Stochastic Structural Dynamics

Lecture-16

Random vibration analysis of MDOF systems-4

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VIBRATION ANALYSIS OF CONTINUOUS SYSTEMS

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2} + \varepsilon(x)\frac{\partial^3 y}{\partial x^2 \partial t} \right] + m(x)\ddot{y} + c(x)\dot{y} = f(x,t)$$

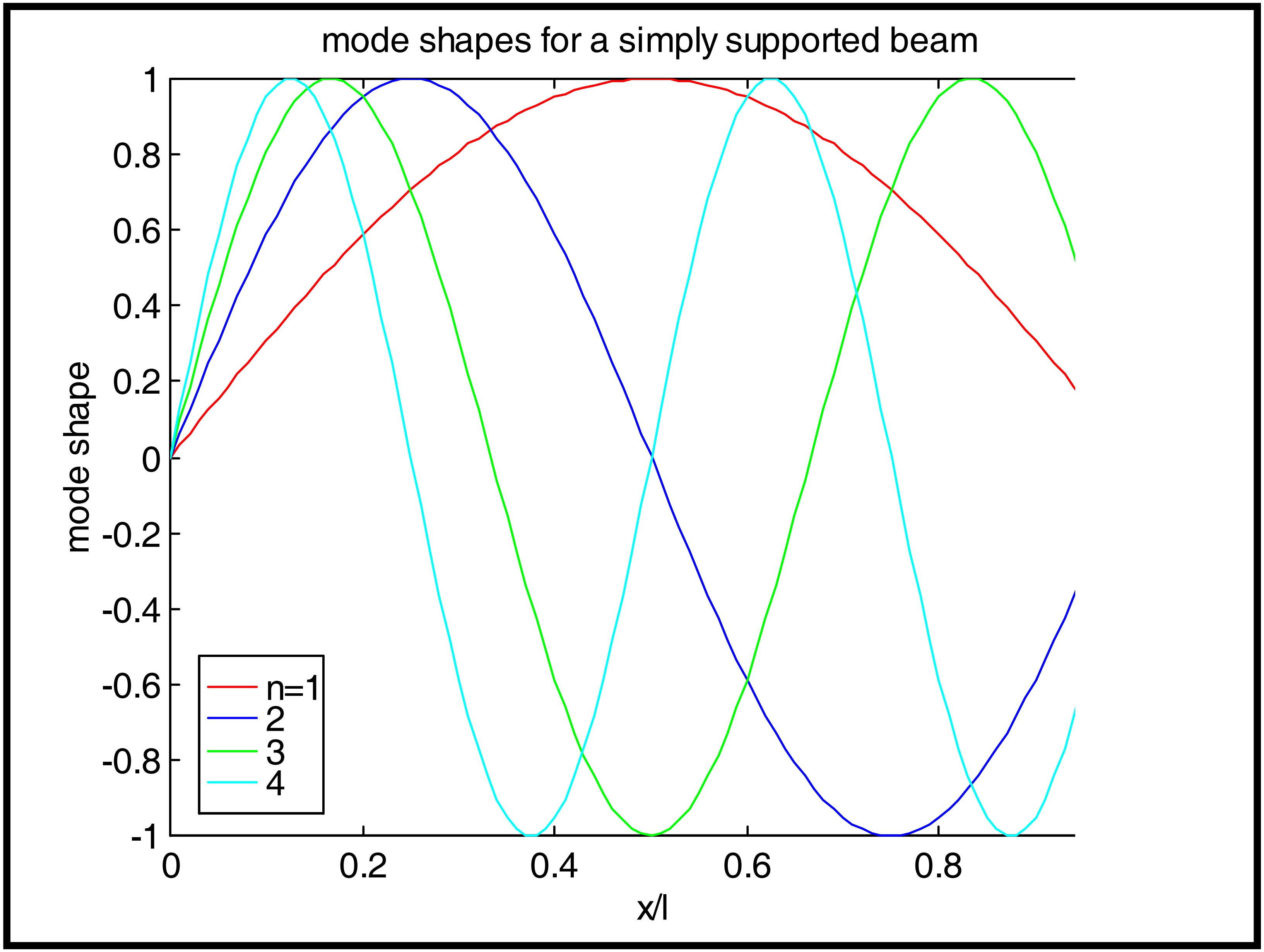
ICS: $y_0(x) = y(x,0) \quad \dot{y}_0(x) = \dot{y}(x,0)$ & BCS as appropriate.

$$\varepsilon(x) = \nu EI(x)$$

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$$

$$\left[EI\varphi_n'' \right]'' = m\omega_n^2 \varphi_n(x)$$

$$\int_0^L EI\varphi_n''\varphi_k'' dx = 0 \quad n \neq k \qquad \int_0^L m\varphi_n\varphi_k dx = 0 \quad n \neq k$$


mode shapes for a simply supported beam
mode shape
x/l
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
n=1
2
3
4

$$\ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = p_n(t);$$

$$2\eta_n\omega_n = (\alpha + \nu\omega_n^2);$$

$$p_n(t) = \frac{\int_0^L \varphi_n(x) f(x,t)dx}{\int_0^L \varphi_n^2(x)m(x)dx} \qquad n = 1, 2, \cdots\infty$$

$$y(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \left\{ \exp(-\eta_n\omega_n t)[A_n \cos\omega_{dn}t + B_n \sin\omega_{dn}t] + \int_0^t h_n(t-\tau)p_n(\tau)d\tau \right\}$$

•Displacement: $y(x,t) = \sum_{n=1}^{N\to\infty} a_n(t)\phi_n(x)$

•Slope: $y'(x,t) = \sum_{n=1}^{N\to\infty} a_n(t)\phi_n'(x)$

•Bending moment: $EI(x)\,y''(x,t) = \sum_{n=1}^{N\to\infty} a_n(t)\,EI(x)\phi_n''(x)$

•Shear force: $\left[EI(x)\,y''(x,t)\right]' = \sum_{n=1}^{N\to\infty} a_n(t)\left[EI(x)\phi_n''(x)\right]'$

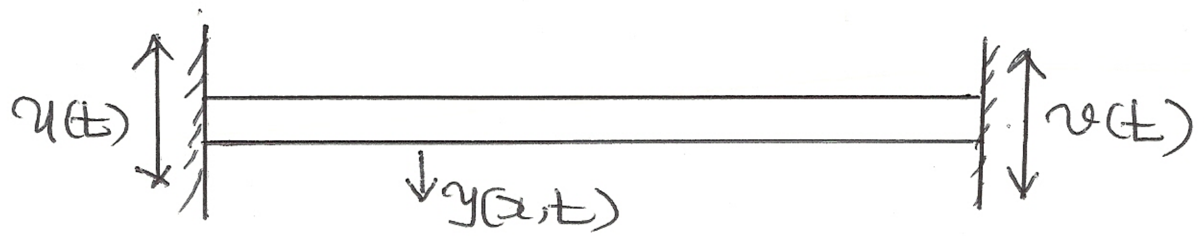
Other quantities

•Bending stress

•Shear stress

•Principal stresses

A clamped beam under differential support displacements



$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$
$$y(0,t) = u(t);\; y'(0,t) = 0$$
$$y(l,t) = v(t);\; y'(l,t) = 0$$
$$y(x,0) = 0;\; \dot{y}(x,0) = 0$$

The BCS are time dependent.
Modal expansion method cannot be used directly.

Introduce a new dependent variable

$$y(x,t) = w(x,t) + h_1(x)u(t) + h_2(x)v(t)$$

$$y(0,t) = w(0,t) + h_1(0)u(t) + h_2(0)v(t) = u(t)$$

Select $w(0,t) = 0;\ h_1(0) = 1;\ \&\ h_2(0) = 0$

$$y'(0,t) = w'(0,t) + h_1'(0)u(t) + h_2'(0)v(t) = 0$$

Select $w'(0,t) = 0;\ h_1'(0) = 1;\ \&\ h_2'(0) = 0$

$$y(l,t) = w(l,t) + h_1(l)u(t) + h_2(l)v(t) = v(t)$$

Select $w(l,t) = 0;\ h_1(l) = 0;\ \&\ h_2(l) = 1$

$$y'(l,t) = w'(l,t) + h_1'(l)u(t) + h_2'(l)v(t) = 0$$

Select $w'(l,t) = 0;\ h_1'(l) = 1;\ \&\ h_2'(l) = 0$

$$EI\left[w^{iv}+h_1^{iv}u+h_2^{iv}u\right]+m\left[\ddot{w}+h_1\ddot{u}+h_2\ddot{v}\right]+c\left[\dot{w}+h_1\dot{u}+h_2\dot{v}\right]=0$$

Select

$$h_1^{iv}=0$$

$$h_2^{iv}=0$$

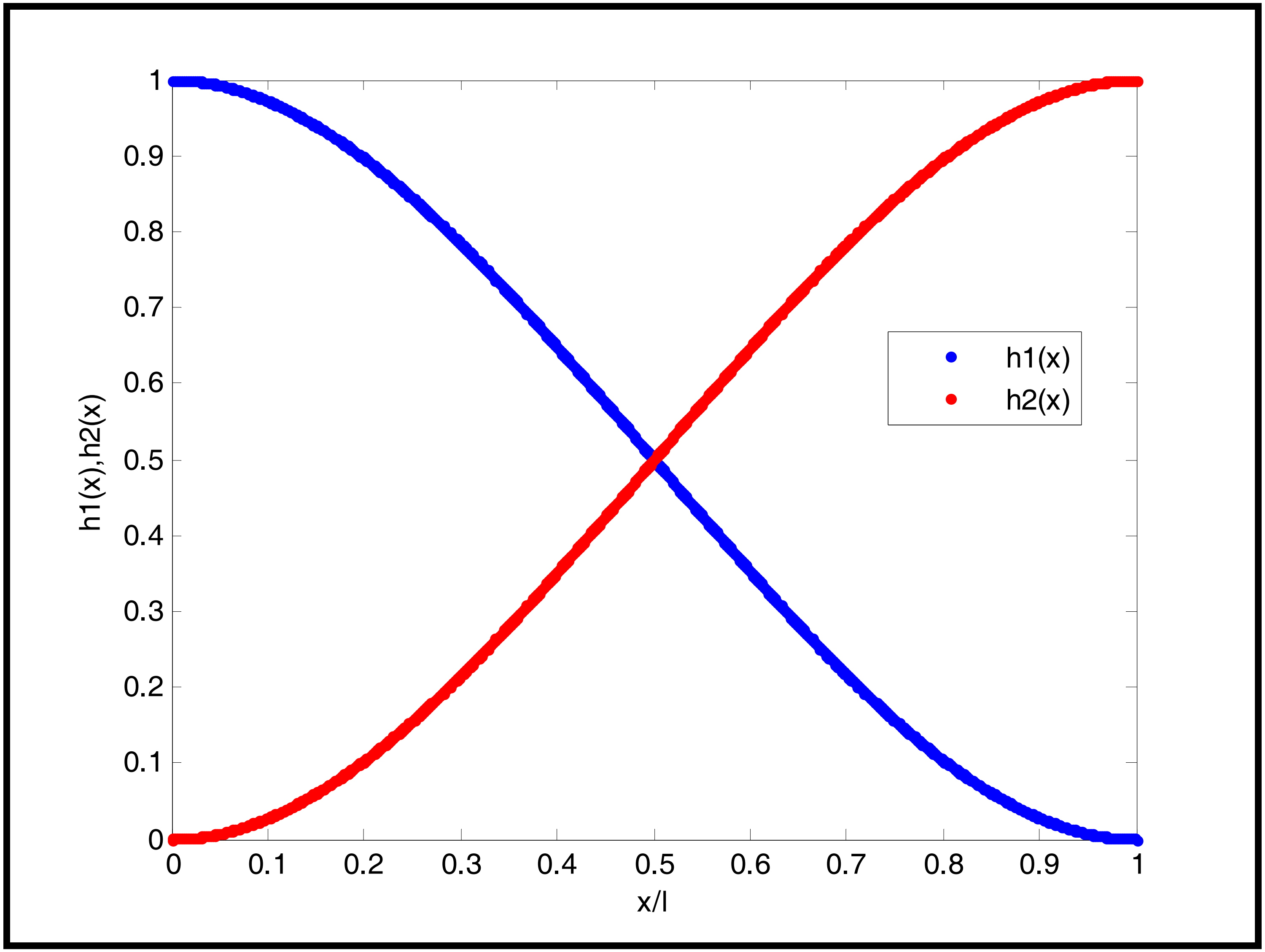
$$h_1(x)=ax^3+bx^2+cx+d$$

$$h_1(0)=1;\,h_1(l)=0;\,h_1'(0)=0;\,h_1'(l)=0;$$

$$h_1(x)=1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3}$$

Similarly

$$h_2(x)=\frac{3x^2}{L^2}-\frac{2x^3}{L^3}$$



h1(x)
h2(x)
h1(x),h2(x)
x/l
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

$$EI\left[w^{iv}+h_1^{iv}u+h_2^{iv}u\right]+m\left[\ddot{w}+h_1\ddot{u}+h_2\ddot{v}\right]+c\left[\dot{w}+h_1\dot{u}+h_2\dot{v}\right]=0$$

$$\Rightarrow EIw^{iv}+m\ddot{w}+c\dot{w}=$$

$$-m\left[\ddot{u}\left(1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3}\right)+\ddot{v}\left(\frac{3x^2}{L^2}-\frac{2x^3}{L^3}\right)\right]$$

$$-c\left[\dot{u}\left(1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3}\right)+\dot{v}\left(\frac{3x^2}{L^2}-\frac{2x^3}{L^3}\right)\right]=f(x,t)$$

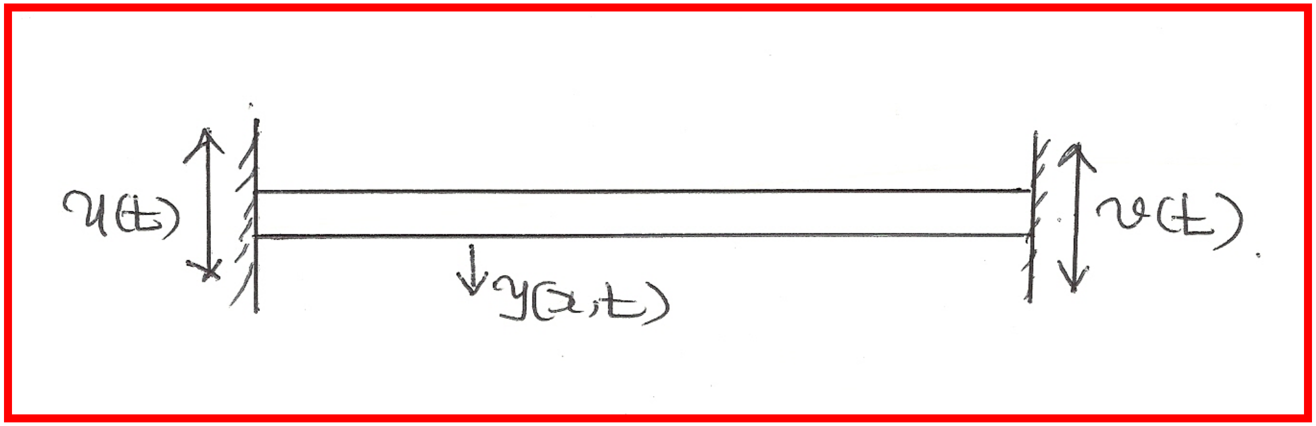
$$w(0,t)=0;w'(0,t)=0$$

$$w(l,t)=0;w'(l,t)=0$$

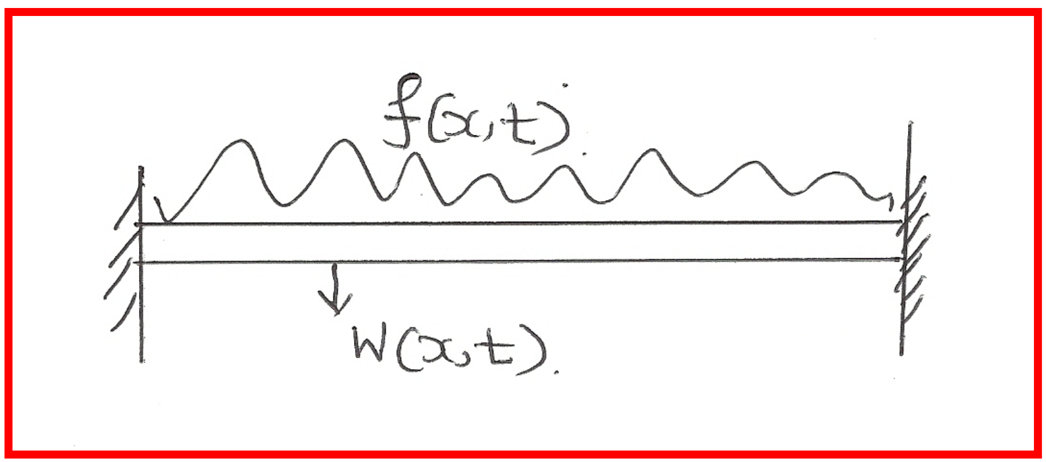
$$w(x,0)=-h_1(x)u(0)-h_2(x)v(0)$$

$$w(x,0)=-h_1(x)\dot{u}(0)-h_2(x)\dot{v}(0)$$

Eigenfunction expansion method can now be used.



$$y(x,t) = w(x,t) + \left[h_1(x)u(t) + h_2(x)v(t) \right]$$



$$y(x,t) = w(x,t) + \left[h_1(x) u(t) + h_2(x) v(t) \right]$$

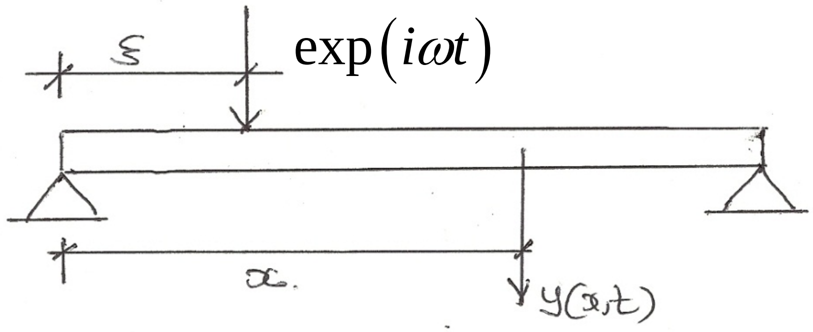
Total response

Dynamic response

Pseudo - dynamic response

$$\begin{Bmatrix} u^T \\ u_g \end{Bmatrix} = \begin{Bmatrix} u(t) \\ 0 \end{Bmatrix} + \begin{Bmatrix} u^p(t) \\ u_g(t) \end{Bmatrix}$$

Harmonically driven beam: Green's functions in frequency domain



$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\nu EI\left(x\right)\frac{\partial^3 y}{\partial x^2\partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=\exp\left(i\omega t\right)\delta\left(x-\xi\right)$$

ICS: $y_0(x)=y(x,0) \quad \dot{y}_0(x)=\dot{y}(x,0)$

BCS: $y\left(0,t\right)=0; EIy''\left(0,t\right)=0; y\left(L,t\right)=0; EIy''\left(L,t\right)=0$

$$\lim_{t\to\infty} y\left(x,t\right)=?$$

$$y(x,t)=\sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$$

$$\left[EI\varphi_n''\right]'' = m\omega_n^2\varphi_n(x)$$

$$\int_0^L EI\varphi_n''\varphi_k'' dx = 0 \quad n \neq k \qquad \int_0^L m\varphi_n\varphi_k dx = 0 \quad n \neq k$$

$$\ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = \int_0^L \exp\left(i\omega t\right)\phi_n\left(x\right)\delta\left(x-\xi\right)dx$$

$$= \phi_n\left(\xi\right)\exp\left(i\omega t\right);$$

$$n = 1, 2, \cdots\infty$$

$$\lim_{t\to\infty} a_n(t) \to \frac{\phi_n(\xi)\exp(i\omega t)}{\omega_n^2 - \omega^2 + i2\eta_n\omega\omega_n}$$

$$\Rightarrow$$

$$\lim_{t\to\infty} y(x,t) = \sum_{n=1}^{N\to\infty} \frac{\phi_n(x)\phi_n(\xi)\exp(i\omega t)}{\omega_n^2 - \omega^2 + i2\eta_n\omega\omega_n}$$

$$= G(x,\xi,\omega)\exp(i\omega t)$$

with

$$G(x,\xi,\omega) = \sum_{n=1}^{N\to\infty} \frac{\phi_n(x)\phi_n(\xi)}{\omega_n^2 - \omega^2 + i2\eta_n\omega\omega_n}$$

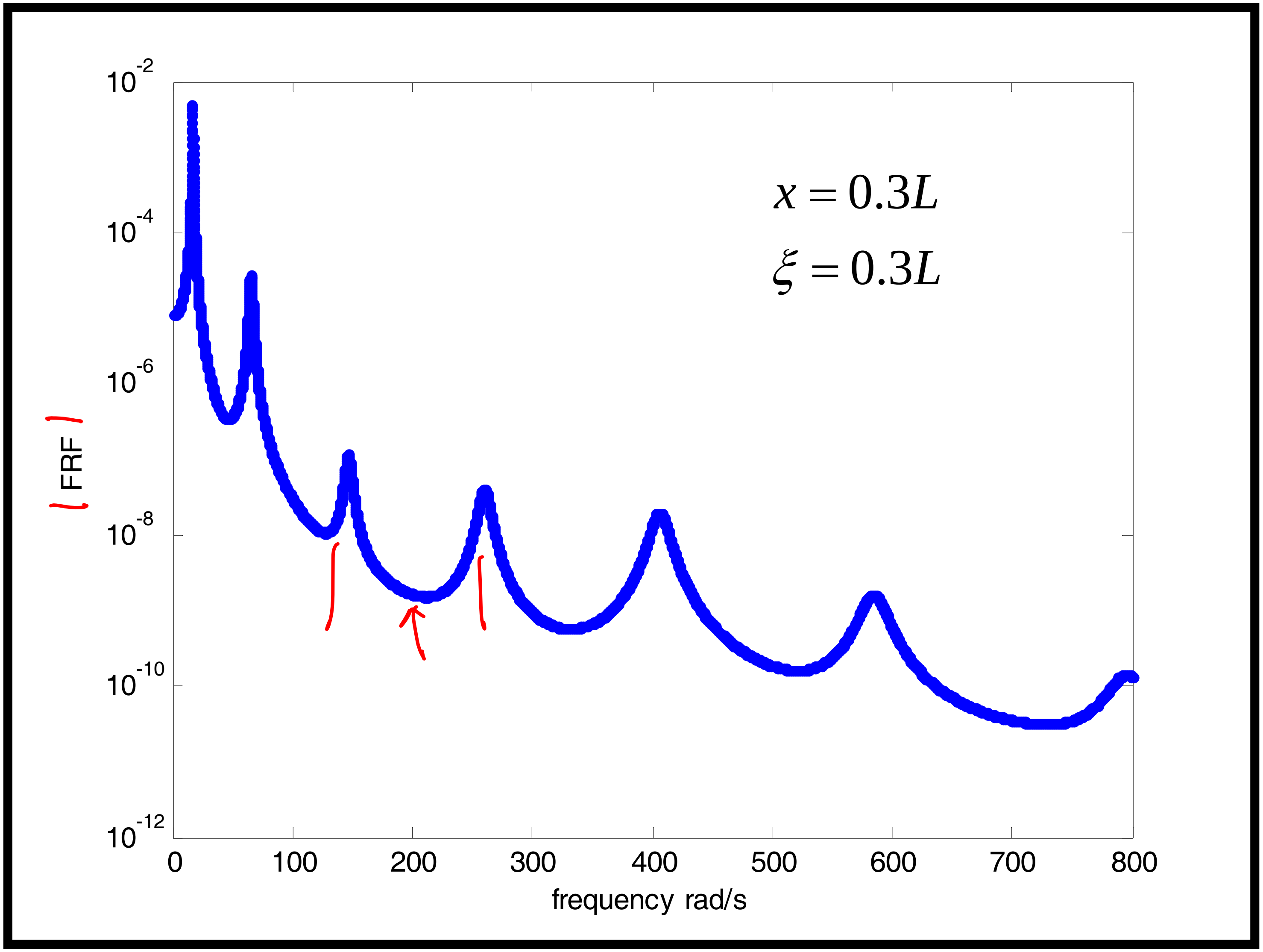
$G(x,\xi,\omega) =$ Green's function

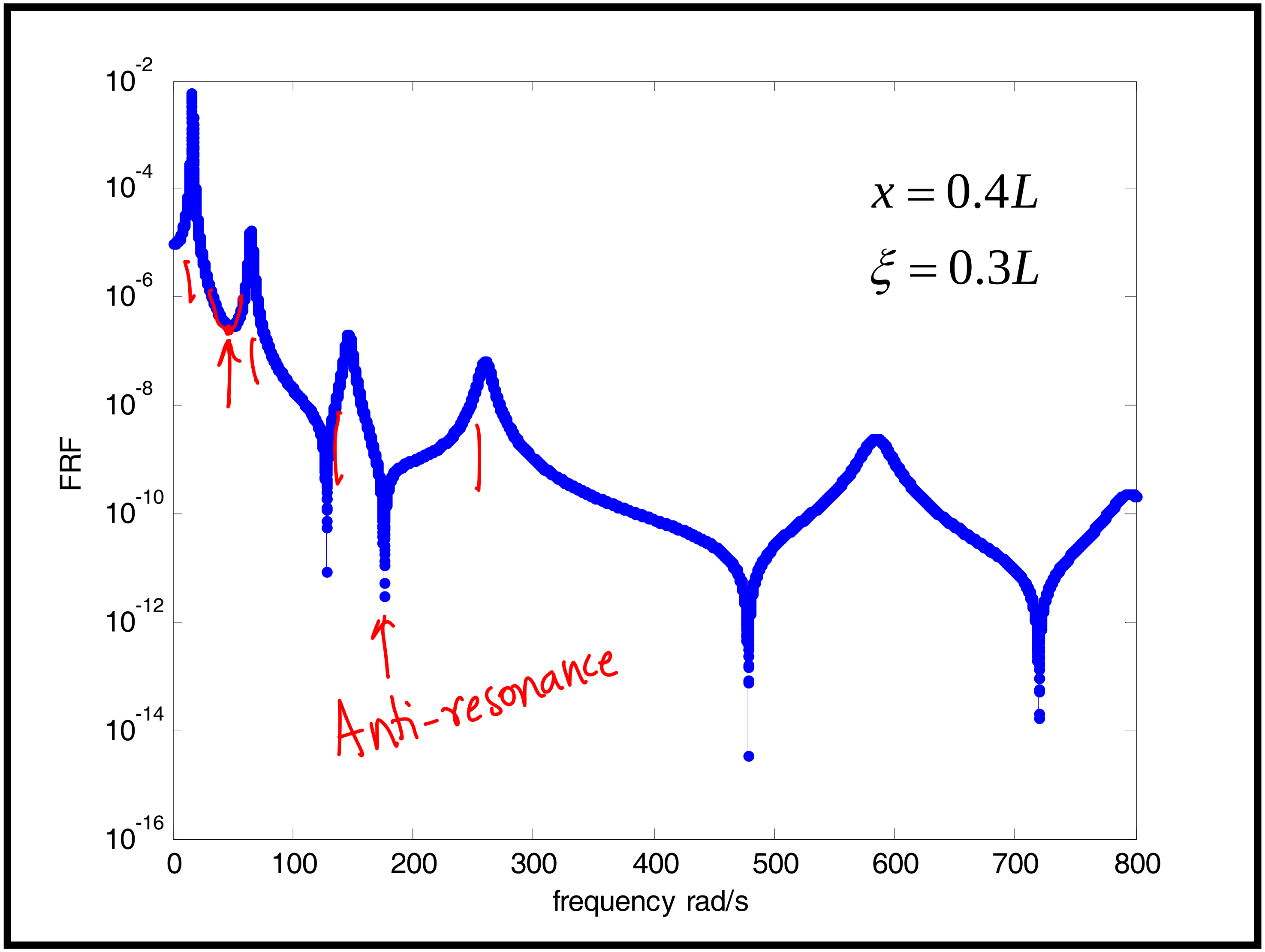
Note

- $G(x,\xi,\omega) = G(\xi,x,\omega)$
- $G(x,\xi,\omega)$ is complex valued
- $G(x,\xi,\omega)$ is the generalization of the FRF discussed earlier

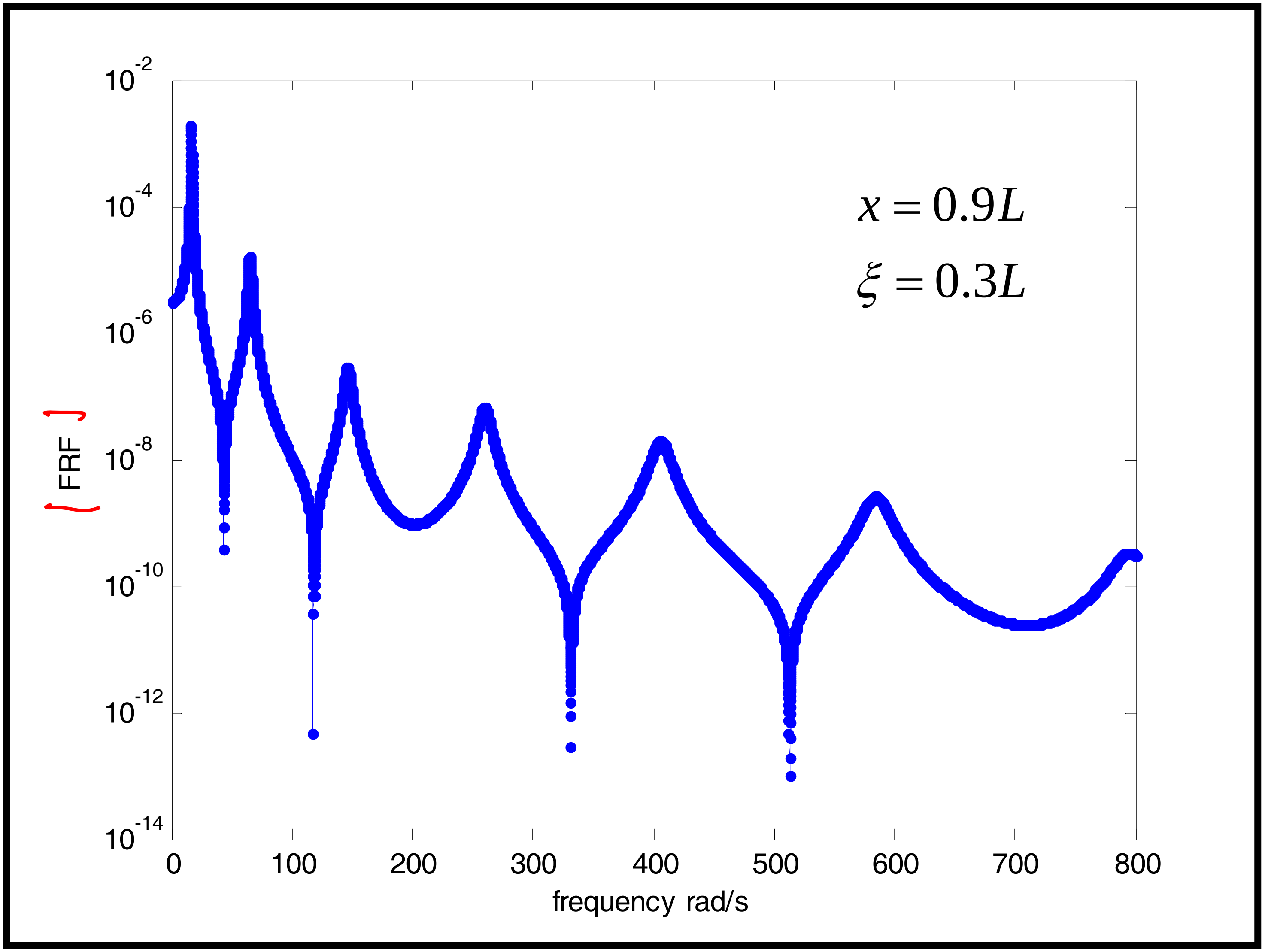
$H_{ij}(\omega)$

x, ξ

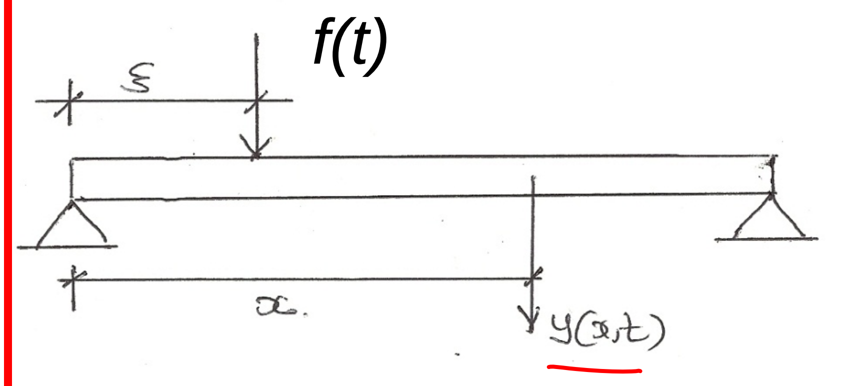

$x = 0.3L$
$\xi = 0.3L$
|FRF|
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
0
100
200
300
400
500
600
700
800
frequency rad/s



10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
FRF
0
100
200
300
400
500
600
700
800
frequency rad/s
x = 0.4L
ξ = 0.3L
Anti-resonance


$x = 0.9L$
$\xi = 0.3L$
FRF
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
0
100
200
300
400
500
600
700
800
frequency rad/s

Response of beam to a general load *f(t)*
Note: the Fourier transform of f(t) is taken to exist



$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\nu EI\left(x\right)\frac{\partial^3 y}{\partial x^2 \partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=f(t)\delta\left(x-\xi\right)$$

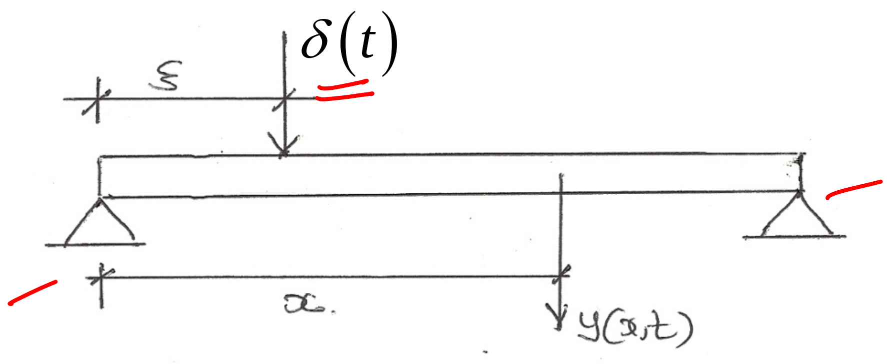
ICS: $y_0(x)=y(x,0) \quad \dot{y}_0(x)=\dot{y}(x,0)$

BCS: $y\left(0,t\right)=0; EIy''\left(0,t\right)=0; y\left(L,t\right)=0; EIy''\left(L,t\right)=0$

$Y(\omega) = H(\omega)F(\omega)$

$$Y\left(x,\omega\right)=G\left(x,\xi,\omega\right)F\left(\omega\right)$$

Beam driven by impulse excitation:
Green's functions in time domain



$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\nu EI(x)\frac{\partial^3 y}{\partial x^2 \partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=\delta(t-0)\delta(x-\xi)$$

ICS: $y_0(x)=y(x,0)=0 \quad \dot{y}_0(x)=\dot{y}(x,0)=0$

BCS: $y(0,t)=0; EIy''(0,t)=0; y(L,t)=0; EIy''(L,t)=0$

$$y(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$$

$$\left[EI\varphi_n''\right]'' = m\omega_n^2\varphi_n(x)$$

$$\int_0^L EI\varphi_n''\varphi_k''dx = 0 \quad n \neq k \qquad \int_0^L m\varphi_n\varphi_k dx = 0 \quad n \neq k$$

$$\ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = \int_0^L \phi_n(x)\delta(t)\delta(x-\xi)dx$$

$$= \phi_n(\xi)\delta(t);$$

$$n = 1, 2, \cdots \infty$$

$$a_n(t) = \phi_n(\xi) h_n(t)$$

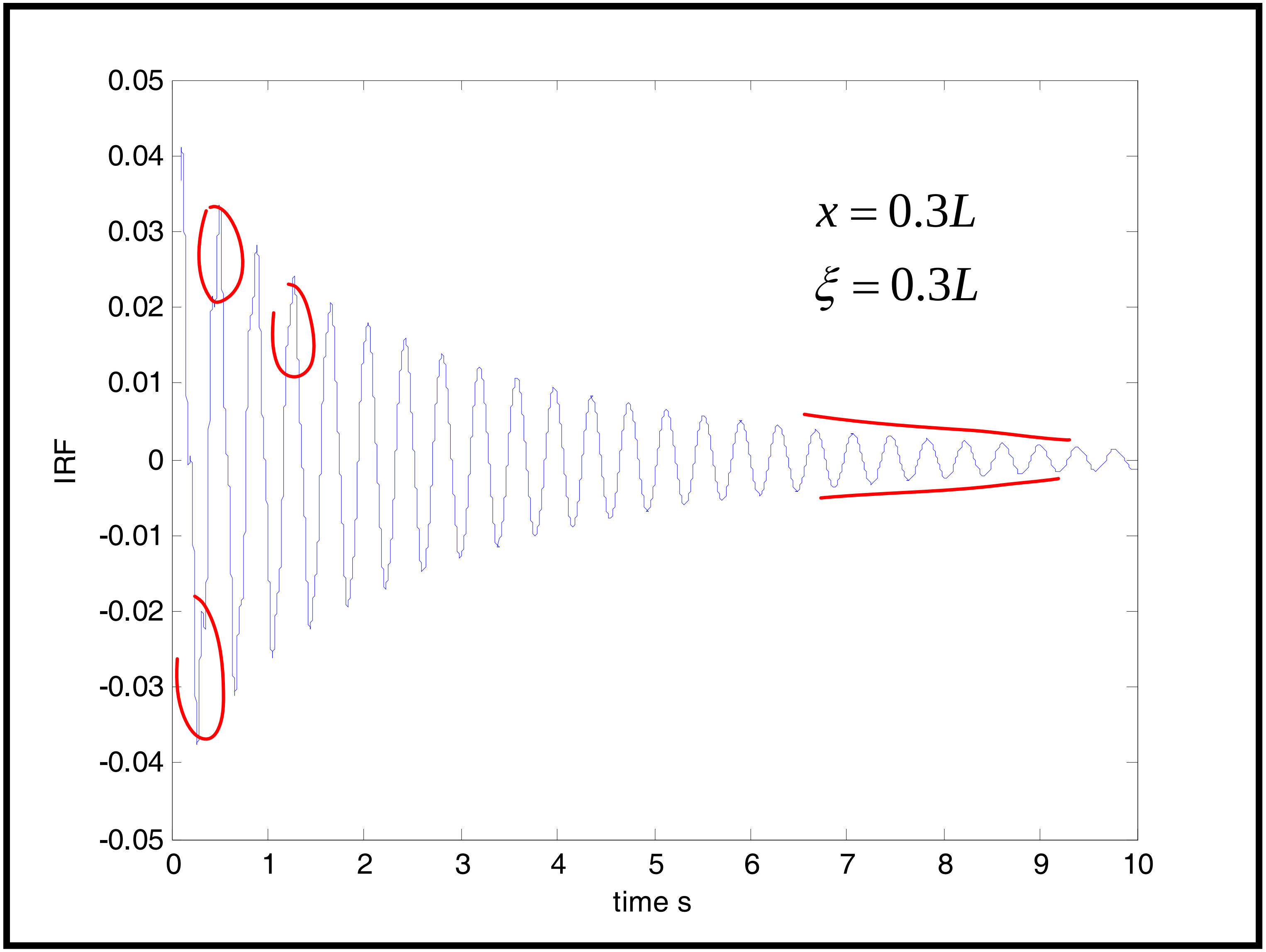
$$= \frac{\phi_n(\xi)}{M_n \omega_{dn}} \exp(-\eta_n \omega_n t) \sin(\omega_{dn} t)$$

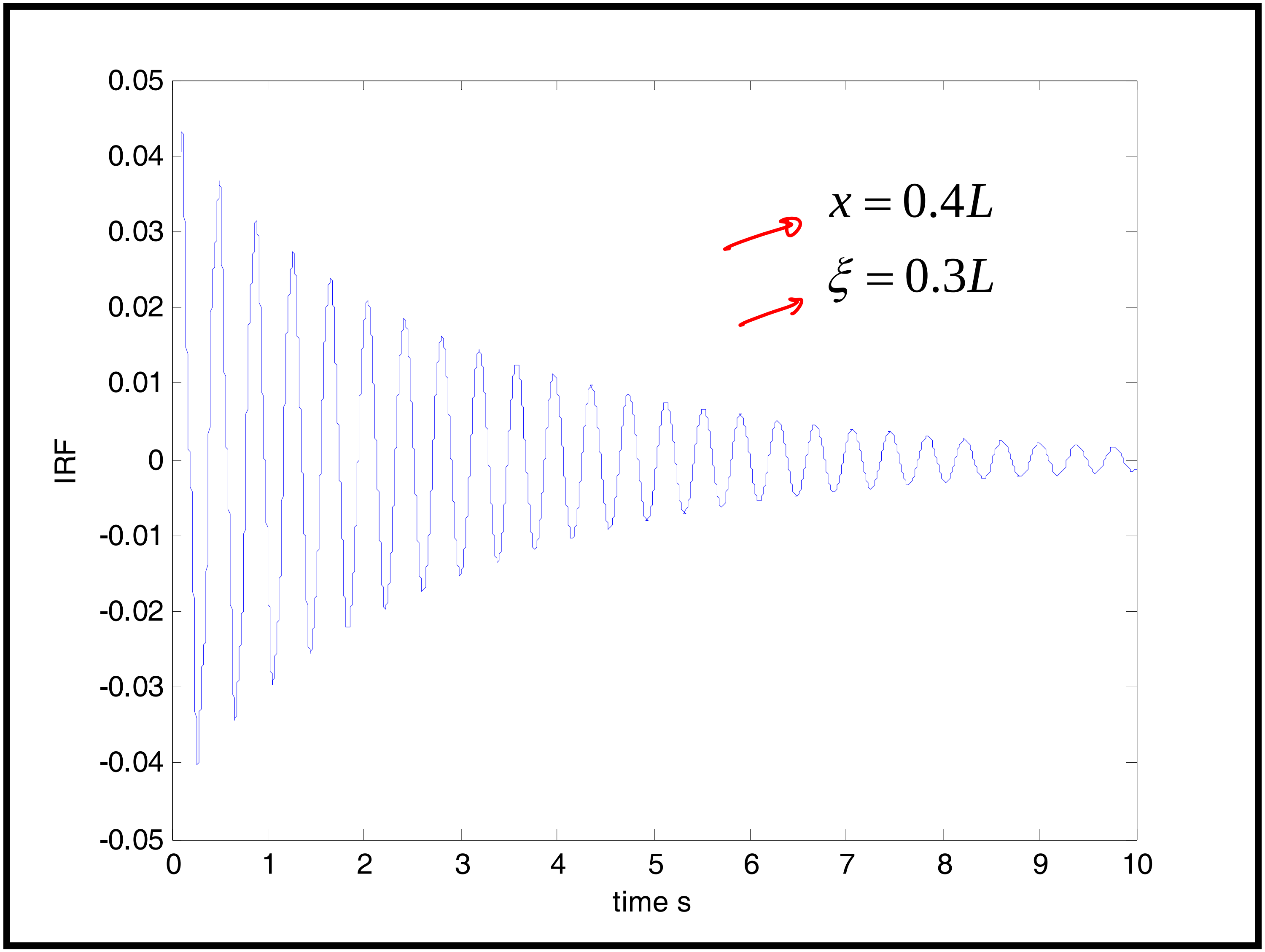
$M_n = 1 \quad \int_0^L m \phi_n^2(x)\,dx = 1$

$$y(x,t) = \sum_{n=1}^{N\to\infty} \frac{\phi_n(\xi)\phi_n(x)}{\omega_{dn}} \exp(-\eta_n \omega_n t) \sin(\omega_{dn} t)$$

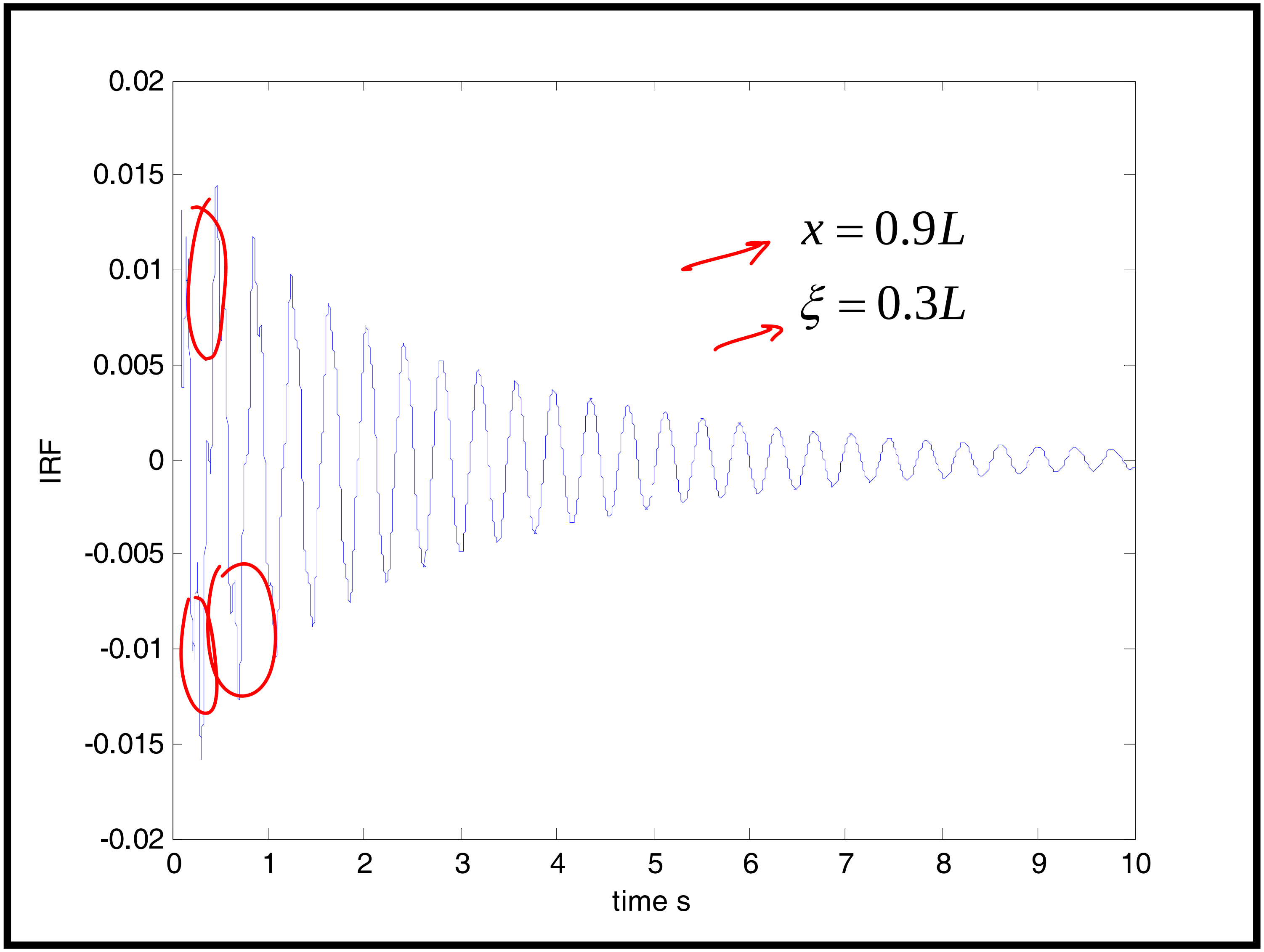
$$= g(x, \xi, t)$$

$$g(x, \xi, t) = g(\xi, x, t)$$

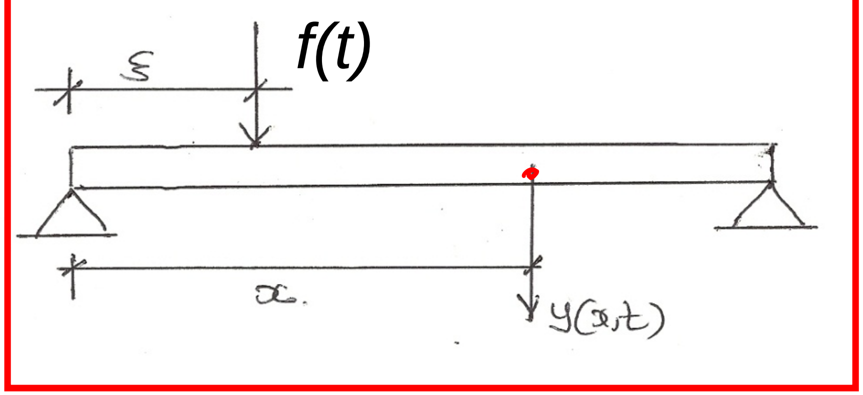

0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
IRF
0
1
2
3
4
5
6
7
8
9
10
time s
$x = 0.3L$
$\xi = 0.3L$



0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
IRF
0
1
2
3
4
5
6
7
8
9
10
time s
$x = 0.4L$
$\xi = 0.3L$


0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
IRF
0
1
2
3
4
5
6
7
8
9
10
time s
$x = 0.9L$
$\xi = 0.3L$

Response of beam to a concentrated load *f(t)*



$$x(t)=\int_0^t h(t-\tau)f(\tau)d\tau$$

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\nu EI\left(x\right)\frac{\partial^3 y}{\partial x^2\partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=f(t)\delta\left(x-\xi\right)$$

ICS: $y_0(x)=y(x,0)=0 \quad \dot{y}_0(x)=\dot{y}(x,0)=0$

BCS: $y\left(0,t\right)=0; EIy''\left(0,t\right)=0; y\left(L,t\right)=0; EIy''\left(L,t\right)=0$

hij

$$y\left(x,t\right)=\int_0^t g\left(x,\xi,t-\tau\right)f\left(\tau\right)d\tau$$

Exercise

Show that

$$g(x,\xi,t) \Leftrightarrow G(x,\xi,\omega)$$

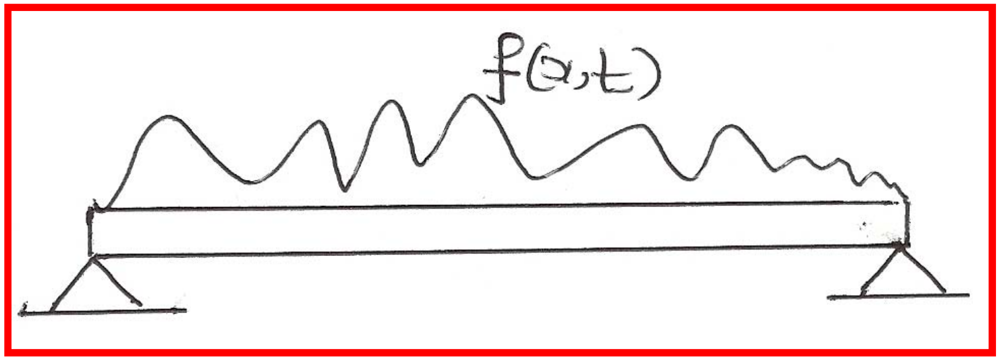
h(t) ⟺ H(ω)

That is

$$G(x,\xi,\omega) = \int_{-\infty}^{\infty} g(x,\xi,t)\exp(i\omega t)\,dt$$

$$g(x,\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x,\xi,\omega)\exp(-i\omega t)\,d\omega$$

Response of beam to a general load *f(x,t)*



$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\nu EI\left(x\right)\frac{\partial^3 y}{\partial x^2 \partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=f(x,t)$$

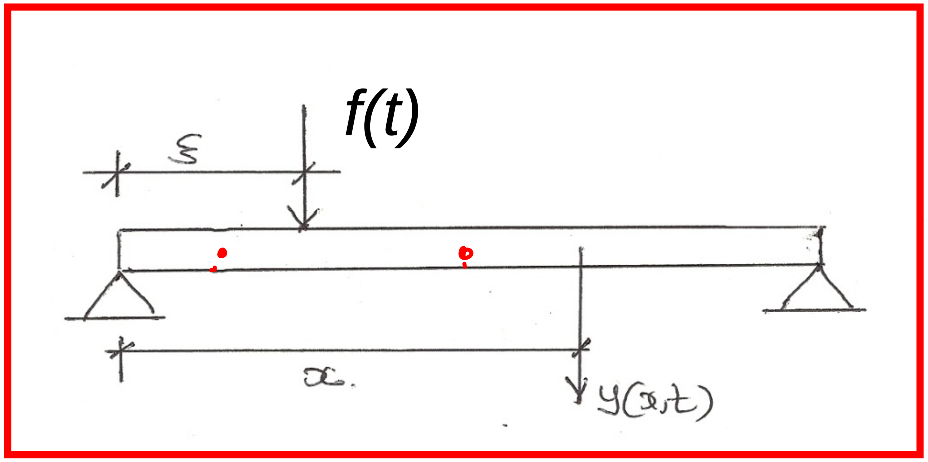
ICS: $y_0(x)=y(x,0)=0 \quad \dot{y}_0(x)=\dot{y}(x,0)=0$

BCS: $y\left(0,t\right)=0; EIy''\left(0,t\right)=0; y\left(L,t\right)=0; EIy''\left(L,t\right)=0$

$$y(x,t)=\int_0^L\int_0^t g(x,\xi,t-\tau)f(\xi,\tau)\,d\xi\,d\tau$$

$$Y(x,\omega)=\int_0^L G(x,\xi,\omega)F(\xi,\omega)\,d\xi$$

Response of beam to a concentrated random load *f(t)*



$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\nu EI\left(x\right)\frac{\partial^3 y}{\partial x^2 \partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=f(t)\delta\left(x-\xi\right)$$

$$\left\langle f(t)\right\rangle=0;\left\langle f\left(t\right)f\left(t+\tau\right)\right\rangle=R_{ff}\left(\tau\right)\Longleftrightarrow S_{ff}(\omega)$$

ICS: $y_0(x)=y(x,0)=0 \quad \dot{y}_0(x)=\dot{y}(x,0)=0$

BCS: $y\left(0,t\right)=0; EIy''\left(0,t\right)=0; y\left(L,t\right)=0; EIy''\left(L,t\right)=0$

$$y(x,t)=\int_0^t g(x,\xi,t-\tau)f(\tau)\,d\tau$$

$$\langle y(x,t)\rangle=\int_0^t g(x,\xi,t-\tau)\langle f(\tau)\rangle\,d\tau=0$$

$$\langle y(x_1,t_1)\,y(x_2,t_2)\rangle=\int_0^{t_1}\int_0^{t_2} g(x_1,\xi,t_1-\tau_1)\,g(x_2,\xi,t_2-\tau_2)\langle f(\tau_1)\,f(\tau_2)\rangle\,d\tau_1 d\tau_2$$

$$=\int_0^{t_1}\int_0^{t_2} g(x_1,\xi,t_1-\tau_1)\,g(x_2,\xi,t_2-\tau_2)R_{ff}(\tau_1,\tau_2)\,d\tau_1 d\tau_2$$

$$=\int_0^{t_1}\int_0^{t_2} g(x_1,\xi,t_1-\tau_1)\,g(x_2,\xi,t_2-\tau_2)R_{ff}(\tau_1-\tau_2)\,d\tau_1 d\tau_2$$

$$\left\langle y(x_1,t_1)\,y(x_2,t_2)\right\rangle = \int_0^{t_1}\int_0^{t_2} g(x_1,\xi,t_1-\tau_1)\,g(x_2,\xi,t_2-\tau_2)\,R_{ff}(\tau_1-\tau_2)\,d\tau_1 d\tau_2$$

$$= \int_0^{t_1}\int_0^{t_2} g(x_1,\xi,t_1-\tau_1)\,g(x_2,\xi,t_2-\tau_2)\left[\frac{1}{\pi}\int_0^{\infty} S_{ff}(\omega)\cos\omega(\tau_1-\tau_2)\,d\omega\right]d\tau_1 d\tau_2$$

$$= \int_0^{\infty} S_{ff}(\omega)\,H(x_1,x_2,\xi,t_1,t_2,\omega)\,d\omega$$

$$H(x_1,x_2,\xi,t_1,t_2,\omega) = \int_0^{t_1}\int_0^{t_2} g(x_1,\xi,t_1-\tau_1)\,g(x_2,\xi,t_2-\tau_2)\cos\omega(\tau_1-\tau_2)\,d\tau_1 d\tau_2$$

$$\left\langle y^2(x,t)\right\rangle = \int_0^{\infty} S_{ff}(\omega)\,H(x,x,\xi,t,t,\omega)\,d\omega$$

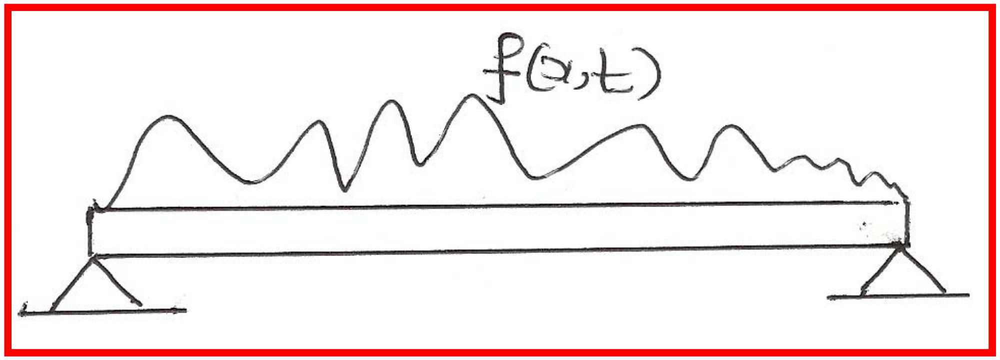
Steady state response

$$Y_T(x,\xi,\omega) = G(x,\xi,\omega)\,F_T(\omega)$$

$$S_{YY}(x,\xi,\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \left|Y_T(x,\xi,\omega)\right|^2 \right\rangle$$

$$= \left|G(x,\xi,\omega)\right|^2 S_{FF}(\omega)$$

Beam excited by space-time white noise forcing
(Rain on the roof excitation)



$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\nu EI\left(x\right)\frac{\partial^3 y}{\partial x^2\partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=f(x,t)$$

$$\left\langle f(x,t)\right\rangle=0;\left\langle f\left(x,t\right)f\left(x+\xi,t+\tau\right)\right\rangle=I_0 m\left(x\right)\delta\left(\xi\right)\delta\left(\tau\right)$$

$$\text{ICS:}\quad y_0(x)=y(x,0)=0\quad \dot{y}_0(x)=\dot{y}(x,0)=0$$

$$\text{BCS: } y\left(0,t\right)=0; EIy''\left(0,t\right)=0; y\left(L,t\right)=0; EIy''\left(L,t\right)=0$$

$$y(x,t)=\sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$$

$$\left[EI\varphi_n''\right]''=m\omega_n^2\varphi_n(x)$$

$$\int_0^L EI\varphi_n''\varphi_k''dx=0 \quad n\neq k \qquad \int_0^L m\varphi_n\varphi_k dx=0 \quad n\neq k$$

$$\ddot{a}_n+2\eta_n\omega_n\dot{a}_n+\omega_n^2 a_n=\int_0^L \phi_n(x)\, f(x,t)dx;\, n=1,2,\cdots\infty$$

$$a_n(t)=\int_0^t\int_0^L h_n(t-\tau)\phi_n(x)\, f(x,\tau)dx\,d\tau$$

$$a_n(t) = \int_0^t \int_0^L h_n(t-\tau)\phi_n(x) f(x,\tau) dx d\tau$$

$$\langle a_n(t) \rangle = \int_0^t \int_0^L h_n(t-\tau)\phi_n(x) \langle f(x,\tau) \rangle dx d\tau = 0$$

$$\langle a_n(t_1) a_k(t_2) \rangle = \int_0^{t_1} \int_0^{t_2} \int_0^L \int_0^L h_n(t_1-\tau_1)\phi_n(x_1) h_k(t_2-\tau_2)\phi_k(x_2)$$

$$\langle f(x_1,\tau_1) f(x_2,\tau_2) \rangle dx_1 dx_2 d\tau_1 d\tau_2$$

$$= \int_0^{t_1} \int_0^{t_2} \int_0^L \int_0^L h_n(t_1-\tau_1)\phi_n(x_1) h_k(t_2-\tau_2)\phi_k(x_2)$$

$$I_0 m(x_1)\delta(\tau_1-\tau_2)\delta(x_1-x_2) dx_1 dx_2 d\tau_1 d\tau_2$$

$$\left\langle a_n(t_1)a_k(t_2)\right\rangle = \int_0^{t_1}\int_0^{t_2}\int_0^{L}\int_0^{L} h_n(t_1-\tau_1)\phi_n(x_1)h_k(t_2-\tau_2)\phi_k(x_2)$$

$$I_0 m(x_1)\delta(\tau_1-\tau_2)\delta(x_1-x_2)\,dx_1dx_2d\tau_1d\tau_2$$

$$=\int_0^{t_2}\int_0^{L} h_n(t_1-\tau_2)h_k(t_2-\tau_2)I_0 m(x_2)\phi_n(x_2)\phi_k(x_2)\,dx_2d\tau_2$$

$$\int_0^L m(x)\,\phi_n(x)\phi_k(x)\,dx$$

$$=0 \text{ for } n\neq k$$

$$=\int_0^{t_2} h_n^2(t_1-\tau)I_0\,d\tau \text{ for } n=k$$

$$\Rightarrow$$

$$\left\langle a_n^2(t)\right\rangle = \int_0^{t} h_n^2(t-\tau)I_0\,d\tau$$

Generalized coordinates are uncorrelated

$$y(x,t) = \sum_{n=1}^{N\to\infty} a_n(t)\phi_n(x)$$

$$\Rightarrow$$

$$\langle y(x,t)\rangle = \sum_{n=1}^{N\to\infty} \phi_n(x)\langle a_n(t)\rangle = 0$$

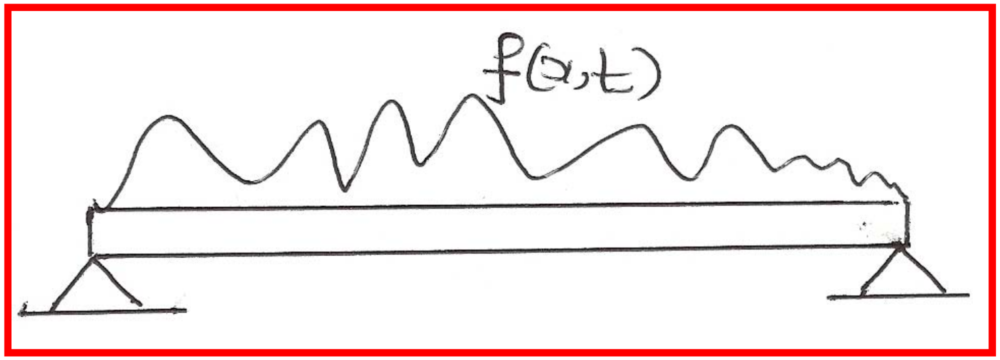
$$\langle y(x_1,t_1)\,y(x_2,t_2)\rangle = \sum_{n=1}^{N\to\infty}\sum_{k=1}^{N\to\infty} \phi_n(x_1)\phi_k(x_2)\langle a_n(t_1)\,a_k(t_2)\rangle$$

$$= \sum_{n=1}^{N\to\infty} \phi_n(x_1)\phi_n(x_2)\langle a_n(t_1)\,a_n(t_2)\rangle$$

$$\Rightarrow$$

$$\langle y^2(x,t)\rangle = \sum_{n=1}^{N\to\infty} \langle a_n^2(t)\rangle\phi_n^2(x)$$

Beam excited by a space-time random process



$$EI\frac{\partial^4 y}{\partial x^4}+m\ddot{y}+c\dot{y}=f(x,t)$$

$$\langle f(x,t)\rangle=0;\left\langle f(x,t)\,f(x+\xi,t+\tau)\right\rangle=I_0\delta(\xi)R(\tau)$$

$$\text{ICS:}\quad y_0(x)=y(x,0)=0\quad \dot{y}_0(x)=\dot{y}(x,0)=0$$

$$\text{BCS:}\; y(0,t)=0;\,EIy''(0,t)=0;\,y(L,t)=0;\,EIy''(L,t)=0$$

$$y(x,t)=\sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

$$=\sum_{n=1}^{\infty}\phi_n(x)\int_0^t\int_0^L h_n(t-\tau)\phi_n(s)\,f(s,\tau)\,dsd\tau$$

$$\Rightarrow \langle y(x,t)\rangle=\sum_{n=1}^{\infty}\phi_n(x)\int_0^t\int_0^L h_n(t-\tau)\phi_n(s)\langle f(s,\tau)\rangle dsd\tau=0$$

$$\langle y(x_1,t_1)\,y(x_2,t_2)\rangle=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\phi_n(x_1)\phi_k(x_2)\int_0^{t_1}\int_0^{t_2}\int_0^L\int_0^L h_n(t_1-\tau_1)h_k(t_2-\tau_2)$$

$$\phi_n(s_1)\phi_k(s_2)\langle f(s_1,\tau_1)\,f(s_2,\tau_2)\rangle ds_1ds_2d\tau_1d\tau_2$$

$$=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\phi_n(x_1)\phi_k(x_2)\int_0^{t_1}\int_0^{t_2}\int_0^L\int_0^L h_n(t_1-\tau_1)h_k(t_2-\tau_2)$$

$$\phi_n(s_1)\phi_k(s_2)\delta(s_1-s_2)R(\tau_1-\tau_2)ds_1ds_2d\tau_1d\tau_2$$

$$\left\langle y(x_1,t_1)\, y(x_2,t_2)\right\rangle = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\phi_n(x_1)\phi_k(x_2)\int_0^{t_1}\int_0^{t_2}\int_0^{L}\int_0^{L} h_n(t_1-\tau_1)\, h_k(t_2-\tau_2)$$

$$\phi_n(s_1)\phi_k(s_2)\,\delta(s_1-s_2)\, R(\tau_1-\tau_2)\, ds_1 ds_2 d\tau_1 d\tau_2$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\phi_n(x_1)\phi_k(x_2)\int_0^{t_1}\int_0^{t_2}\int_0^{L} h_n(t_1-\tau_1)\, h_k(t_2-\tau_2)\,\phi_n(s_2)\,\phi_k(s_2)\, R(\tau_1-\tau_2)\, ds_2 d\tau_1 d\tau_2$$

Recall

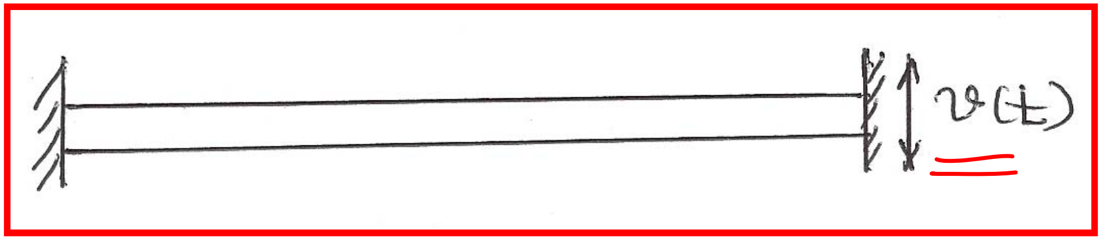
$$\int_0^L m\varphi_n\varphi_k\, dx = \delta_{nk} \Rightarrow //$$

m constant

c constant

$$\left\langle y(x_1,t_1)\, y(x_2,t_2)\right\rangle = \sum_{n=1}^{\infty}\phi_n(x_1)\phi_n(x_2)(1/m)\int_0^{t_1}\int_0^{t_2} h_n(t_1-\tau_1)\, h_n(t_2-\tau_2) R(\tau_1-\tau_2)\, d\tau_1 d\tau_2$$

Beam under random support motions



$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$

$$y(0,t) = 0;\ y'(0,t) = 0$$

$$y(l,t) = v(t);\ y'(l,t) = 0$$

$$y(x,0) = 0;\ \dot{y}(x,0) = 0$$

$$\langle v(t) \rangle = 0$$

$$\langle v(t)v(t+\tau) \rangle = R_{vv}(\tau) \Leftrightarrow S_{vv}(\omega)$$

Introduce a new dependent variable

$$y(x,t) = w(x,t) + h(x)v(t)$$

$$y(0,t) = w(0,t) + h(0)v(t) = 0$$

Select $w(0,t) = 0$; $h(0) = 0$

$$y'(0,t) = w'(0,t) + h'(0)v(t) = 0$$

Select $w'(0,t) = 0$; $h'(0) = 0$

$$y(l,t) = w(l,t) + h(l)v(t) = v(t)$$

Select $w(l,t) = 0$; $h(l) = 1$

$$y'(l,t) = w'(l,t) + h'(l)v(t) = 0$$

Select $w'(l,t) = 0$; $h'(l) = 0$

$$EI\left[w^{iv} + h^{iv} v \right] + m\left[\ddot{w} + h\ddot{v} \right] + c\left[\dot{w} + h\dot{v} \right] = 0$$

Select

$$h^{iv} = 0$$

$$h(x) = ax^3 + bx^2 + cx + d$$

$$h(0) = 0; h(l) = 1; h'(0) = 0; h'(l) = 0;$$

$$h(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$

$$EI\left[w^{iv}+h^{iv}v\right]+m\left[\ddot{w}+h\ddot{v}\right]+c\left[\dot{w}+h\dot{v}\right]=0$$

$$\Rightarrow EIw^{iv}+m\ddot{w}+c\dot{w}=$$

$$-m\left[\ddot{v}\left(\frac{3x^2}{L^2}-\frac{2x^3}{L^3}\right)\right]-c\left[\dot{v}\left(\frac{3x^2}{L^2}-\frac{2x^3}{L^3}\right)\right]=f(x,t)$$

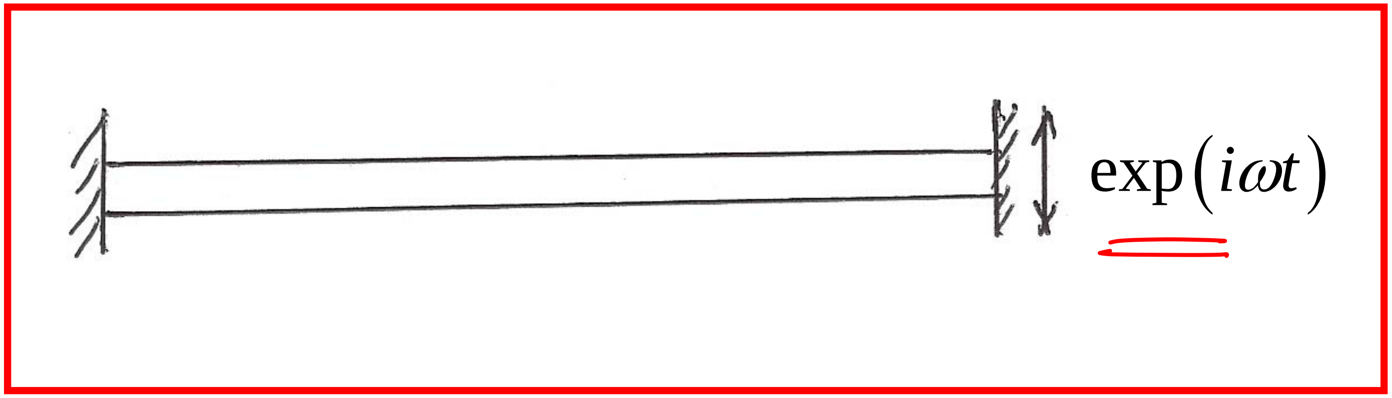
$$w(0,t)=0;w'(0,t)=0$$

$$w(l,t)=0;w'(l,t)=0$$

$$w(x,0)=-h(x)v(0)$$

$$w(x,0)=-h(x)\dot{v}(0)$$

Alternative approach for steady state response analysis



$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$
$$y(0,t) = 0;\, y'(0,t) = 0$$
$$y(l,t) = \exp(i\omega t);\, y'(l,t) = 0$$
$$y(x,0) = 0;\, \dot{y}(x,0) = 0$$

$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$

$$y(x,t) = \phi(x)\exp(i\omega t)$$

$$\Rightarrow$$

$$EI\phi^{iv} - m\omega^2\phi + i\omega c\phi = 0$$

$$\phi(0) = 0; \phi'(0) = 0; \phi(l) = 1; \phi'(l) = 0$$

$$\Rightarrow$$

$$\phi^{iv} - \lambda^4\phi = 0; \lambda^4 = \frac{m\omega^2 - i\omega c}{EI}$$

$$\phi(x) = a(\cos\lambda x + \cosh\lambda x) + b(\cos\lambda x - \cosh\lambda x)$$
$$+c(\sin\lambda x + \sinh\lambda x) + d(\sin\lambda x - \sinh\lambda x)$$

$$\phi'(x) = a\lambda(-\sin\lambda x + \sinh\lambda x) + b\lambda(-\sin\lambda x - \sinh\lambda x)$$
$$+c\lambda(\cos\lambda x + \cosh\lambda x) + d\lambda(\cos\lambda x - \cosh\lambda x)$$

$$y(l,t) = \phi(l)e^{i\omega t} = e^{i\omega t}$$
$$\phi(l) = 1$$

$$\phi(0)=0; \phi'(0)=0; \phi(l)=1; \phi'(l)=0$$

$$\phi(x)=a(\cos\lambda x+\cosh\lambda x)+b(\cos\lambda x-\cosh\lambda x)$$

$$+c(\sin\lambda x+\sinh\lambda x)+d(\sin\lambda x-\sinh\lambda x)$$

$$\phi'(x)=a\lambda(-\sin\lambda x+\sinh\lambda x)+b\lambda(-\sin\lambda x-\sinh\lambda x)$$

$$+c\lambda(\cos\lambda x+\cosh\lambda x)+d\lambda(\cos\lambda x-\cosh\lambda x)$$

$$\phi(0)=0 \Rightarrow a=0$$

$$\phi'(0)=0 \Rightarrow c=0$$

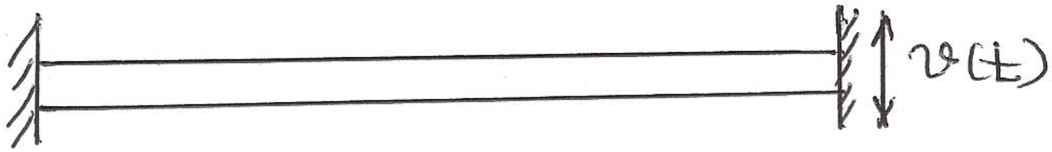
$$\phi(l)=1 \Rightarrow b(\cos\lambda l-\cosh\lambda l)+d(\sin\lambda l-\sinh\lambda l)$$

$$\phi'(l)=0 \Rightarrow b\lambda(-\sin\lambda l-\sinh\lambda l)+d\lambda(\cos\lambda l-\cosh\lambda l)=0$$

$b \,\&\, d$ can thus be determined.

$$\Rightarrow y(x,t)=\left[b(\cos\lambda x-\cosh\lambda x)+d(\sin\lambda x-\sinh\lambda x)\right]\exp(i\omega t)$$

Random support motions



PSD function of $y(x,t)$

$$S_{YY}(x,\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle Y_T(x,\omega)Y_T^*(x,\omega)\right\rangle$$

$$Y_T(x,\omega)=\phi(x,\omega)V_T(\omega)$$

$$\Rightarrow$$

$$S_{YY}(x,\omega)=\left|\phi(x,\omega)\right|^2 S_{VV}(\omega)$$

Merits of studying continuous systems

- **Means of first cut models for tall buildings, soil layers and line like structures such as chimneys and towers.**
- **For certain problems, continuous models may simplify the problem: for example, continuous models for lattice structures such as towers.**
- **If loads are rapidly fluctuating or when high frequency vibration is of interest it may be preferable to use continuous models.**
- **Exact solutions are possible for a class of problems. This is of educational value and also helps in assessing approximate methods of analysis.**

Limitations

- For each structural type, we need to develop a separate theory (axially vibrating bar, beam, arch, plate, shell…).
- Built-up structures (like building structures or water tanks) are difficult (if not impossible) to study using continuous system models.