Stochastic Structural Dynamics

Lecture-9

Random processes-4
Random vibrations of sdof systems-1

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Recall

- Gaussian random process
- Poisson process
- Simple Random walk
- Wiener process
- Brownian motion
- Random pulse process
- Gaussian white noise process

Mean square derivative

2

Two random processes: $U(t)$ and $V(t)$

Description of $U(t)$

- $P\left[U(t) \le u_1\right]$
- $P\left[U(t_1) \le u_1 \bigcap U(t_2) \le u_2\right]$

$\vdots$

- $P\left[\bigcap_{i=1}^{n} U(t_i) \le u_i\right]$

$p_{\tilde{U}}\left(\tilde{u};\tilde{t}\right)$

$$m_U(t) = \int_{-\infty}^{\infty} u p_U(u;t)\,du$$

$$C_{UU}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[u_1 - m_U(t_1)\right]\left[u_2 - m_U(t_2)\right] p_{UU}(u_1,u_2;t_1,t_2)\,du_1 du_2$$

$\vdots$

Description of $V(t)$

$\bullet P\left[V(t) \le v_1\right]$

$\bullet P\left[V(t_1) \le v_1 \bigcap V(t_2) \le v_2\right]$

$\vdots$

$\bullet P\left[\bigcap_{i=1}^{n} V(t_i) \le v_i\right]$

$p_{\tilde{V}}\left(\tilde{v};\tilde{t}\right)$

$$m_V(t) = \int_{-\infty}^{\infty} v p_V(v;t)\,dv$$

$$C_{VV}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[v_1 - m_V(t_1)\right]\left[v_2 - m_V(t_2)\right] p_{VV}(v_1,v_2;t_1,t_2)\,dv_1 dv_2$$

$\vdots$

Joint description of $U(t)$ and $V(t)$

$\langle U(t) \rangle = 0 \quad \langle V(t) \rangle = 0$

$$\bullet P\left[U(t_1) \le u_1 \bigcap V(t_2) \le v_2\right]$$

⋮

$$\bullet P\left[\left\{\bigcap_{i=1}^{n} U(t_i) \le u_i\right\} \cap \left\{\bigcap_{j=1}^{m} V(s_j) \le v_j\right\}\right]$$

$$p_{\tilde{U}\tilde{V}}\left(\tilde{u}, \tilde{v}; \tilde{t}, \tilde{s}\right)$$

$$C_{UV}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[u_1 - m_U(t_1)\right]\left[v_2 - m_V(t_2)\right] p_{UV}(u_1, v_2; t_1, t_2)\, du_1 dv_2$$

$C_{UV}(t_1, t_2)$

$= \langle U(t_1) V(t_2) \rangle$

⋮

Cross covariance function

Joint stationarity

$U(t)$ and $V(t)$ are said to be jointly stationary (wide sense)

$$m_U(t) = m_U$$

$$m_V(t) = m_V$$

$$C_{UU}(t, t+\tau) = C_{UU}(\tau)$$

$$C_{VV}(t, t+\tau) = C_{VV}(\tau)$$

$$C_{UV}(t, t+\tau) = C_{UV}(\tau)$$

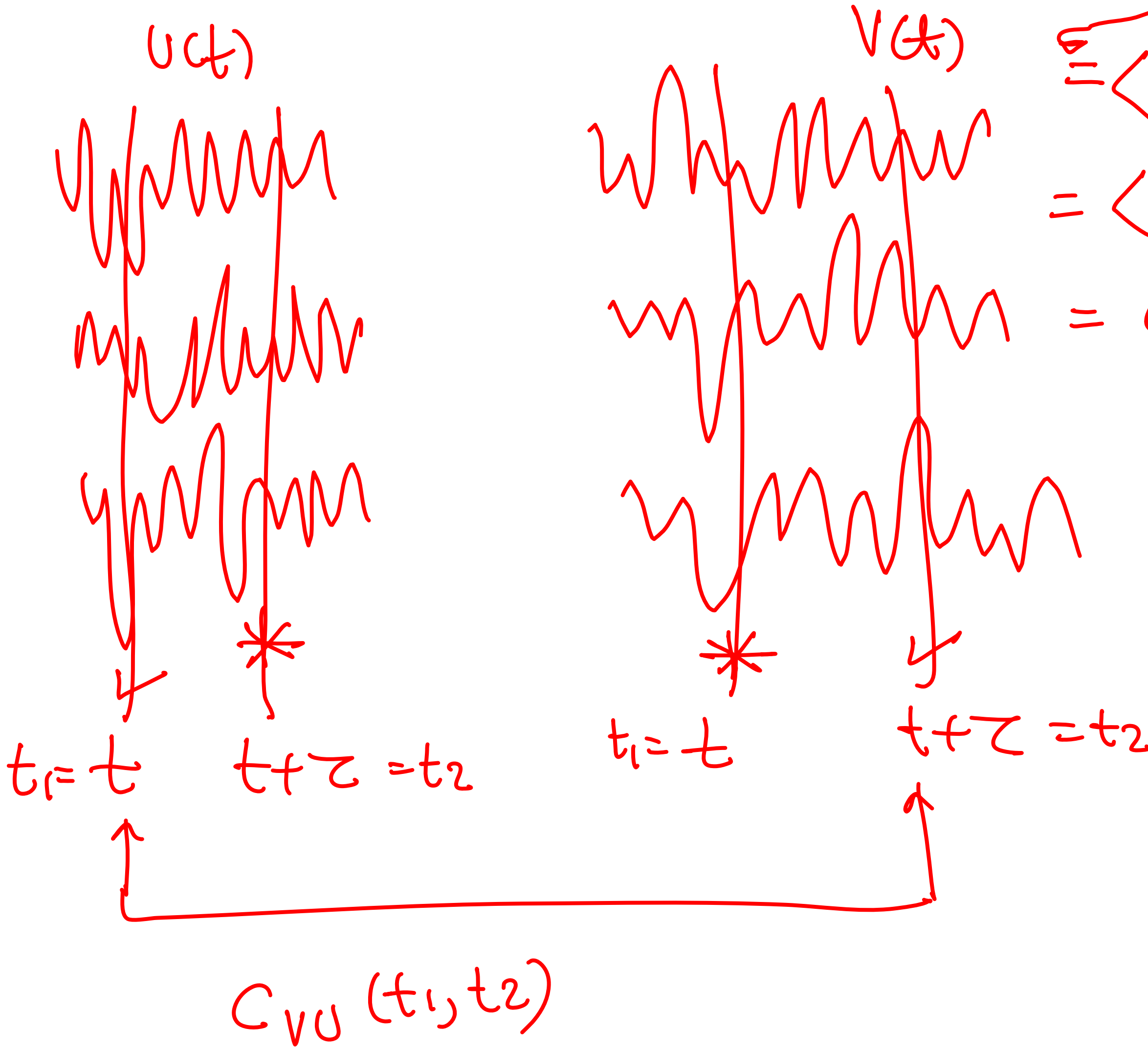
Definition for strong sense stationarity could also be provided in terms of joint pdf-s.

$$C_{UV} = \langle U(t_1)\, V(t_2) \rangle = C_{UV}(t_1, t_2)$$

$$= \langle V(t_2)\, U(t_1) \rangle$$

$$= \langle V(t+\tau)\, U(t) \rangle$$

$$= C_{VU}(t_2, t_1)$$



Covariance matrix

$$C\left(t_1,t_2\right)=\begin{bmatrix} C_{UU}\left(t_1,t_2\right) & C_{UV}\left(t_1,t_2\right) \\ C_{VU}\left(t_1,t_2\right) & C_{VV}\left(t_1,t_2\right) \end{bmatrix}$$

$$C\left(\tau\right)=\begin{bmatrix} C_{UU}\left(\tau\right) & C_{UV}\left(\tau\right) \\ C_{VU}\left(\tau\right) & C_{VV}\left(\tau\right) \end{bmatrix}$$

$$C_{UV}\left(\tau\right)=\left\langle U\left(t\right)V\left(t+\tau\right)\right\rangle=\left\langle V\left(t+\tau\right)U\left(t\right)\right\rangle$$

$$\Rightarrow C_{UV}\left(\tau\right)=C_{VU}\left(-\tau\right)$$

PSD matrix

$$S(\omega) = \begin{bmatrix} S_{UU}(\omega) & S_{UV}(\omega) \\ S_{VU}(\omega) & S_{VV}(\omega) \end{bmatrix}$$

$$S_{UU}(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle U_T(\omega) U_T^*(\omega) \right\rangle \Rightarrow S_{UU}(\omega) = S_{UU}(-\omega)$$

$$S_{UV}(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle U_T(\omega) V_T^*(\omega) \right\rangle \Rightarrow$$

$$S_{UV}(-\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle U_T(-\omega) V_T^*(-\omega) \right\rangle = \lim_{T\to\infty} \frac{1}{T} \left\langle U_T^*(\omega) V_T(\omega) \right\rangle = S_{VU}(\omega)$$

$$S_{UV}^*(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle U_T^*(\omega) V_T(\omega) \right\rangle = \lim_{T\to\infty} \frac{1}{T} \left\langle U_T(-\omega) V_T^*(-\omega) \right\rangle = S_{UV}(-\omega)$$

$$S_{UV}(\omega) = \int_{-\infty}^{\infty} R_{UV}(\tau)\exp(i\omega\tau)\,d\tau$$

$$R_{UV}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{UV}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$S_{UV}(\omega) = \left|S_{UV}(\omega)\right|\exp\left[-i\phi(\omega)\right]$$

$$\left|S_{UV}(\omega)\right| = \text{amplitude of cross PSD function}$$

$$\phi(\omega) = \text{phase spectrum}$$

$$p_{UV}(\omega) = \mathrm{Re}\left[S_{UV}(\omega)\right] = \text{co-spectrum}$$

$$q_{UV}(\omega) = \mathrm{Im}\left[S_{UV}(\omega)\right] = \text{quadrature spectrum}$$

Complex coherency function

$$\mathrm{coh}_{UV}(\omega) = \frac{S_{UV}(\omega)}{\sqrt{S_{UU}(\omega) S_{VV}(\omega)}}$$

$$\mathrm{coh}_{UV}(\omega) = \left|\mathrm{coh}_{UV}(\omega)\right| \exp\left(-i\theta(\omega)\right)$$

Coherency

$$\left|\mathrm{coh}_{UV}(\omega)\right| = \frac{\left|S_{UV}(\omega)\right|}{\sqrt{S_{UU}(\omega) S_{VV}(\omega)}}$$

$$0 \le \left|\mathrm{coh}_{UV}(\omega)\right| \le 1$$

$$\left|\mathrm{coh}_{UV}(\omega)\right| = 0$$

$\Rightarrow$ lack of linear dependency between two processes

Two processes are linearly related

$$\left|\mathrm{coh}_{UV}(\omega)\right| = 1$$

Example

Let $U(t)$ and $V(t)$ be defined as

$$U(t) = S(t)$$

$$V(t) = S(t+\alpha)$$

$S(t)$=stationary Gaussian random process with zero mean.

$$C_{UV}(\tau) = \langle U(t)V(t+\tau)\rangle = \langle S(t)S(t+\alpha+\tau)\rangle = C_{SS}(\alpha+\tau)$$

$$\Rightarrow S_{UV}(\omega) = \int_{-\infty}^{\infty} C_{UV}(\tau)\exp(-i\omega\tau)d\tau = \int_{-\infty}^{\infty} C_{SS}(\alpha+\tau)\exp(-i\omega\tau)d\tau$$

Substitute $\beta=\alpha+\tau \Rightarrow$

$$S_{UV}(\omega) = \int_{-\infty}^{\infty} C_{SS}(\beta)\exp(-i\omega\tau+i\omega\alpha)d\beta = \exp(i\omega\alpha)S_{SS}(\omega)$$

$\Rightarrow$

$$\mathrm{coh}_{UV}(\omega) = \exp(i\omega\alpha)$$

Exercise

Let $U(t)$ and $V(t)$ be defined as

$$U(t) = S(t)$$

$$V(t) = S(t) + W(t)$$

$S(t)$=stationary Gaussian random process with zero mean.

$W(t)$=zero mean Gaussian white noise independent of $S(t)$

with $\langle W(t)W(t+\tau)\rangle = 2\pi S_0 \delta(\tau)$

Determine $\mathrm{coh}_{UV}(\omega)$

Non stationarity

Example

$X(t) = e(t)S(t)$

$e(t) =$ deterministic envelope function

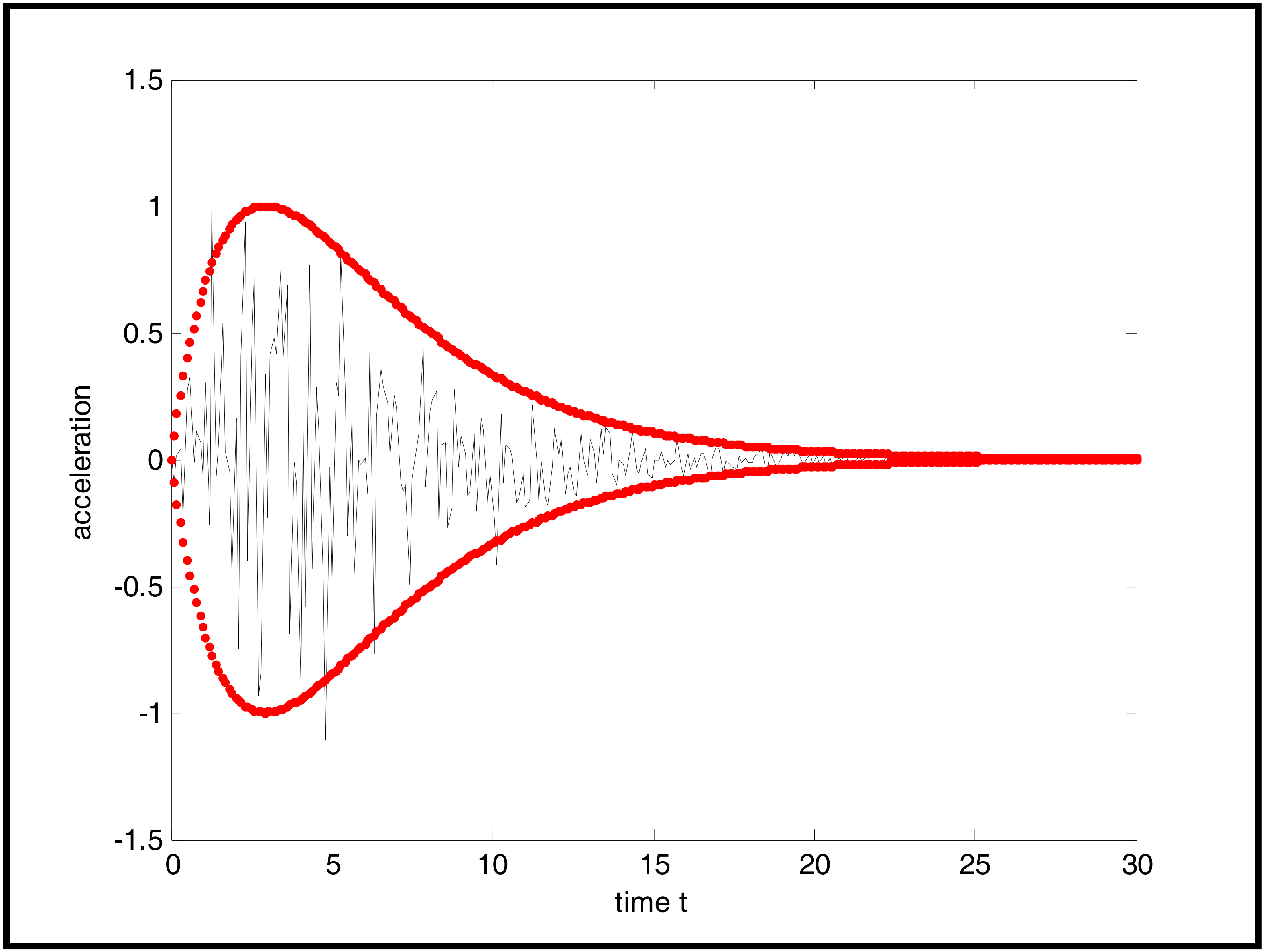
$S(t)$=zero mean stationary Gaussian random process

$$e(t) = A_0\left[\exp(-\alpha t) - \exp(-\beta t)\right]; \alpha > \beta > 0$$

$$\langle X(t) \rangle = \langle e(t)S(t) \rangle = 0$$

$$\langle X(t)X(t+\tau) \rangle = \langle e(t)S(t)e(t+\tau)S(t+\tau) \rangle = e(t)e(t+\tau)R_{SS}(\tau)$$

$$\sigma_X^2(t) = e^2(t)\sigma_S^2$$


1.5
1
0.5
0
-0.5
-1
-1.5
acceleration
0
5
10
15
20
25
30
time t

Markov Property

Let $X(t)$ be a random process with continuous state and continuous parameter (time t).

Let $t_1 < t_2 < \cdots < t_n$ be n time instants.

This defines n random variables

$X(t_1), X(t_2), \cdots, X(t_n)$.

$X(t)$ is said to possess Markov property if

$$P\left[X(t_n) \le x_n \mid X(t_{n-1}) \le x_{n-1}, X(t_{n-2}) \le x_{n-2}, \cdots, X(t_1) \le x_1\right]$$

$$= P\left[X(t_n) \le x_n \mid X(t_{n-1}) \le x_{n-1}\right]$$

for any n and any choice of $t_1 < t_2 < \cdots < t_n$.

$\Rightarrow$

$$P_X\left(x_n,t_n \mid x_{n-1},t_{n-1};x_{n-2},t_{n-2};\cdots,x_1,t_1\right)=P_X\left(x_n,t_n \mid x_{n-1},t_{n-1}\right)$$

$$p_X\left(x_n,t_n \mid x_{n-1},t_{n-1};x_{n-2},t_{n-2};\cdots,x_1,t_1\right)=p_X\left(x_n,t_n \mid x_{n-1},t_{n-1}\right)$$

Description of a Markov process

- $p\left(x_1,t_1\right)$
- $p\left(x_2,t_2;x_1,t_1\right)=p\left(x_2,t_2 \mid x_1,t_1\right)p\left(x_1,t_1\right)$
- $p\left(x_3,t_3;x_2,t_2;x_1,t_1\right)=p\left(x_3,t_3 \mid x_2,t_2;x_1,t_1\right)p\left(x_2,t_2 \mid x_1,t_1\right)p\left(x_1,t_1\right)$

$=p\left(x_3,t_3 \mid x_2,t_2\right)p\left(x_2,t_2 \mid x_1,t_1\right)p\left(x_1,t_1\right)$

$\vdots$

- $p\left(x_n,t_n;x_{n-1},t_{n-1};\cdots;x_1,t_1\right)=\prod_{\nu=2}^{n}p\left(x_\nu,t_\nu \mid x_{\nu-1},t_{\nu-1}\right)p\left(x_1,t_1\right)$

Transistion probability density function

tpdf : $p\left(x_{\nu}, t_{\nu} \mid x_{\nu-1}, t_{\nu-1}\right)$

- $p\left(x_1, t_1\right)$ and $p\left(x_{\nu}, t_{\nu} \mid x_{\nu-1}, t_{\nu-1}\right)$ completely specify a Markov process

Chapman - Kolmogorov - Smoluchowski Equation

$t = t_1$ $\quad$ $t = \tau$ $\quad$ $t = t_2$

$$p(x_2,t_2;x_1,t_1) = p(x_2,t_2 \mid x_1,t_1)\, p(x_1,t_1)$$

$$= \int p(x_2,t_2;x,\tau;x_1,t_1)\,dx$$

$$= \int p(x_2,t_2 \mid x,\tau;x_1,t_1)\, p(x,\tau \mid x_1,t_1)\, p(x_1,t_1)\,dx$$

$$\Rightarrow$$

$$p(x_2,t_2 \mid x_1,t_1) = \int p(x_2,t_2 \mid x,\tau;x_1,t_1)\, p(x,\tau \mid x_1,t_1)\,dx$$

$$= \int p(x_2,t_2 \mid x,\tau)\, p(x,\tau \mid x_1,t_1)\,dx$$

Consistency condition for the process to be Markov

$$p\left(x_2, t_2 \mid x_1, t_1\right) = \int p\left(x_2, t_2 \mid x, \tau\right) p\left(x, \tau \mid x_1, t_1\right) dx$$

Review of dynamics of sdof systems

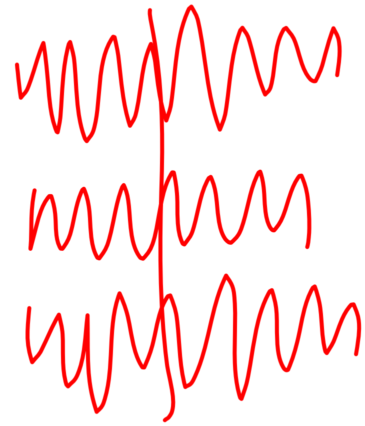
Equation of motion

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

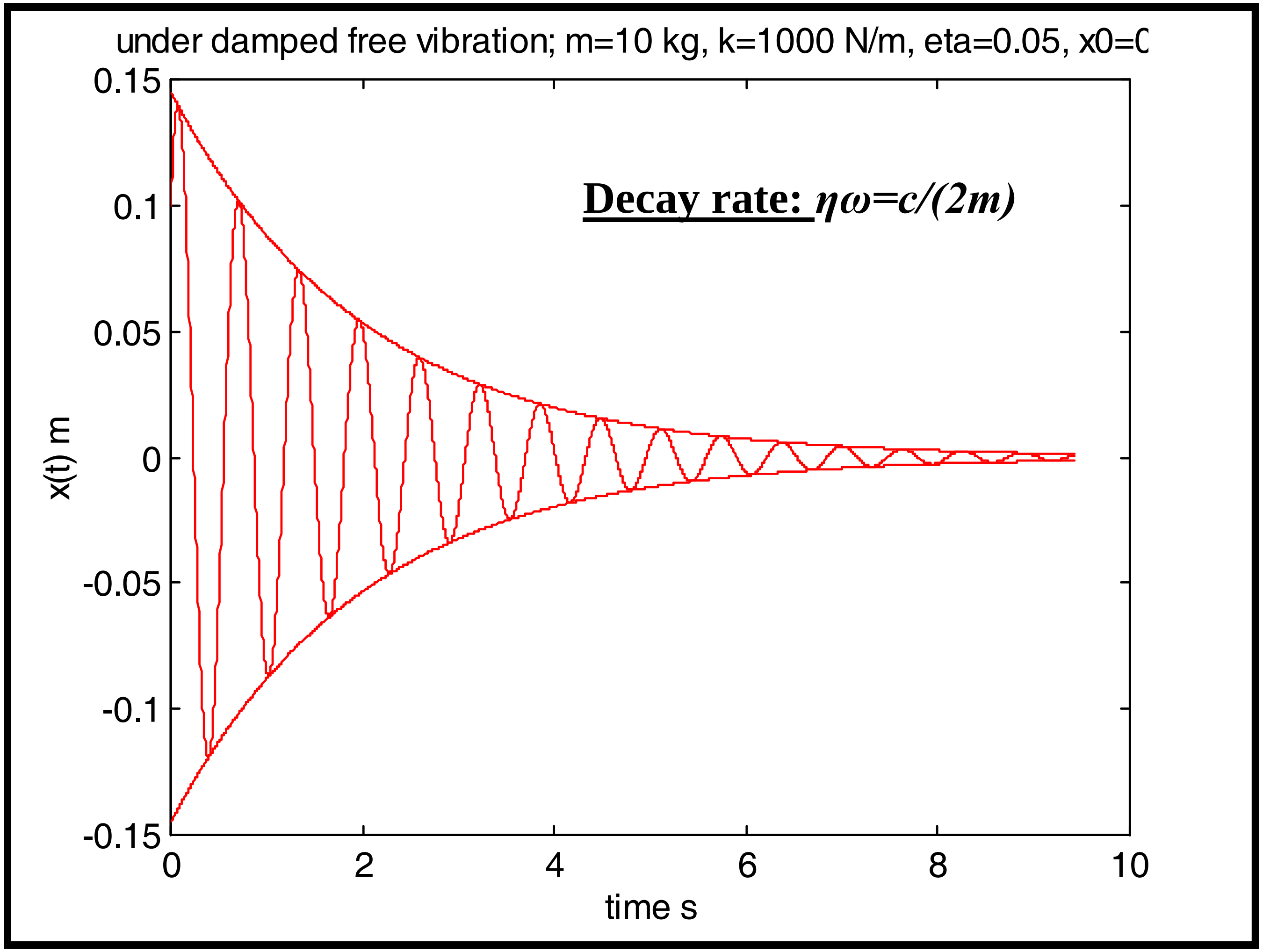
$$x(0) = x_0$$

$$\dot{x}(0) = \dot{x}_0$$

$\dot{(\ )} = \frac{d}{dt}$



Assumption
System is underdamped



Forced vibration under harmonic input

- Leads to the concept of resonance
- Serves as building block for more general problems

$$m\ddot{x} + c\dot{x} + kx = P\cos\lambda t$$

$$x(0) = x_0;\ \dot{x}(0) = \dot{x}_0$$

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = (P/m)\cos\lambda t$$

$$x(t) = CF + PI = x_{cf}(t) + x_{pi}(t)$$

$$x_{cf}(t) = \exp(-\eta\omega t)(A\cos\omega_d t + B\sin\omega_d t)$$

$$x_{pi}(t) = C\cos\lambda t + D\sin\lambda t$$

$$C = \frac{(P/m)(\omega^2 - \lambda^2)}{(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2}; D = \frac{(P/m)2\eta\omega\lambda}{(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2}$$

$$x(t) = \exp(-\eta\omega t)(A\cos\omega_d t + B\sin\omega_d t) + \frac{(P/m)\cos(\lambda t - \theta)}{[(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2]^{\frac{1}{2}}}$$

$$x_0 = A + \frac{(P/m)\cos(\theta)}{[(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2]^{\frac{1}{2}}}$$

$$\dot{x}_0 = -\eta\omega A + B\omega_d + \frac{(P/m)\lambda\sin(\theta)}{[(\omega^2 - \lambda^2)^2 + (2\eta\omega\lambda)^2]^{\frac{1}{2}}}$$

Remarks:

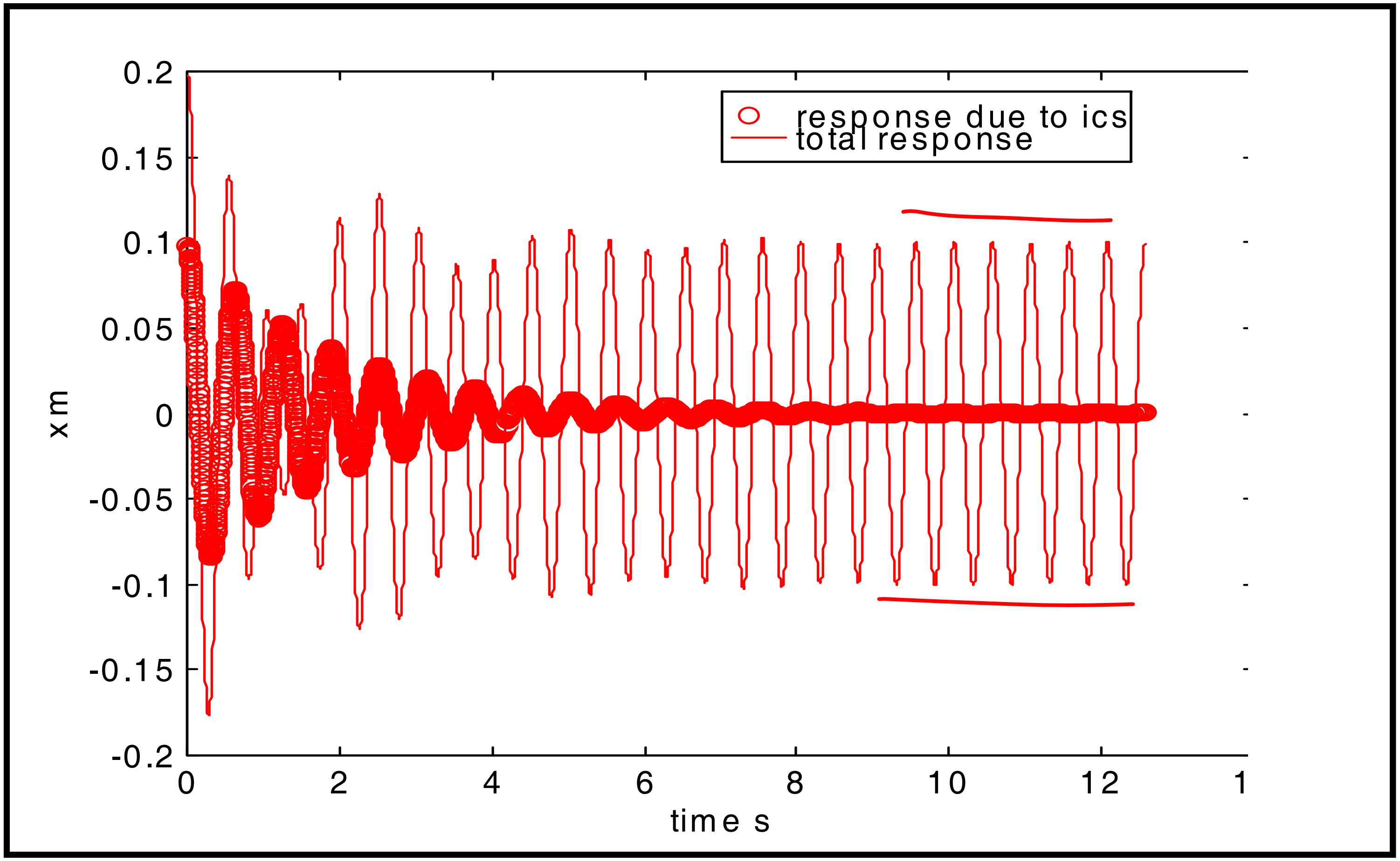
- Response is aperiodic for small *t*; as *t* becomes large, response becomes periodic (harmonic at the driving frequency)

- Transient phase:
  - response is aperiodic,
  - influence of ics present.

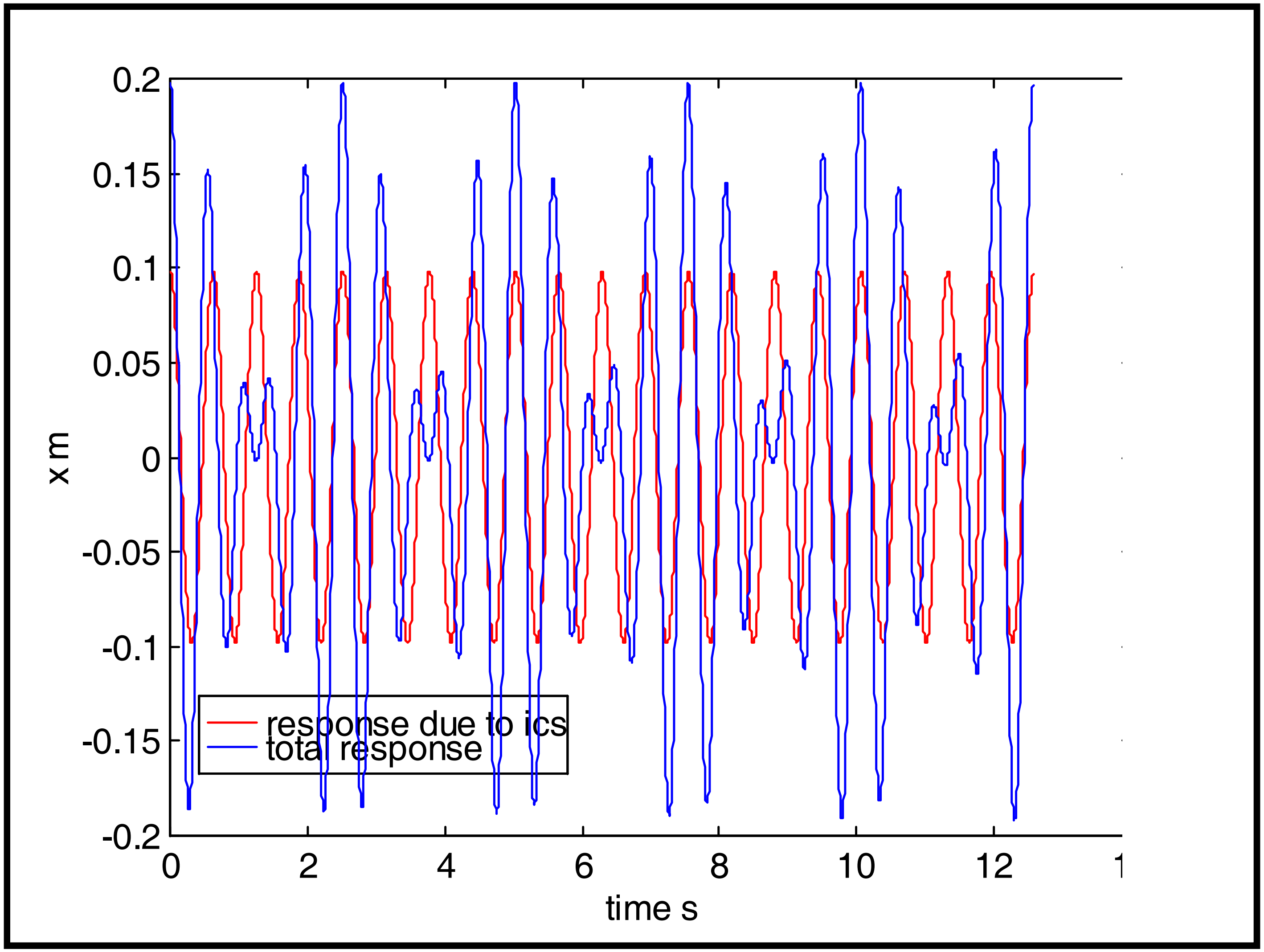
Remarks:

- Steady state:
    - for large t, response becomes harmonic and $\leftarrow e^{-\eta\omega t}$
    - influence of ics vanish
- Time required to reach steady state is governed by $\eta\omega$.
- For η=0, the effect of ics never dies and response consists of two harmonics

Damped system



Undamped system



Nature of steady state response

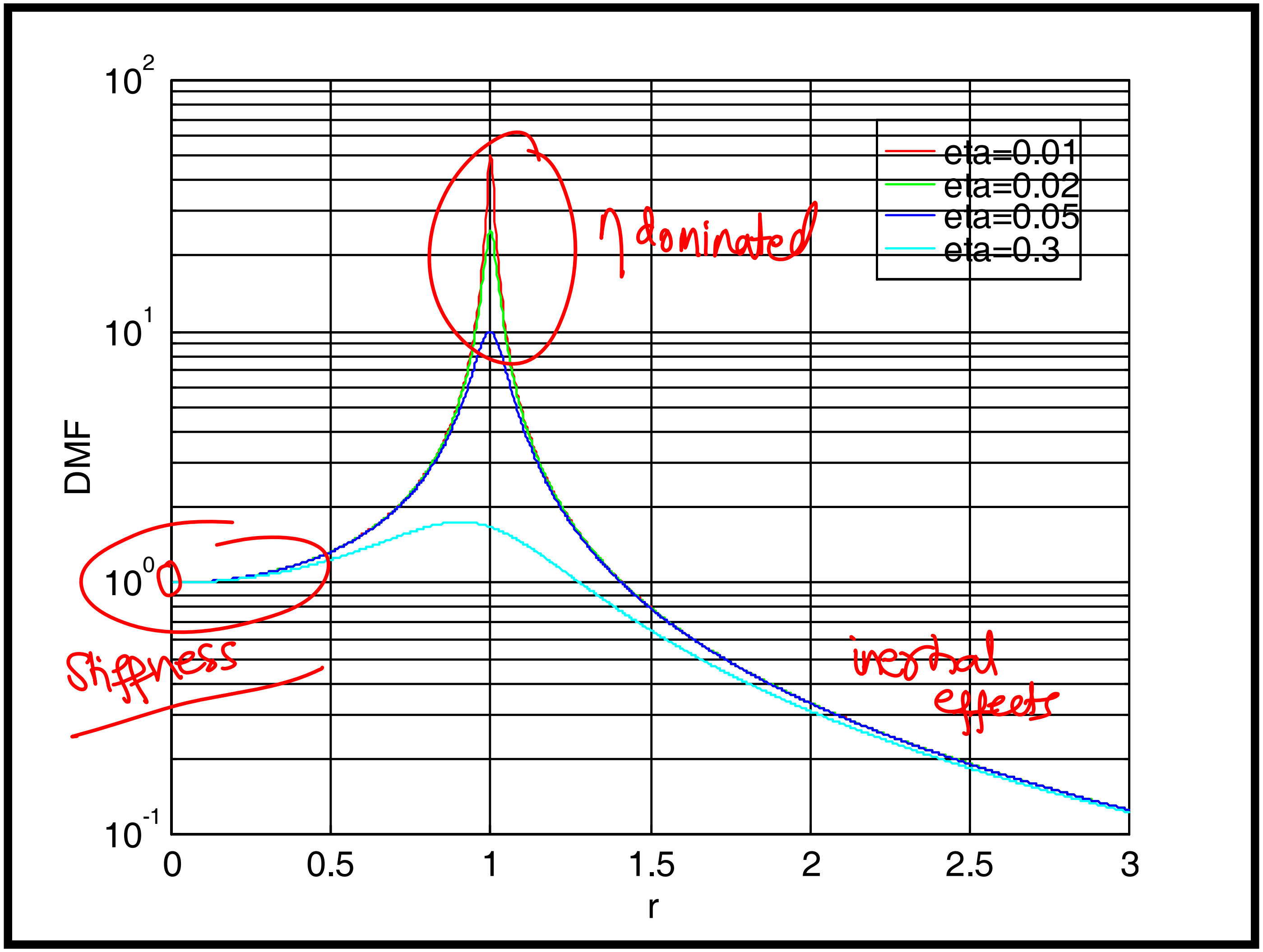
$$\lim_{t\to\infty} x(t) \to \frac{(P/m)\cos(\lambda t-\theta)}{\left[(\omega^2-\lambda^2)^2+(2\eta\omega\lambda)^2\right]^{\frac{1}{2}}}$$

$$= \frac{(P/k)\cos(\lambda t-\theta)}{\left[(1-r^2)^2+(2\eta r)^2\right]^{\frac{1}{2}}} = X\cos(\lambda t-\theta); r = \frac{\lambda}{\omega}$$

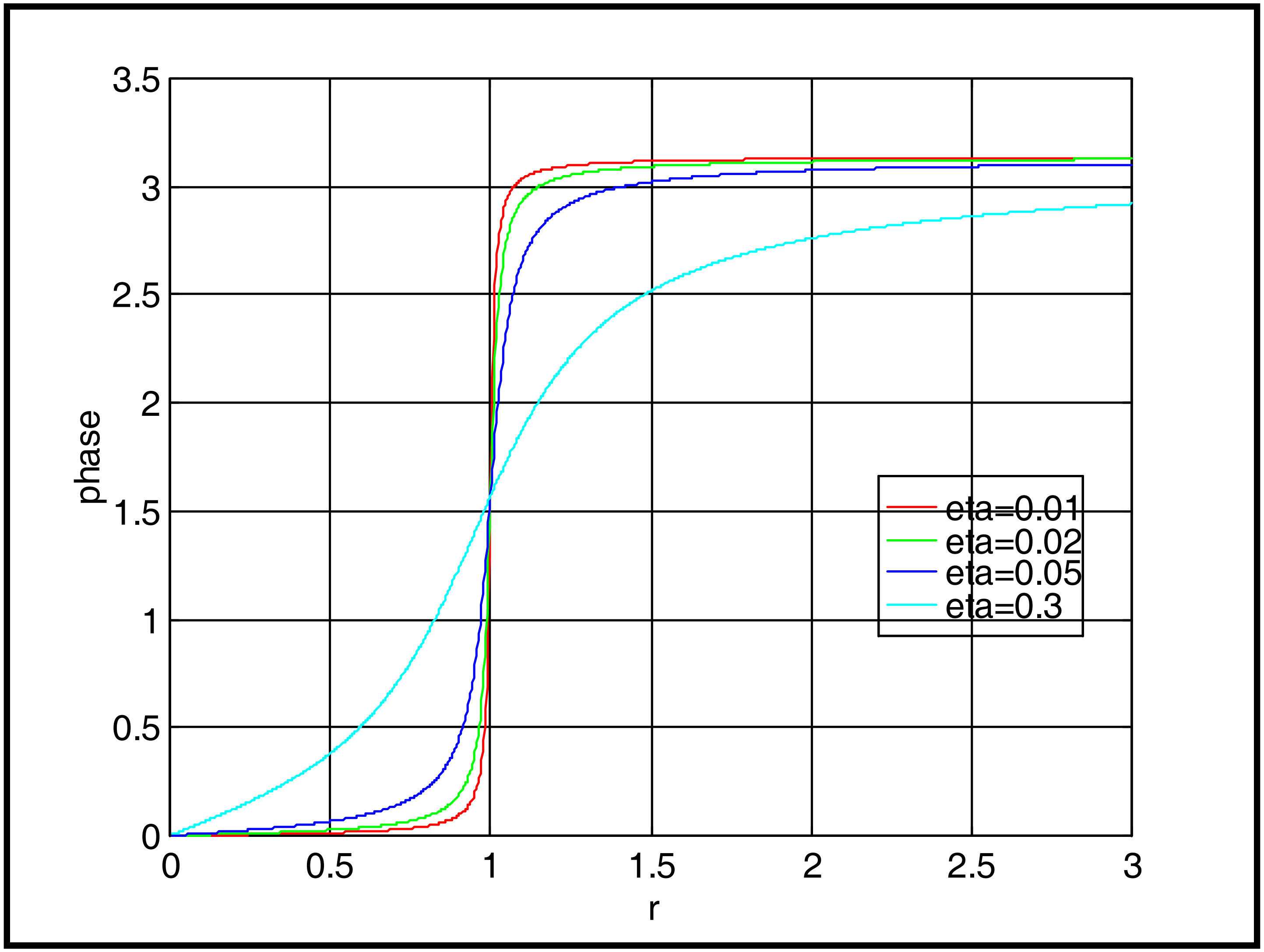
$$\frac{X}{(P/k)} = DMF = \frac{1}{\left[(1-r^2)^2+(2\eta r)^2\right]^{\frac{1}{2}}}$$

$$\theta = \tan^{-1}\left(\frac{2\eta r}{1-r^2}\right)$$

- **DMF=dynamic magnification factor (modification?)**
- **(P/k)=static response under force P**



DMF
r
eta=0.01
eta=0.02
eta=0.05
eta=0.3
η dominated
Stiffness
inertial effects



3.5
3
2.5
2
1.5
1
0.5
0
phase
0
0.5
1
1.5
2
2.5
3
r
eta=0.01
eta=0.02
eta=0.05
eta=0.3

Remarks

- **For *r*=0, DMF=1.**
    - **No dynamic amplification.**
    - **Response is stiffness controlled**
- **For large r, DMF approaches zero.**
- **Inertial effects dominate the response.**
- **In the neighborhood of *r=1*, DMF is high.**
- **At *r*=1, DMF=1/(2*η*).**
  - **Response is controlled by *η*.**
- **For** $r = \sqrt{(1-2\eta^2)}$ **the DMF takes its highest value of**

$$(DMF)_{\max} = 1/\left[2\eta\sqrt{(1-\eta^2)}\right]$$

•

This condition is termed as the resonant condition.

- **At *r*=1, the phase angle changes rapidly from values closer to zero *(r<1)* to values closer to pi (*r>1*).**

Resonance in undamped system

$$\ddot{x} + \omega^2 x = (P/m)\cos\lambda t$$

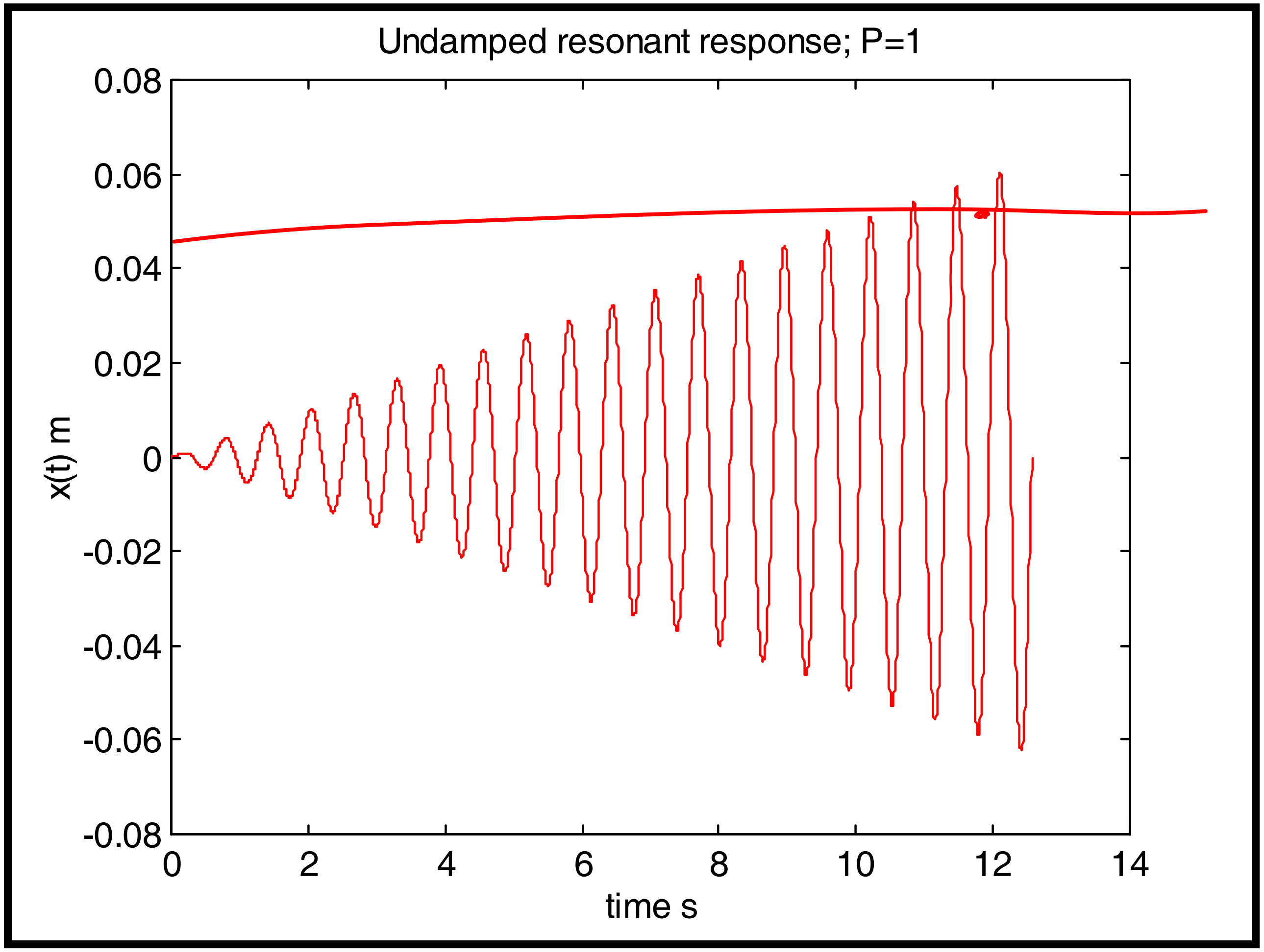
$$x(t) = A\cos\omega t + B\sin\omega t + \frac{(P/m)\cos\lambda t}{\omega^2 - \lambda^2}$$

$$x(0) = 0; \dot{x}(0) = 0$$

$$x(t) = (P/m)\frac{\cos\lambda t - \cos\omega t}{\omega^2 - \lambda^2}$$

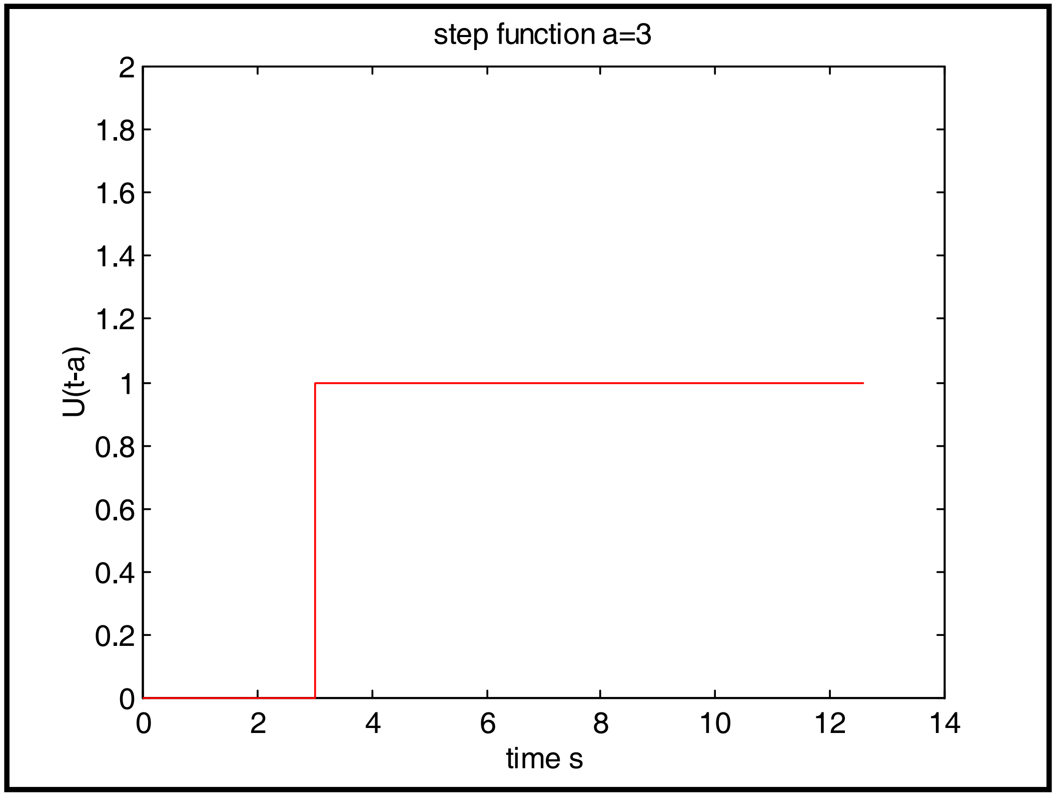
$$\lim_{\lambda\to\omega} x(t) \to \frac{Pt}{2m\omega}\sin\omega t$$

$$\lim_{\substack{\lambda\to\omega \\ t\to\infty}} x(t) \to \infty$$


Undamped resonant response; P=1
x(t) m
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0
2
4
6
8
10
12
14
time s

Heaveside’s step function

$$U(t-a)=0 \quad t<a$$
$$=1 \quad t>a$$
$$=1/2 \quad t=a$$



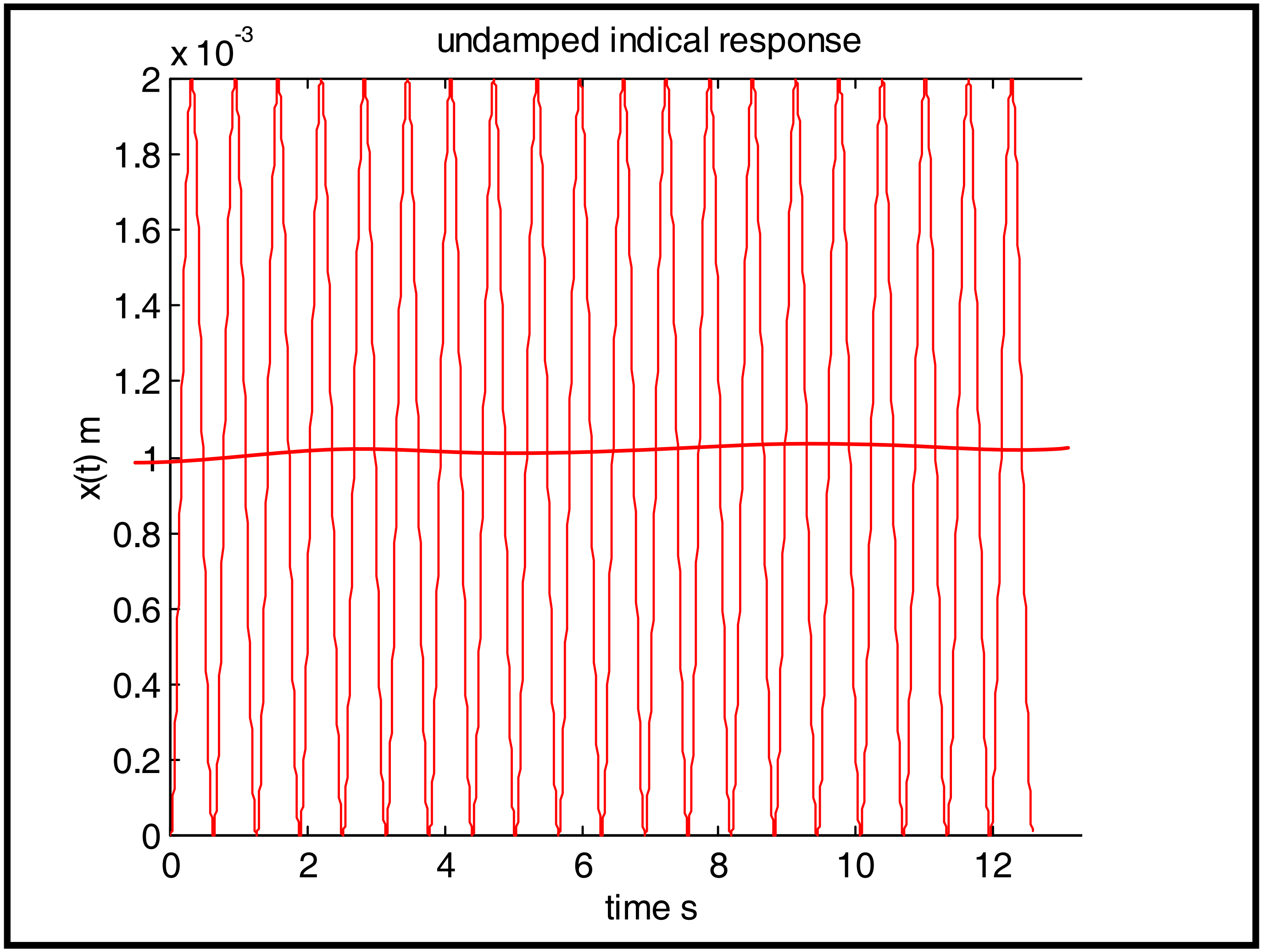
Indicial response analysis

$$m\ddot{x} + c\dot{x} + kx = U(t)$$

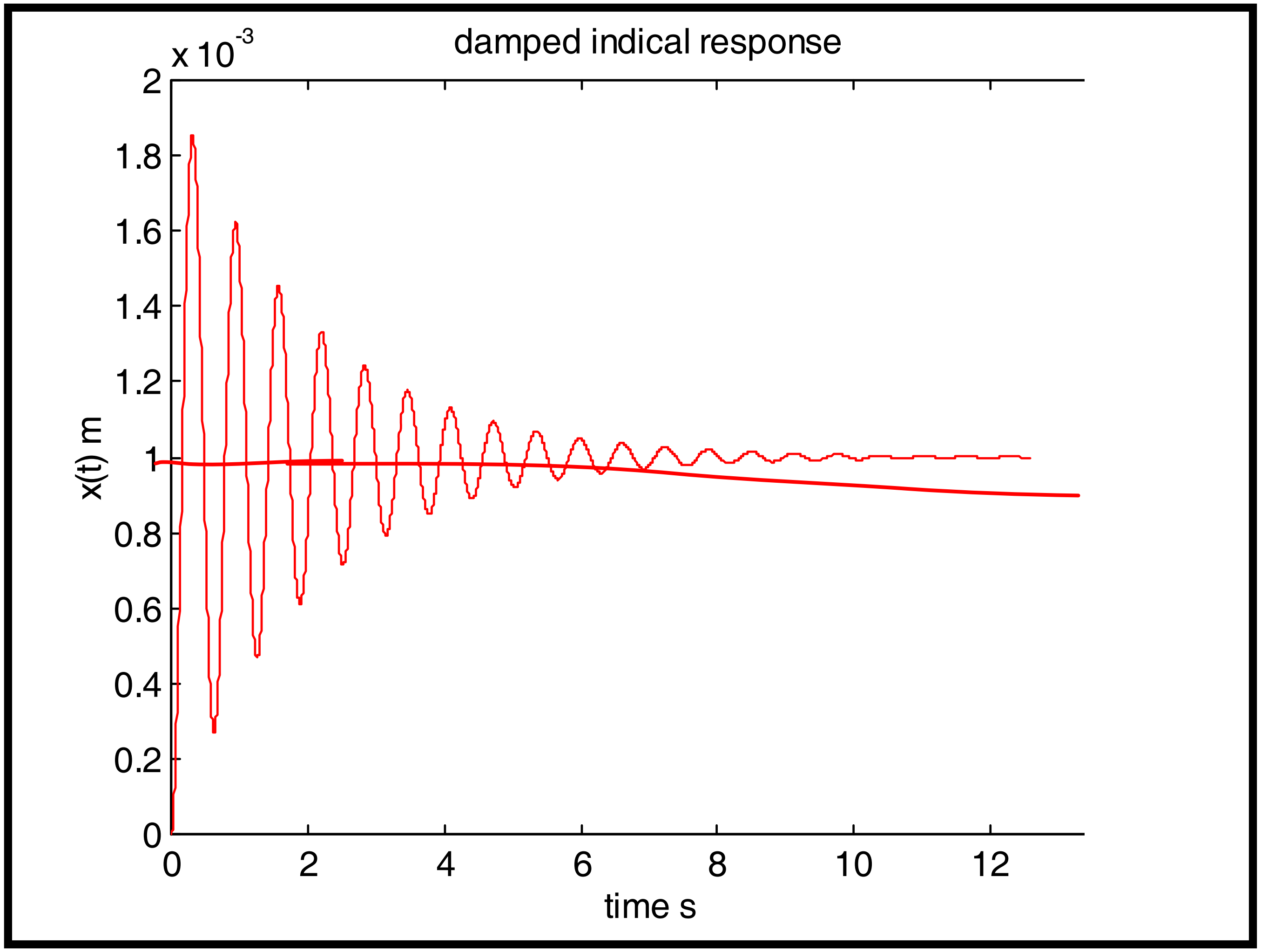
$$x(0) = 0; \dot{x}(0) = 0$$

$$x(t) = \exp(-\eta\omega t)(A\cos\omega_d t + B\sin\omega_d t) + (1/k)$$

$$x(t) = (1/k)\left[1 - \exp(-\eta\omega t)\left(\cos\omega_d t + \frac{\eta}{\sqrt{(1-\eta^2)}}\sin\omega_d t\right)\right] = G(t)$$



undamped indical response
x 10^-3
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
x(t) m
0
2
4
6
8
10
12
time s


damped indical response
x 10^-3
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
x(t) m
0
2
4
6
8
10
12
time s

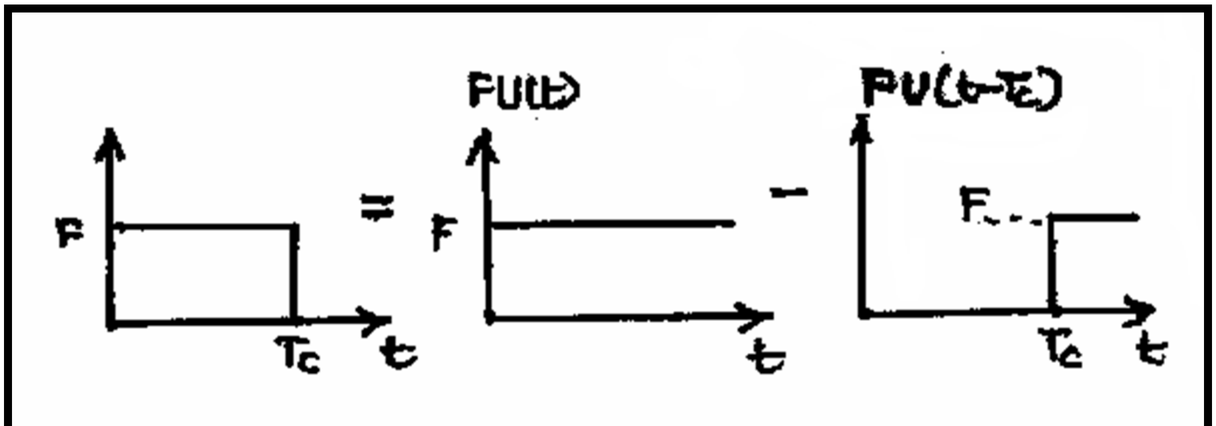
Response to a box input

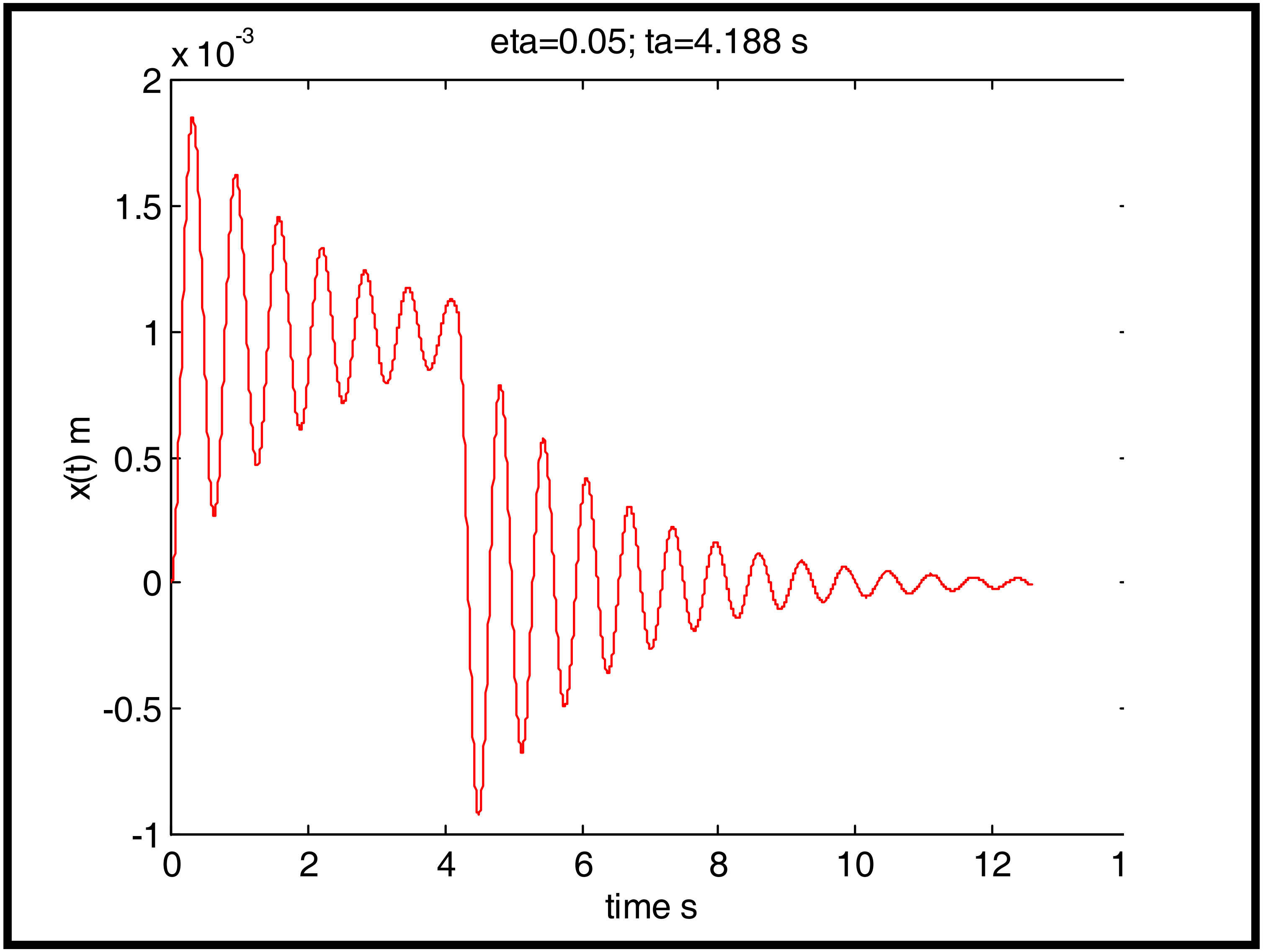
$$m\ddot{x} + c\dot{x} + kx = U(t) - U(t-a)$$

$$x(0) = 0; \quad \dot{x}(0) = 0$$

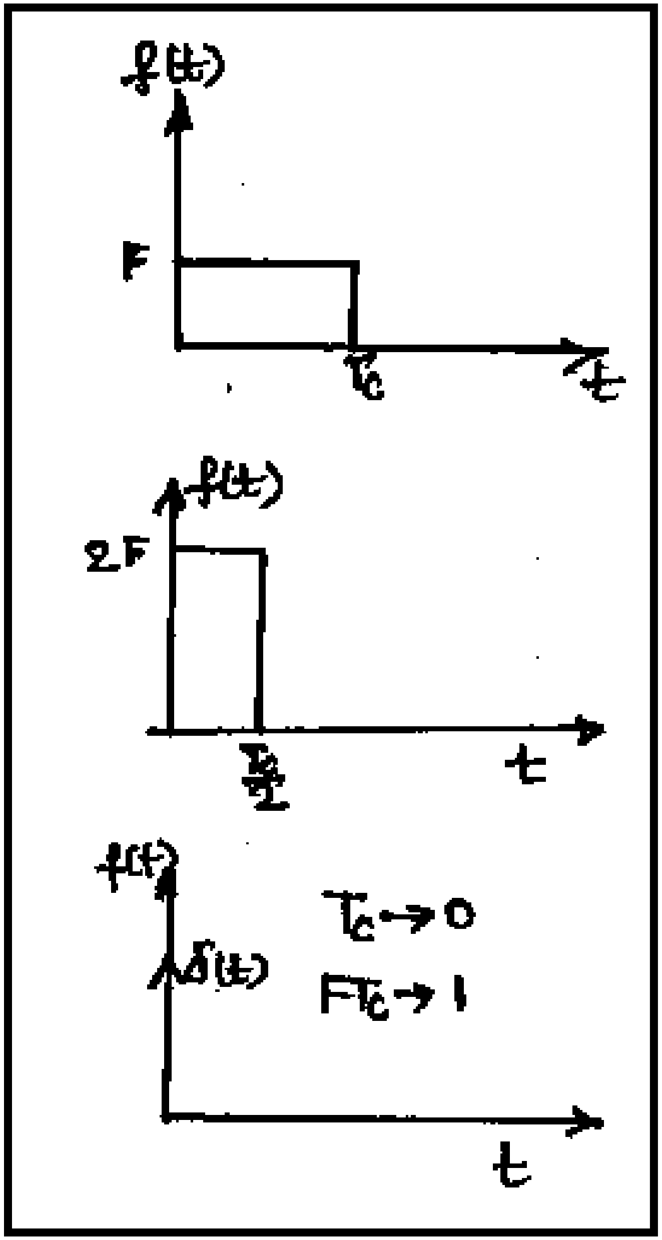
$$x(t) = G(t) \quad t < a$$

$$x(t) = G(t) - G(t-a) \quad t > a$$




eta=0.05; ta=4.188 s
x 10^-3
2
1.5
1
0.5
0
-0.5
-1
x(t) m
0
2
4
6
8
10
12
1
time s

Response to an unit impulse at t=0



Dirac's delta function

$$\delta(t-a)=0 \quad t\neq a$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

$$\int_{-\infty}^{\infty} \delta(t-a)dt = 1$$

Response to an unit impulse at t=0

$$f(t) = F[U(t) - U(t - T_c)]$$

$$x(t) = FG(t) \quad t < T_c$$

$$x(t) = F[G(t) - G(t - T_c)] \quad t > T_c$$

$$f(t) = FT_c \frac{[U(t) - U(t - T_c)]}{T_c}$$

$$\lim_{\substack{T_c \to 0 \\ FT_c \to 1}} f(t) = \frac{dU}{dt} = \delta(t)$$

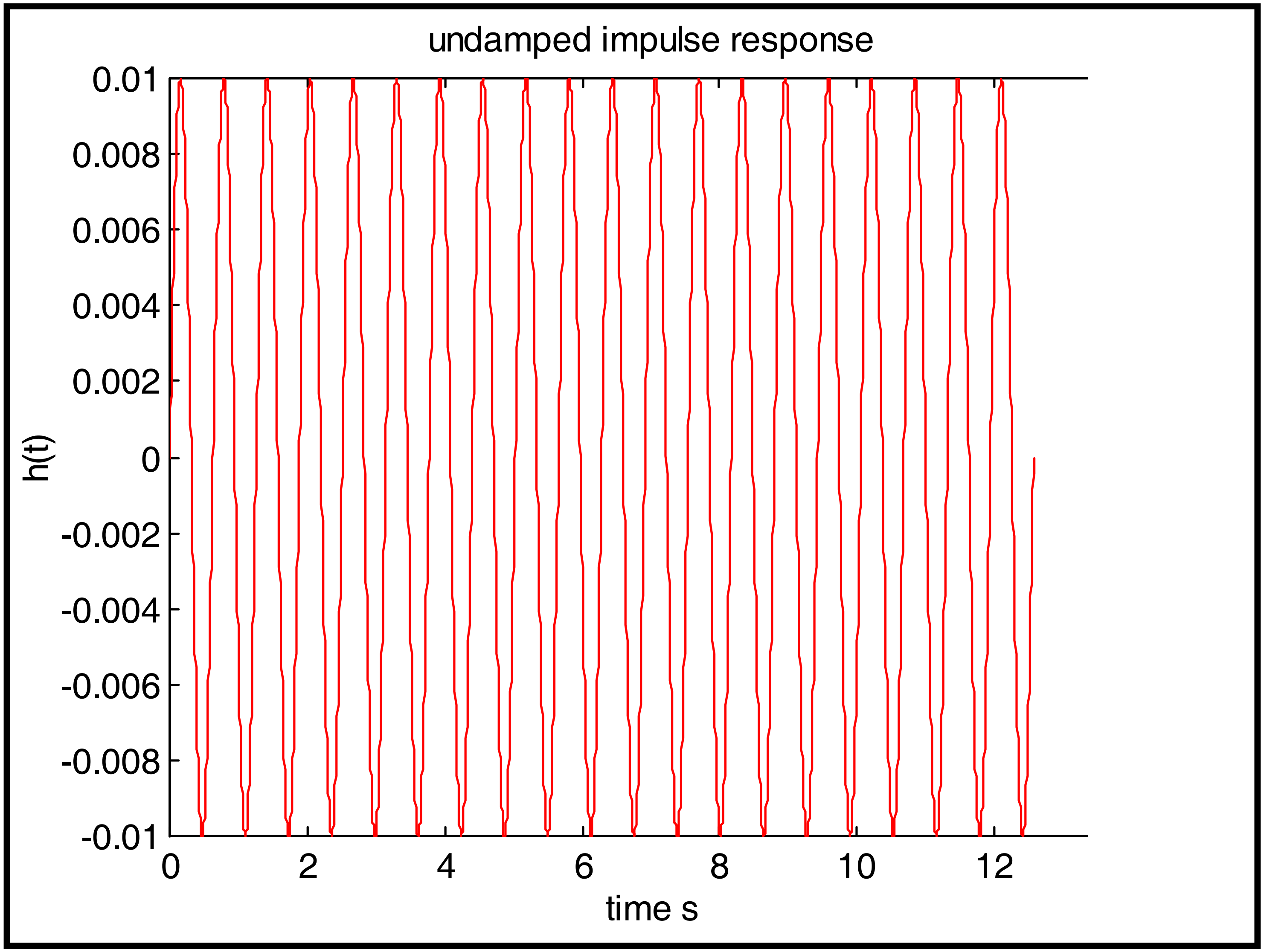
$$x(t) = FT_c \frac{[G(t) - G(t - T_c)]}{T_c}$$

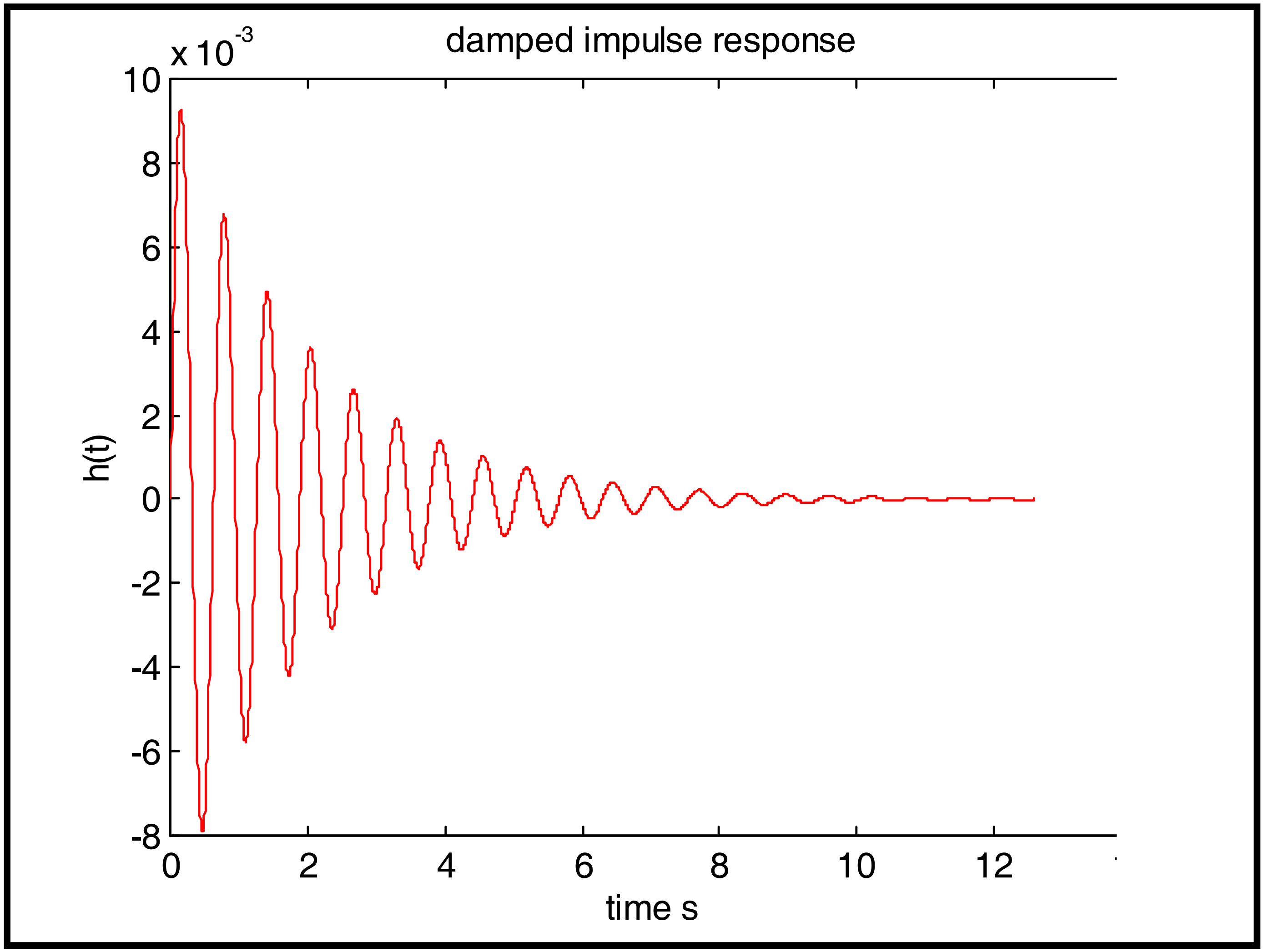
$$\lim_{\substack{T_c \to 0 \\ FT_c \to 1}} x(t) = \frac{dG}{dt} = h(t)$$

Impulse response function *h(t)*

$$G(t) = (1/k)\left[1 - \exp(-\eta\omega t)\left(\cos\omega_d t + \frac{\eta}{\sqrt{(1-\eta^2)}}\sin\omega_d t\right)\right]$$

$$h(t) = \frac{dG}{dt} = \frac{1}{m\omega_d}\exp(-\eta\omega t)\sin\omega_d t$$


undamped impulse response
0.01
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
-0.008
-0.01
h(t)
0
2
4
6
8
10
12
time s


damped impulse response
x 10^-3
10
8
6
4
2
0
-2
-4
-6
-8
h(t)
0
2
4
6
8
10
12
time s