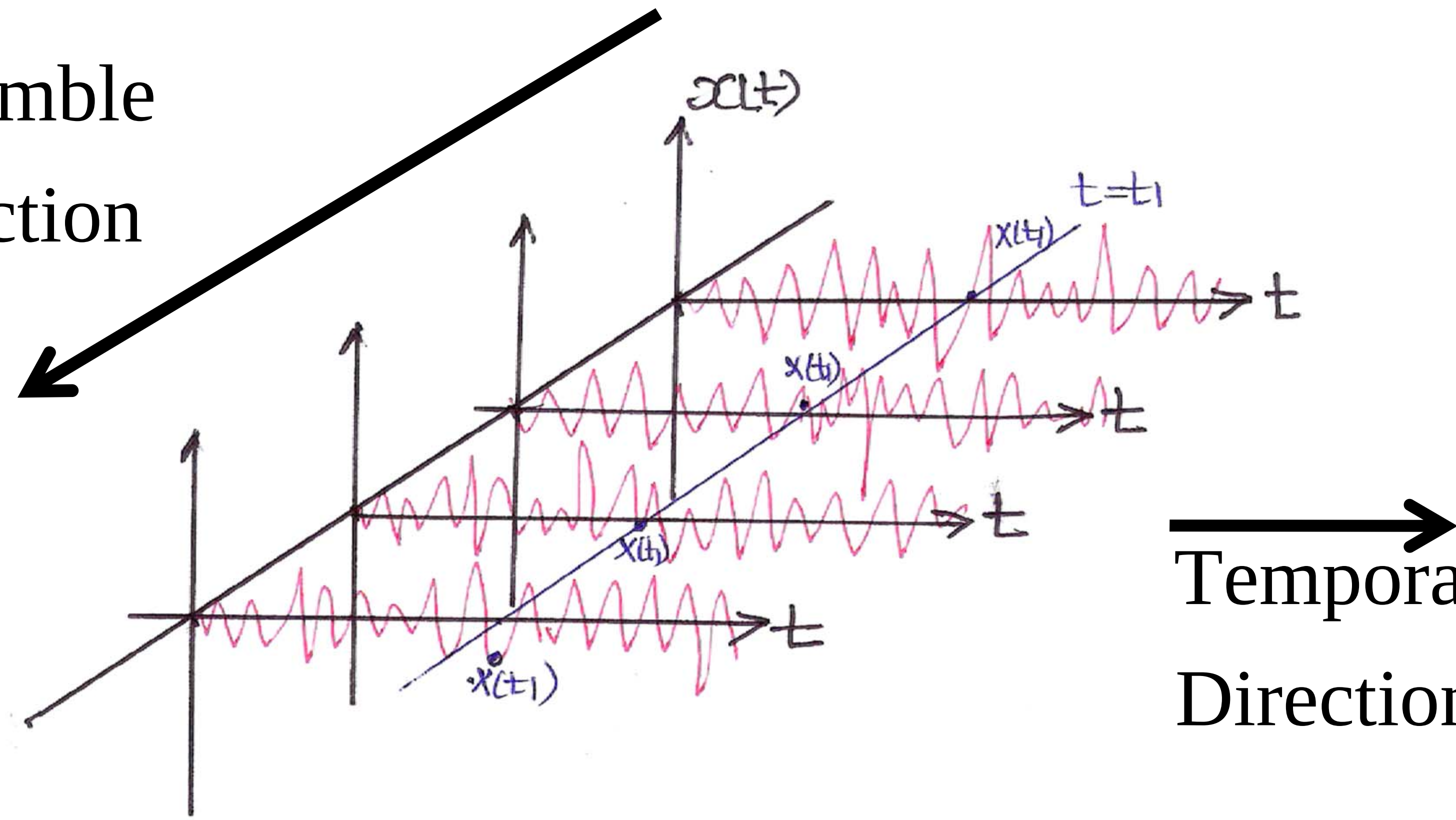
Acceleration



time

Ergodicty of a random process

Basic notion
Equivalence of temporal and ensemble averages



Let $x(t)$ be a sample realization of the random process $X(t)$. We define the time average of a given function of $X(t)$, $g[X(t)]$ by

$$\mathrm{T}_{\mathrm{av}}\{g[X(t)]\} = \frac{1}{T}\int_0^T g[x(t)]\mathrm{d}t$$

If $X(t)$ is an ergodic random process, then $< g[X(t)] >= \mathrm{T}_{\mathrm{av}}\{g[X(t)]\}$.

Definitions

- **Ergodicity in mean** $X(t)$ is ergodic in mean if

$$\mathrm{T_{av}}\{X(t)\} = \frac{1}{T}\int_0^T x(t)\mathrm{d}t = < X(t) >$$

- **Ergodicity in the mean square** $X(t)$ is ergodic in meansquare if

$$\mathrm{T_{av}}\{X^2(t)\} = \frac{1}{T}\int_0^T x^2(t)\mathrm{d}t = < X^2(t) >$$

- **Ergodicity in autocorrelation** $X(t)$ is said to be ergodic in auto-correlation if

$$\mathrm{T_{av}}\{X(t)X(t1+\tau)\} = \frac{1}{T}\int_0^T x(t)x(t+\tau)\mathrm{d}t = < X(t)X(t+\tau) > = R_X(\tau)$$

Remaraks

1. The above list of definitions of ergodicity are not exhaustive: several other similar definitions can be constructed by considering other descriptors of the random process.

2. Ergodic processes are necessarily stationary in nature; a stationary random process need not be ergodic.

3. Physically, ergodicity means that a sufficiently long record of a stationary random process contains all the statistical information about the random phenomenon.

Ergodicity in mean

Let *X(t)* be a stationary random process with specified joint pdf structure

$$\eta_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

$\Rightarrow \eta_T$ is a random variable

$$E\left[\eta_T\right] = \frac{1}{2T}\int_{-T}^{T} E[X(t)]dt = E\left[x(t)\right] = \eta$$

$$\sigma_{\eta_T}^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E\left[\left\{X(t_1)-\eta\right\}\left\{X(t_2)-\eta\right\}\right]dt_1 dt_2$$

$$= \frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right)[R(\tau)-\eta^2]d\tau$$