Stochastic Structural Dynamics

Lecture-24

Markov Vector Approach-4

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Solution of FPK equations

$$M\ddot{X} + C\dot{X} + KX = \Gamma W(t); t \geq 0; X(0) = X_0; \dot{X}(0) = \dot{X}_0$$

$$X(t) \sim N \times 1;\ \Gamma \sim n \times m;\ W(t) \sim m \times 1$$

$$\langle W(t) \rangle = 0; \langle W(t) W^t(t+\tau) \rangle = \left[2D_{ij}\right]_{m\times m} \delta(\tau)$$

Transient & Steady state solutions

$$\dot{x} + \beta(x) = w(t); t \geq 0 \,\&\, x(0) = x_0$$

$$\langle w(t) \rangle = 0; \langle w(t_1) w(t_2) \rangle = 2D\delta(t_1 - t_2)$$

$$\ddot{x} + \dot{x}f(H) + g(x) = w(t); t \geq 0; x(0) = x_0; \dot{x}(0) = \dot{x}_0.$$

$$\langle w(t) \rangle = 0; \langle w(t) w(t+\tau) \rangle = 2D\delta(\tau)$$

Steady state solutions

$$m_j \ddot{X}_j + m_j \beta_j \dot{X}_j + \frac{\partial U}{\partial X_j} = W_j(t); t \geq 0; X_j(0) = x_{j0} \,\&\, \dot{X}_j(0) = \dot{x}_{j0}$$

$$U = \frac{1}{2} X^t K X; \langle W_j(t) \rangle = 0; \langle W_j(t) W_k(t+\tau) \rangle = 2D_{jk}\delta_{jk}\delta(\tau);$$

$$\frac{m_j \beta_j}{2D_{jj}} = \gamma \forall j = 1, 2, \cdots, n$$

Moment equations

$$dX(t) = f\left[t, X(t)\right]dt + G\left[t, X(t)\right]dB(t); t \geq 0; X(0) = X_0$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[f_j(x,t)\, p\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^t\right)_{ij} p\right]$$

$$\frac{d}{dt}\left\langle h\left[X(t),t\right]\right\rangle = \left\langle \frac{\partial h}{\partial t}\right\rangle + \sum_{j=1}^{n}\left\langle f_j(X,t)\frac{\partial h}{\partial X_j}\right\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle \left(GDG^t\right)_{ij} \frac{\partial^2 h}{\partial X_i \partial X_j}\right\rangle$$

$$\dot{X} + \beta X = w(t); t \geq 0; X(0) = X_0$$

$$\ddot{X} + 2\eta\omega\dot{X} + \omega^2 X = w(t); t \geq 0; X(0) \,\&\, \dot{X}(0) \text{ specified}$$

Example

$$\dot{X} + \beta X + \alpha X^3 = w(t); t \geq 0; X(0) = X_0$$

$$dX(t) = -\beta X dt - \alpha X^3 dt + dB(t)$$

$m_k = \langle X^k(t) \rangle$

$$f = -\beta X - \alpha X^3$$

$$\frac{d}{dt}\langle h[X(t),t]\rangle = \left\langle \frac{\partial h}{\partial t} \right\rangle + \left\langle -\left(\beta X + \alpha X^3\right) \frac{\partial h}{\partial X_j} \right\rangle + D \left\langle \frac{\partial^2 h}{\partial X^2} \right\rangle$$

$$\dot{m}_1 = \left\langle \left[-\beta X - \alpha X^3\right] \right\rangle = -\beta m_1 - \alpha m_3$$

$$\dot{m}_2 = \left\langle \left[-\beta X - \alpha X^3\right] 2X \right\rangle + 2D = -2\beta m_2 - 2\alpha m_4 + 2D$$

$$\dot{m}_3 = \left\langle \left[-\beta X - \alpha X^3\right] 3X^2 \right\rangle + 6Dm_1 = -3\beta m_3 - 3\alpha m_5 + 6Dm_1$$

$$\dot{m}_4 = \left\langle \left[-\beta X - \alpha X^3\right] 4X^3 \right\rangle + 12Dm_2 = -4\beta m_4 - 4\alpha m_6 + 12Dm_2$$

$\vdots$

Remarks

- Moment equations form an infinite hierarchy of equations which at no stage provide sufficient number of equations to solve for the moments: CLOSURE PROBLEM
- This is the characteristic feature of moment equations of non linear systems driven by random excitations
- Closure approximations
    - Assume that the higher order moments, beyond a given order, are related to the lower order ones as though they obey an adhocly specified pdf (Ex: Gaussian closure)
    - Neglect cumulants beyond a specified order

Note : This problem occurs for non - white excitation also

$m\ddot{x}+c\dot{x}+kx+\alpha x^3 = f(t); t\geq 0; x(0)\,\&\,\dot{x}(0)$ specified

Taking expectations on both sides

$$m\langle\ddot{x}(t)\rangle+c\langle\dot{x}(t)\rangle+k\langle x(t)\rangle+\alpha\langle x^3(t)\rangle=\langle f(t)\rangle$$

Equation for mean response contains 3rd order expectations.

$$m\langle x(t_1)\ddot{x}(t)\rangle+c\langle x(t_1)\dot{x}(t)\rangle+k\langle x(t_1)x(t)\rangle+\alpha\langle x(t_1)x^3(t)\rangle=\langle x(t_1)f(t)\rangle$$

$$m\frac{\partial^2 R_{xx}(t,t_1)}{\partial t^2}+c\frac{\partial R_{xx}(t,t_1)}{\partial t}+kR_{xx}(t,t_1)+\alpha\langle x(t_1)x^3(t)\rangle=R_{xf}(t_1,t)$$

Equation for ACF of $x(t)$ contains $\alpha\langle x(t_1)x^3(t)\rangle$

⋮

Closure problem

Example

$$\ddot{X} + a_2 \dot{X} + \left[a_1 + w_1(t) \right] X = w_2(t);$$

$t \geq 0; X(0)$ & $\dot{X}(0)$ specified.

$$dX_1 = X_2 dt$$

$$dX_2 = -a_2 X_2 dt - a_1 X_1 dt - X_1 dB_1(t) + dB_2(t)$$

$$\begin{Bmatrix} dX_1 \\ dX_2 \end{Bmatrix} = \begin{Bmatrix} X_2 \\ -a_2 X_2 - a_1 X_1 \end{Bmatrix} dt + \begin{bmatrix} 0 & 0 \\ -X_1 & 1 \end{bmatrix} \begin{Bmatrix} dB_1(t) \\ dB_2(t) \end{Bmatrix}$$

$$\left\langle dB_i(t) dB_j(t+\tau) \right\rangle = 2 D_{ij} \delta_{ij} \delta(\tau); i, j = 1, 2$$

$$GDG^t = \begin{bmatrix} 0 & 0 \\ -X_1 & 1 \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -X_1 & 1 \end{bmatrix}^t = \begin{bmatrix} 0 & 0 \\ 0 & D_{11} X_1^2 + D_{22} \end{bmatrix}$$

$$\begin{Bmatrix} dX_1 \\ dX_2 \end{Bmatrix} = \begin{Bmatrix} X_2 \\ -a_2 X_2 - a_1 X_1 \end{Bmatrix} dt + \begin{bmatrix} 0 & 0 \\ -X_1 & 1 \end{bmatrix} \begin{Bmatrix} dB_1(t) \\ dB_2(t) \end{Bmatrix}$$

$$M_{rs} = \langle x^r(t)\, \dot{x}^s(t) \rangle$$

$$\dot{m}_{10} = \langle X_2 \rangle = m_{01}$$

$$\dot{m}_{01} = \langle [-a_2 X_2 - a_1 X_1] \rangle = -a_2 m_{01} - a_1 m_{10}$$

$$\dot{m}_{20} = \langle X_2 2 X_1 \rangle = 2 m_{11}$$

$$\dot{m}_{11} = \langle X_2 X_2 + [-a_2 X_2 - a_1 X_1] X_1 \rangle = m_{02} - a_2 m_{11} - a_1 m_{20}$$

$$\dot{m}_{20} = \langle [-a_2 X_2 - a_1 X_1] 2 X_2 \rangle + \langle X_1^2 D_{11} + D_{22} \rangle$$

02 ✓

$$= -2a_2 m_{02} - 2a_1 m_{11} + D_{11} m_{20} + D_{22}$$

I order moments

$$\dot{m}_{10} = \langle X_2 \rangle = m_{01}$$

$$\dot{m}_{01} = \langle [-a_2 X_2 - a_1 X_1] \rangle = -a_2 m_{01} - a_1 m_{10}$$

II order moments

$$\dot{m}_{20} = \langle X_2 2 X_1 \rangle = 2 m_{11}$$

$$\dot{m}_{11} = \langle X_2 X_2 + [-a_2 X_2 - a_1 X_1] X_1 \rangle = m_{02} - a_2 m_{11} - a_1 m_{20}$$

$$\dot{m}_{20} = \langle [-a_2 X_2 - a_1 X_1] 2 X_2 \rangle + \langle X_1^2 D_{11} + D_{22} \rangle$$

$$= -2 a_2 m_{02} - 2 a_1 m_{11} + D_{11} m_{20} + D_{22}$$

Moment equations are closed & hence can be evaluated exactly (numerically at least)

Steady state

$$\dot{m}_{10} = 0 \Rightarrow m_{01} = 0$$

$$\dot{m}_{01} = 0 \Rightarrow -a_2 m_{01} - a_1 m_{10} = 0$$

$$m_{01} = 0 \,\&\, m_{10} = 0$$

$$\dot{m}_{20} = 0 \Rightarrow 2m_{11} = 0 \Rightarrow m_{11} = 0$$

$$\dot{m}_{11} = 0 \Rightarrow m_{02} - a_2 m_{11} - a_1 m_{20} = 0 \Rightarrow m_{02} - a_1 m_{20} = 0$$

$$\dot{m}_{20} = 0 \Rightarrow -2a_2 m_{02} - 2a_1 m_{11} + D_{11} m_{20} + D_{22} = 0$$

$$-2a_2 m_{02} + D_{11} m_{20} + D_{22} = 0$$

$$\vdots$$

Question: Are the steady state moments realizable?

That is. are these moments stable?

Strategy : perturb the solutions and see if perturbations die out.

The moment equations can also be used to study stability of the system in terms of response moments.

$$\dot{Y} = AY + B$$

$$\tilde{Y} = Y + \nu$$

$$\nu(t) = \text{small perturbation}$$

$$\Rightarrow$$

$$\dot{Y} + \dot{\nu} = A(Y + \nu) + B$$

$$\Rightarrow \dot{\nu} = A\nu \Rightarrow \nu(t) = \nu_0 \exp(st) \Rightarrow A\nu_0 = sI\nu_0$$

$$|A - sI| = 0$$

Perturbations do not grow in time if the real part of the eigenvalues of A are all ≤ 0.

Equations for two time moments

Consider $n=2$

$$dX(t)=f\left[t,X(t)\right]dt+G\left[t,X(t)\right]dB(t);t\geq 0;X(0)=X_0$$

$$\frac{\partial p}{\partial t}=-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[f_j(x,t)p\right]+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\left[\left(GDG^t\right)_{ij}p\right]$$

$$p=p\left(\tilde{x};t\,|\,\tilde{x}_0;t_0\right)=p\left(x_1,x_2;t\,|\,x_{01},x_{02};t_0\right)$$

Consider

$$\left\langle X_1^m(t)X_1^n(t_0)\right\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^m\eta_1^n p\left(x_1,\eta_1;t,t_0\right)dx_1d\eta_1$$

$$p\left(x_1,\eta_1;t,t_0\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p\left(x_1,x_2,\eta_1,\eta_2;t,t,t_0t_0\right)dx_1dx_2d\eta_1d\eta_2$$

$$\left\langle X_1^m(t)\,X_1^n(t_0)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^m \eta_1^n\, p\left(x_1,\eta_1;t,t_0\right) dx_1 d\eta_1$$

$$p\left(x_1,\eta_1;t,t_0\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\left(x_1,x_2,\eta_1,\eta_2;t,t,t_0 t_0\right) dx_2 d\eta_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\left(x_1,x_2;t,t_0 \mid \eta_1,\eta_2;t,t_0\right) p\left(\eta_1,\eta_2;t,t_0\right) dx_2 d\eta_2$$

$$\left\langle X_1^m(t)\,X_1^n(t_0)\right\rangle =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^m \eta_1^n\, p\left(x_1,x_2;t,t_0 \mid \eta_1,\eta_2;t,t_0\right) p\left(\eta_1,\eta_2;t,t_0\right) dx_1 d\eta_1 dx_2 d\eta_2$$

$$v\left(x_1,x_2;t,t_0\right) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_1^n\, p\left(x_1,x_2;t,t_0 \mid \eta_1,\eta_2;t,t_0\right) p\left(\eta_1,\eta_2;t,t_0\right) d\eta_1 d\eta_2$$

$$\left\langle X_1^m(t)\, X_1^m(t_0)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^m v(x_1, x_2; t, t_0)\, dx_1 dx_2$$

$$v(x_1, x_2; t, t_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_1^n\, p(x_1, x_2; t, t_0 \mid \eta_1, \eta_2; t, t_0)\, p(\eta_1, \eta_2; t, t_0)\, d\eta_1 d\eta_2$$

Consider the FPK equation

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[f_j(x,t)\, p\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^t\right)_{ij} p\right]$$

It follows that

$$\frac{\partial v}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[f_j(x,t)\, v\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^t\right)_{ij} v\right]$$

$$\lim_{t\to t_0} v(x_1, x_2; t, t_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_1^n\, \delta(x_1 - \eta_1)\, \delta(x_2 - \eta_2)\, p(\eta_1, \eta_2; t_0, t_0)\, d\eta_1 d\eta_2$$

$$= x_1^n\, p(y_1, y_2; t_0, t_0)$$

Backward Kolmogorov equation and reliability function

References :

- Y K Lin and G C Cai, 1995, Probabilistic structural dynamics, McGraw Hill, NY]
- D R Cox and H D Miller, 1965, The theory of stochastic processes, Chapman and Hall, NY.

Backward Kolmogorov equation and reliability function

Let $X(t)$ be a scalar Markov process. For $t_1 < t_2 < t_3$ we have

$$p(x_3;t_3 \mid x_1;t_1) = \int_{-\infty}^{\infty} p(x_3;t_3 \mid x_2;t_2)\, p(x_2;t_2 \mid x_1;t_1)\, dx_2$$

$$p(x_3;t_3 \mid x_2;t_2) = p(x_3;t_3 \mid x_1 + x_2 - x_1;t_2)$$

$$= p(x_3;t_3 \mid x_1;t_2) + (x_2 - x_1)\left.\frac{\partial p}{\partial x_2}\right|_{x_2=x_1}$$

$$+\frac{1}{2}(x_2 - x_1)^2 \left.\frac{\partial^2 p}{\partial x_2^2}\right|_{x_2=x_1} + o\left[(x_2 - x_1)^2\right]$$

$$p\left(x_3;t_3\mid x_1;t_1\right)=\int_{-\infty}^{\infty} p\left(x_3;t_3\mid x_2;t_2\right)p\left(x_2;t_2\mid x_1;t_1\right)dx_2$$

$$=p\left(x_3;t_3\mid x_1;t_2\right)+\left.\frac{\partial p}{\partial x_2}\right|_{x_2=x_1}\int_{-\infty}^{\infty}\left(x_2-x_1\right)p\left(x_2;t_2\mid x_1;t_1\right)dx_2$$

$$+\frac{1}{2}\left.\frac{\partial^2 p}{\partial x_2^2}\right|_{x_2=x_1}\int_{-\infty}^{\infty}\left(x_2-x_1\right)^2 p\left(x_2;t_2\mid x_1;t_1\right)dx_2+o\left[\left(x_2-x_1\right)^2\right]$$

$$\Rightarrow\frac{1}{\Delta t}\left[p\left(x_3;t_3\mid x_1;t_2\right)-p\left(x_3;t_3\mid x_1;t_1\right)\right]$$

$$+\frac{1}{\Delta t}\left.\frac{\partial p}{\partial x_2}\right|_{x_2=x_1}\int_{-\infty}^{\infty}\left(x_2-x_1\right)p\left(x_2;t_2\mid x_1;t_1\right)dx_2$$

$$\frac{1}{\Delta t}\left.\frac{\partial^2 p}{\partial x_2^2}\right|_{x_2=x_1}\int_{-\infty}^{\infty}\left(x_2-x_1\right)^2 p\left(x_2;t_2\mid x_1;t_1\right)dx_2+\frac{o\left[\left(x_2-x_1\right)^2\right]}{\Delta t}=0$$

Consider the lim $t_2 \to t_1 \Rightarrow$

$$\frac{\partial p}{\partial t_1} + \alpha_1(x_1, t_1)\frac{\partial p}{\partial x_1} + \frac{1}{2}\alpha_2(x_1, t_1)\frac{\partial^2 p}{\partial x_1^2} = 0$$

$$p \equiv p(x_3, t_3 \mid x_1, t_1)$$

In the standard form the above equation written as

$$\frac{\partial p}{\partial t_0} = -\alpha_1(x_0, t_0)\frac{\partial p}{\partial x_0} - \frac{1}{2}\alpha_2(x_0, t_0)\frac{\partial^2 p}{\partial x_0^2}; p \equiv p(x, t \mid x_0, t_0)$$

is known as the backward Kolmogorov equation.

(Backward $\because$ the independent variable $t_0 < t$)

ICS: $p(x, t_0 \mid x_0, t_0) = \delta(x - x_0)$

[Typical] BCS: $\lim\limits_{x_0 \to \pm\infty} p(x, t \mid x_0, t_0) \to 0$

Vector version of Backward Kolmogorov equation

$$dX(t) = f\left[t, X(t)\right]dt + G\left[t, X(t)\right]dB(t); t \geq 0; X(0) = X_0$$

$$\frac{\partial p}{\partial t_0} = -\sum_{j=1}^{n} f_j\left[t_0, \tilde{x}_0\right]\frac{\partial p}{\partial x_{0j}} - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0, \tilde{x}_0\right)\right]_{ij}\frac{\partial^2 p}{\partial x_{0i}\partial x_{0j}}$$

$$p \equiv p\left(\tilde{x}; t \mid \tilde{x}_0; t_0\right)$$

$$\text{ICS}: p\left(\tilde{x}; t_0 \mid \tilde{x}_0; t_0\right) = \prod_{i=1}^{n}\delta\left(x_i - x_{i0}\right)$$

$$\text{BCS}: \forall j = 1, 2, \cdots, n, \lim_{x_j \to \pm\infty} p\left(\tilde{x}; t \mid \tilde{x}_0; t_0\right) \to 0$$

Remarks

- If we are interested in pdf of $X(t)$ for a given initial condition we use forward equation.
- If we are interested in the pdf of time for first passage across a threshold, we use backward equation.

First passage times

Consider the scalar SDE

$dX(t) = f(X)dt + G(X)dB(t)$ with $X(t_0) = x_0 < x_c$

T = time at which $X(t)$ reaches the critical value x_c for the first time given that $X(t_0) = x_0 < x_c$.

Safe region: $X(t) \leq x_c$

Unsafe region: $X(t) > x_c$

$R(t, x_c; x_0; t_0)$ = Probability that $X(t)$ stays in the safe region during the interval t_0 to t.

$$R(t, x_c; x_0; t_0) = \int_{x_l}^{x_c} p(x; t \mid x_0; t_0)\, dx = P\left[T > t - t_0 \mid X(t_0) = x_0\right]$$

$$R(t, x_c; x_0; t_0) = \int_{x_l}^{x_c} p(x; t \mid x_0; t_0)\, dx = P\left[T > t - t_0 \mid X(t_0) = x_0\right]$$

Remark

The samples of $X(t)$ are such that the first time they reach the level x_c they get absorbed. That is, $X(t) = x_c$ is an absorbing barrier.

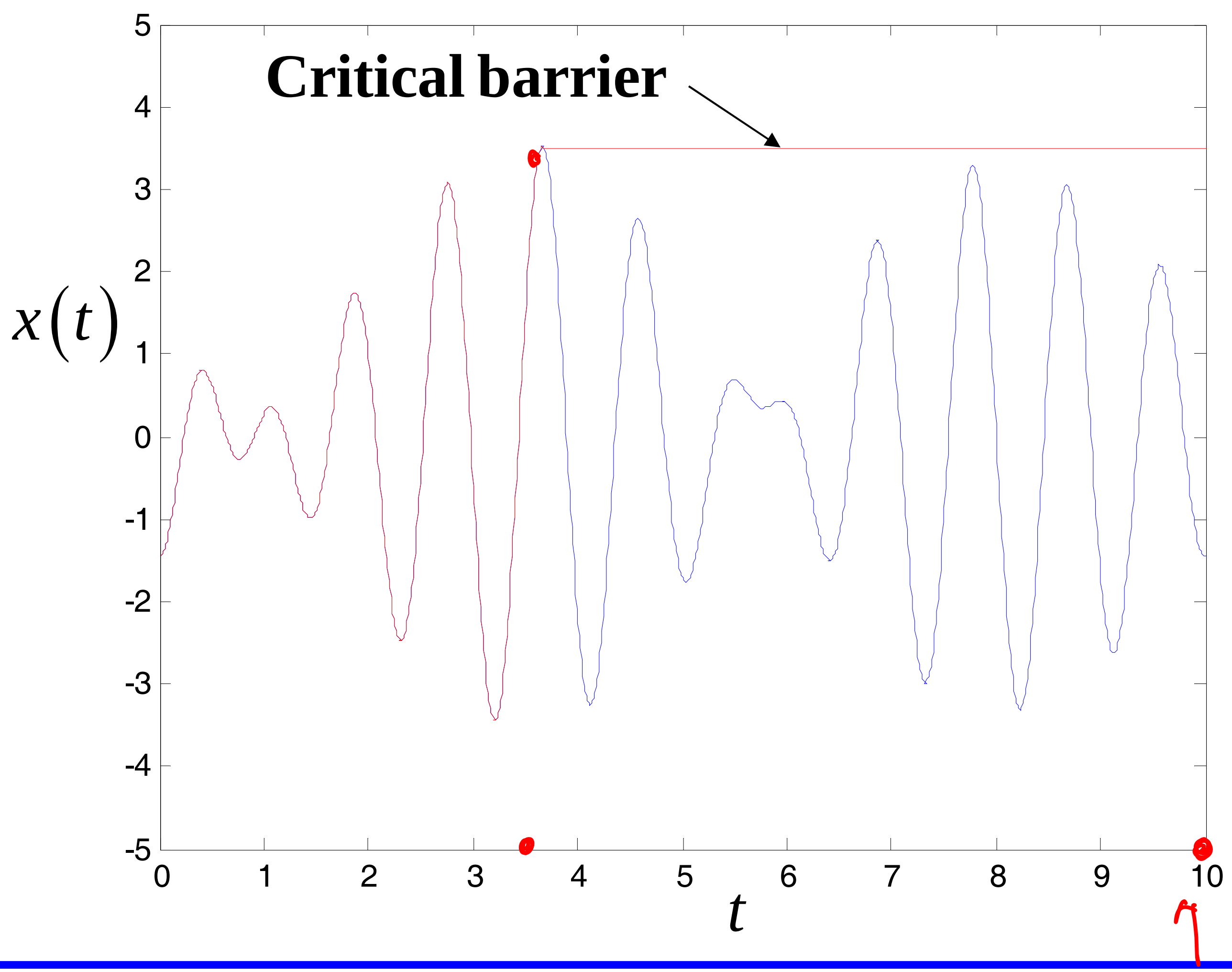
BK-equation

$$\frac{\partial p}{\partial t_0} + f(x_0)\frac{\partial p}{\partial x_0} + G(x_0) D \frac{\partial^2 p}{\partial x_0^2} = 0; \; p \equiv p(x; t \mid x_0; t_0)$$

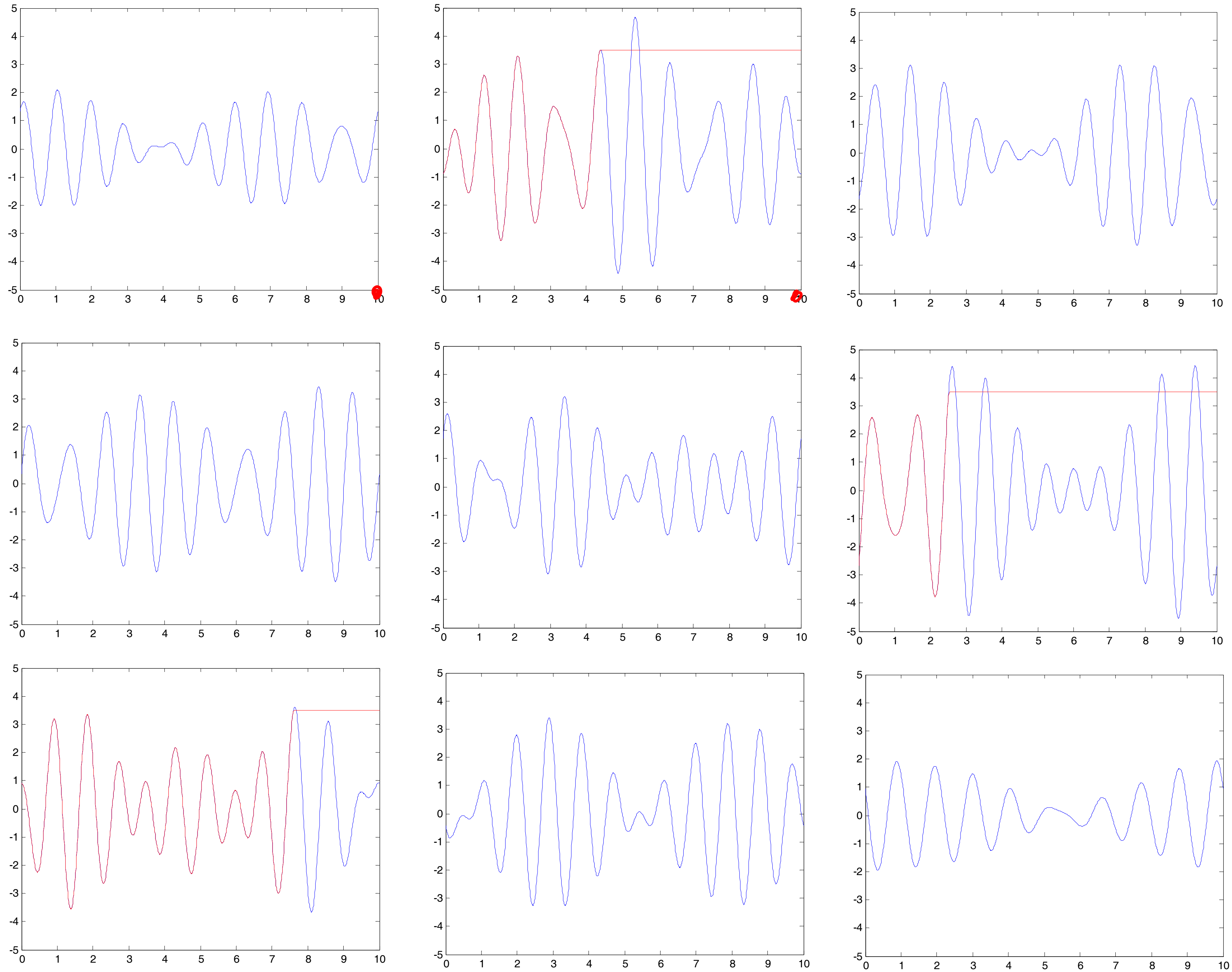
ICS: $p(x; t_0 \mid x_0; t_0) = \delta(x - x_0)$

BCS: $p(x_c; t \mid x_0; t_0) = 0$ [Absorbing boundary]

We consider only those trajectories which have not yet reached the critical barrier.



Critical barrier
x(t)
t
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
1
2
3
4
5
6
7
8
9
10


5 4 3 2 1 0 -1 -2 -3 -4 -5
0 1 2 3 4 5 6 7 8 9 10

$$R\left(t, x_c; x_0; t_0\right) = \int_{x_l}^{x_c} p\left(x; t \mid x_0; t_0\right) dx = P\left[T > t - t_0 \mid X\left(t_0\right) = x_0\right]$$

$$\frac{\partial p}{\partial t_0} + f\left(x_0\right)\frac{\partial p}{\partial x_0} + G\left(x_0\right) D \frac{\partial^2 p}{\partial x_0^2} = 0 \Rightarrow$$

$$\frac{\partial R}{\partial t_0} + f\left(x_0\right)\frac{\partial R}{\partial x_0} + G\left(x_0\right) D \frac{\partial^2 R}{\partial x_0^2} = 0$$

BCS

$$R\left(t_0, x_c; x_0; t_0\right) = 1$$

• [At $t = t_0$, no trajectory has reached the critical barrier]

$$R\left(t, x_c; x_c; t_0\right) = 0$$

• [If the trajectory originates from x_c the probability of failure=1]

$$0 \le R\left(t, x; x_0; t_0\right) \le 1$$

• [Because R is a probability]

Summary

$$R\left(t,x_c;x_0;t_0\right)=P\left[T>t-t_0 \mid X\left(t_0\right)=x_0\right]$$

$$\frac{\partial R}{\partial t_0}+f\left(x_0\right)\frac{\partial R}{\partial x_0}+DG\left(x_0\right)\frac{\partial^2 R}{\partial x_0^2}=0$$

$$R\left(t_0,x_c;x_0;t_0\right)=1$$

$$R\left(t,x_c;x_c;t_0\right)=0$$

$$0\le R\left(t,x;x_0;t_0\right)\le 1$$

Assume: $x_0=z_l$ is reflective

$$\left[\frac{\partial p\left(x;t \mid x_0;t_0\right)}{\partial x_0}\right]_{x_0=x_l}=0\Rightarrow\left(\frac{\partial R}{\partial x_0}\right)_{x_0=x_l}=0$$

Let $\tau = t - t_0$

$$R(t, x_c; x_0; t_0) \equiv R(\tau, x_c; x_0) = P\left[T > \tau \mid X(t_0) = x_0\right]$$

$$-\frac{\partial R}{\partial \tau} + f(x_0)\frac{\partial R}{\partial x_0} + DG(x_0)\frac{\partial^2 R}{\partial x_0^2} = 0$$

$$R(\tau, x_c; x_0)\big|_{\tau=0} = 1$$

$$R(\tau, x_c; x_c) = 0$$

$$0 \le R(\tau, x; x_0) \le 1$$

$$\left(\frac{\partial R}{\partial x_0}\right)_{x_0 = x_l} = 0$$

Moments of first passage time

$$P\left[T \le \tau \mid X\left(t_0\right) = x_0\right] = 1 - P\left[T > \tau \mid X\left(t_0\right) = x_0\right]$$

$$P_T\left(\tau, x_c, x_0\right) = 1 - R\left(\tau, x_c; x_0\right)$$

$$p_T\left(\tau, x_c, x_0\right) = \frac{\partial}{\partial \tau} P_T\left(\tau, x_c, x_0\right) = -\frac{\partial}{\partial \tau} R\left(\tau, x_c; x_0\right)$$

We have

$$-\frac{\partial R}{\partial \tau} + f\left(x_0\right)\frac{\partial R}{\partial x_0} + DG\left(x_0\right)\frac{\partial^2 R}{\partial x_0^2} = 0$$

$$\Rightarrow p_T\left(\tau, x_c, x_0\right) = -f\left(x_0\right)\frac{\partial R}{\partial x_0} - DG\left(x_0\right)\frac{\partial^2 R}{\partial x_0^2}$$

$$\left\langle T^n \right\rangle = \int_0^\infty \tau^n p_T\left(\tau, x_c, x_0\right) d\tau = M_n\left(x_c, x_0\right)$$

$$M_n = \langle T^n \rangle = \int_0^\infty \tau^n p_T(\tau, x_c, x_0)\, d\tau = \int_0^\infty \tau^n \left(-\frac{\partial}{\partial \tau} R(\tau, x_c; x_0) \right) d\tau$$

$$= \left[\tau^n R(\tau, x_c; x_0) \right]_0^\infty + n \int_0^\infty \tau^{n-1} R(\tau, x_c; x_0)\, d\tau$$

$$= n \int_0^\infty \tau^{n-1} R(\tau, x_c; x_0)\, d\tau$$

We have

$$-\frac{\partial R}{\partial \tau} + f(x_0) \frac{\partial R}{\partial x_0} + DG(x_0) \frac{\partial^2 R}{\partial x_0^2} = 0$$

Multiply both sides of this equation by τ^n and

integrate over 0 to $\infty \Rightarrow$

$$\int_0^\infty \tau^n \left(-\frac{\partial R}{\partial \tau} \right) d\tau + \int_0^\infty \tau^n f\left(x_0\right) \left(\frac{\partial R}{\partial x_0} \right) d\tau + D \int_0^\infty \tau^n G\left(x_0\right) \left(\frac{\partial^2 R}{\partial x_0^2} \right) d\tau = 0$$

$$M_n + \frac{1}{n+1} f\left(x_0\right) \frac{d}{dx_0} M_{n+1} + D \frac{G\left(x_0\right)}{n+1} \frac{d^2}{dx_0^2} M_{n+1} = 0$$

$$\left(n+1\right) M_n + f\left(x_0\right) \frac{d}{dx_0} M_{n+1} + DG\left(x_0\right) \frac{d^2}{dx_0^2} M_{n+1} = 0$$

$$M_n\left(x_c, x_0\right)|_{x_0 = x_c} = 0$$

$$M_n\left(x_c, x_0\right)|_{x_0 = x_l} < \infty$$

$$n = 1, 2, \ldots$$

$$(n+1)M_n + f(x_0)\frac{d}{dx_0}M_{n+1} + DG(x_0)\frac{d^2}{dx_0^2}M_{n+1} = 0$$

$$n = 0 \Rightarrow M_0 = 1$$

$$1 + f(x_0)\frac{dM_1}{dx_0} + DG(x_0)\frac{d^2M_1}{dx_0^2} = 0$$

$$M_1(x_c, x_0)|_{x_0 = x_c} = 0$$

$$M_1(x_c, x_0)|_{x_0 = x_l} < \infty$$

$$n = 1$$

$$f(x_0)\frac{dM_2}{dx_0} + DG(x_0)\frac{d^2M_2}{dx_0^2} = -2M_1$$

$$M_2(x_c, x_0)|_{x_0 = x_c} = 0$$

$$M_2(x_c, x_0)|_{x_0 = x_l} < \infty$$

Vi+ GPV eqns

Generalized Pontryagin Equations

$$f\left(x_0\right)\frac{d}{dx_0}M_{n+1}+DG\left(x_0\right)\frac{d^2}{dx_0^2}M_{n+1}=-\left(n+1\right)M_n$$

$$M_{n+1}\left(x_c,x_0\right)|_{x_0=x_c}=0$$

$$M_{n+1}\left(x_c,x_0\right)|_{x_0=x_l}<\infty$$

$$n=1,2,\cdots$$

Generalization to multi - dimensional problems

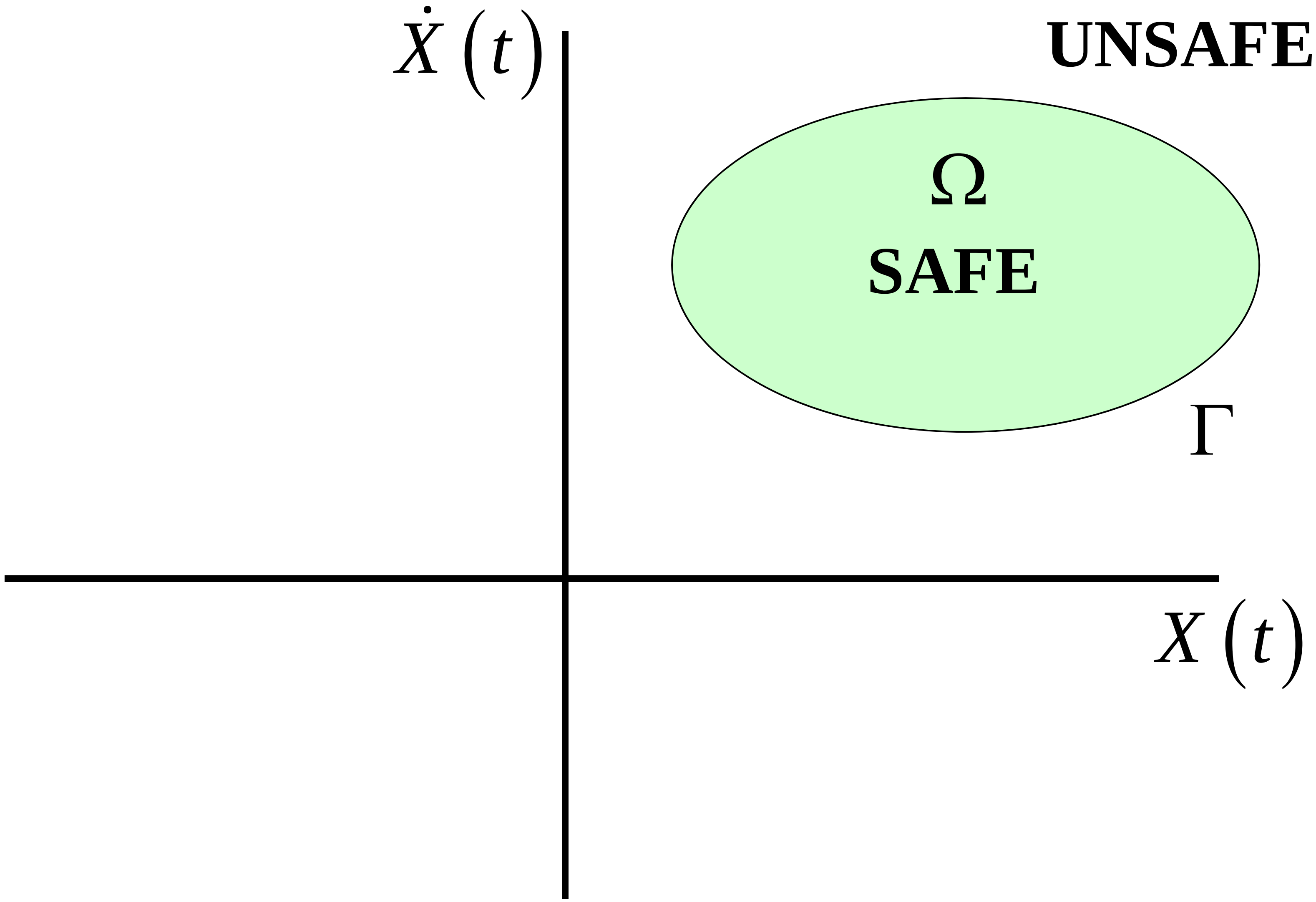
$$dX(t) = f\left[t, X(t)\right]dt + G\left[t, X(t)\right]dB(t); t \geq 0; X(0) = X_0$$

$$\frac{\partial p}{\partial t_0} = -\sum_{j=1}^{n} f_j\left[t_0, \tilde{x}_0\right]\frac{\partial p}{\partial x_{0j}} - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0, \tilde{x}_0\right)\right]_{ij}\frac{\partial^2 p}{\partial x_{0i}\partial x_{0j}}$$

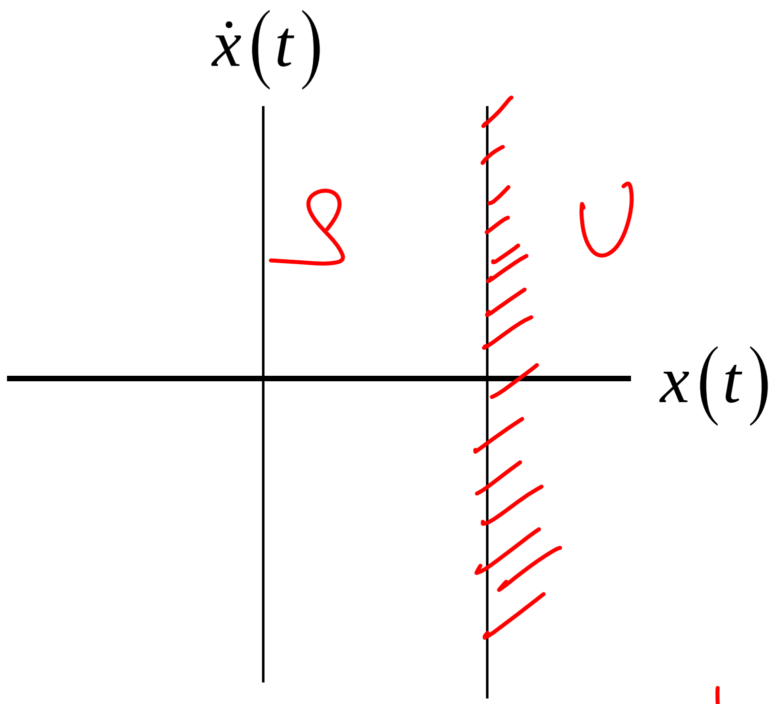
$$p \equiv p\left(\tilde{x}; t \mid \tilde{x}_0; t_0\right)$$

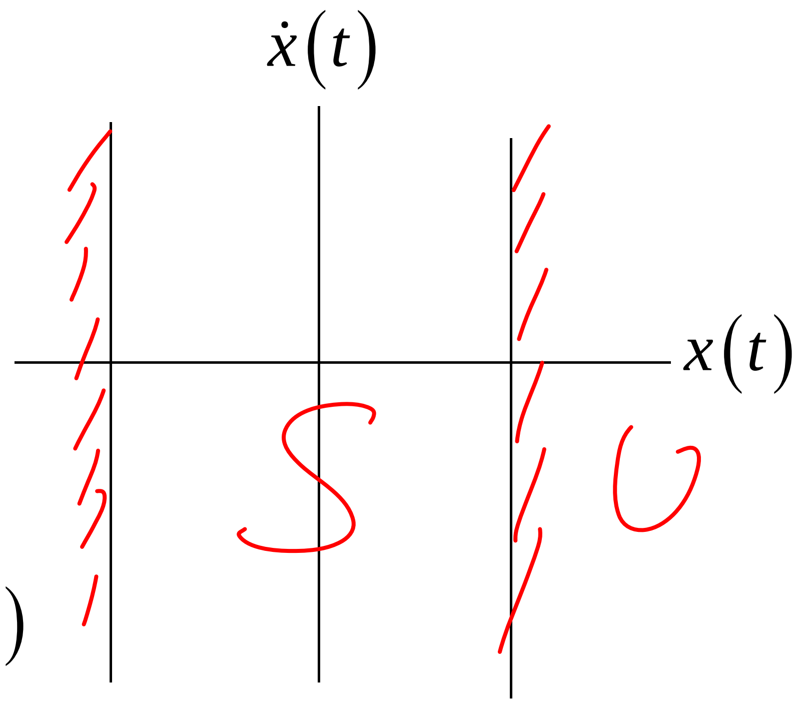
Safe region : $X(t) \in \Omega$; Γ = Limits of safe region

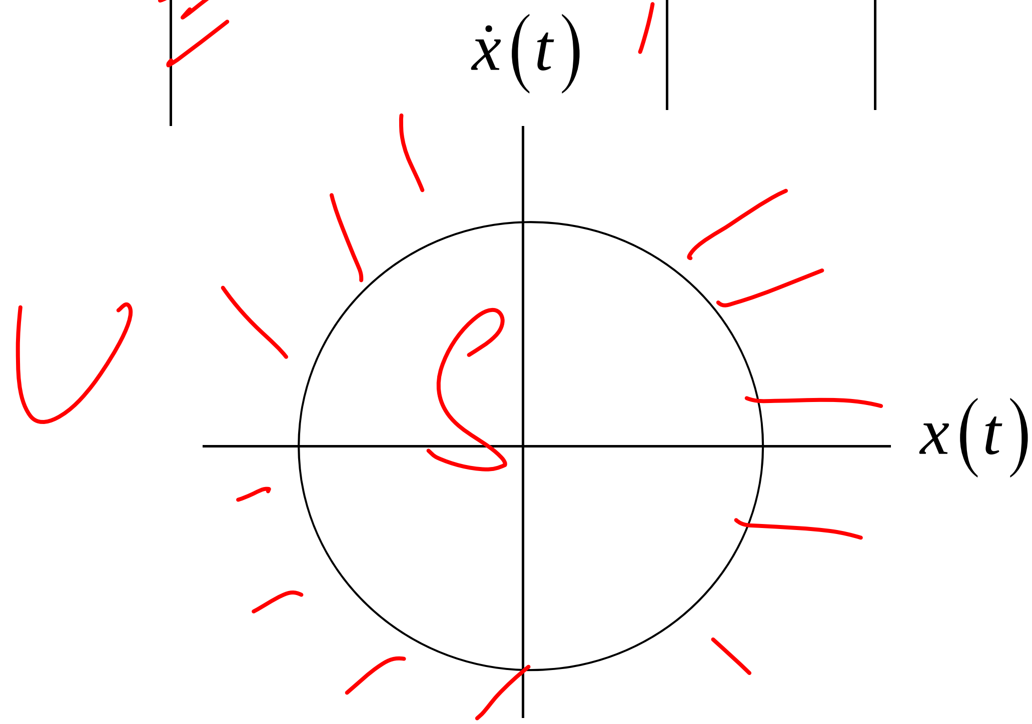
$$R\left(t, \Gamma; t_0, \Omega\right) = \int_{\Omega} p\left(\tilde{x}; t \mid \tilde{x}_0; t_0\right)d\tilde{x} = P\left[T > t - t_0 \mid X\left(t_0\right) = \tilde{x}_0\right]$$



$\dot{X}(t)$
UNSAFE
Ω
SAFE
Γ
$X(t)$


$\dot{x}(t)$
$x(t)$
S
U


$\dot{x}(t)$
$x(t)$
S
U


$\dot{x}(t)$
$x(t)$
S
U

$$R\left(t,\Gamma;t_0,\Omega\right)=\int_{\Omega} p\left(\tilde{x};t\,|\,\tilde{x}_0;t_0\right)d\tilde{x}=P\left[T>t-t_0\,|\,X\left(t_0\right)=\tilde{x}_0\right]$$

$$\frac{\partial R}{\partial t_0}=-\sum_{j=1}^{n} f_j\left[t_0,\tilde{x}_0\right]\frac{\partial R}{\partial x_{0j}}-\sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0,\tilde{x}_0\right)\right]_{ij}\frac{\partial^2 R}{\partial x_{0i}\partial x_{0j}}$$

$$R\left(t_0,\Gamma;t_0,\Omega\right)=1$$

• [At $t=t_0$, no trajectory has reached the critical barrier]

$$R\left(t,\Gamma;t_0,\Gamma\right)=0$$

• [If the trajectory originates on Γ the probability of failure=1]

$$0\le R\left(t,\Gamma;t_0,\Omega\right)\le 1$$

• [Because R is a probability]

Let $\tau = t - t_0$

$$R\left(t, \Gamma; x_0; t_0\right) \equiv R\left(\tau, \Gamma; x_0\right) = P\left[T > \tau \mid X\left(t_0\right) = x_0 \in \Omega\right]$$

$$-\frac{\partial R}{\partial \tau} = -\sum_{j=1}^{n} f_j\left[t_0, \tilde{x}_0\right]\frac{\partial R}{\partial x_{0j}} - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0, \tilde{x}_0\right)\right]_{ij}\frac{\partial^2 R}{\partial x_{0i}\partial x_{0j}}$$

$$R\left(\tau, \Gamma; x_0 \in \Omega\right)|_{\tau=0} = 1$$

$$R\left(\tau, \Gamma; x_c \in \Gamma\right) = 0$$

$$0 \le R\left(\tau, \Gamma; x_0 \in \Omega\right) \le 1$$

Moments of first passage time

$$P\left[T \le \tau \mid X\left(t_0\right) = x_0 \in \Omega\right] = 1 - P\left[T > \tau \mid X\left(t_0\right) = x_0 \in \Omega\right]$$

$$R\left(t, \Gamma; \tilde{x}_0 \in \Omega\right) = \int_{\Omega} p\left(\tilde{x}; t \mid \tilde{x}_0; t_0\right) d\tilde{x} = P\left[T > t - t_0 \mid X\left(t_0\right) = \tilde{x}_0 \in \Omega\right]$$

$$P_T\left(\tau, \Gamma, x_0 \in \Omega\right) = 1 - R\left(t, \Gamma; \tilde{x}_0 \in \Omega\right)$$

$$p_T\left(\tau, \Gamma, x_0 \in \Omega\right) = \frac{\partial}{\partial \tau} P_T\left(\tau, \Gamma, x_0 \in \Omega\right) = -\frac{\partial}{\partial \tau} R\left(t, \Gamma; \tilde{x}_0 \in \Omega\right)$$

We have

$$-\frac{\partial R}{\partial \tau} = -\sum_{j=1}^{n} f_j\left[t_0,\tilde{x}_0\right]\frac{\partial R}{\partial x_{0j}} - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0,\tilde{x}_0\right)\right]_{ij}\frac{\partial^2 R}{\partial x_{0i}\partial x_{0j}}$$

$$\Rightarrow p_T\left(\tau,\Gamma,x_0\in\Omega\right) = -\sum_{j=1}^{n} f_j\left[t_0,\tilde{x}_0\right]\frac{\partial R}{\partial x_{0j}} -$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0,\tilde{x}_0\right)\right]_{ij}\frac{\partial^2 R}{\partial x_{0i}\partial x_{0j}}$$

$$\left\langle T^n\right\rangle = \int_0^\infty \tau^n p_T\left(\tau,\Gamma,x_0\in\Omega\right)d\tau = M_n\left(\Gamma,x_0\in\Omega\right)$$

$$\Rightarrow$$

$$\mathbf{L} = -\sum_{j=1}^{n} f_j\left[t_0,\tilde{x}_0\right]\frac{\partial}{\partial x_{0j}} - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0,\tilde{x}_0\right)\right]_{ij}\frac{\partial^2}{\partial x_{0i}\partial x_{0j}}$$

$$\left(n+1\right)M_n + \mathbf{L}M_{n+1} = 0; n = 1,2,\ldots$$

GPU

Generalized Pontryagin Equations

$$(n+1)M_n + \mathbf{L}M_{n+1} = 0; n = 1, 2, \ldots$$

$$\mathbf{L} = -\sum_{j=1}^{n} f_j\left[t_0, \tilde{x}_0\right]\frac{\partial}{\partial x_{0j}} - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[GDG^t\left(t_0, \tilde{x}_0\right)\right]_{ij}\frac{\partial^2}{\partial x_{0i}\partial x_{0j}}$$

$$M_{n+1}\left(\Gamma, x_0 \in \Gamma\right) = 0$$

$$M_{n+1}\left(\Gamma, x_0 \in \Omega\right) < \infty$$

$$n = 1, 2, \cdots$$

Introduction to method of stochastic averaging

- An approximate method for analysing lightly damped nonlinear systems under broad band excitations.
- Leads to simplified models for response envelope and phase processes
- Enables application of Markovian methods to the solution of equations govenring the response envelope and phase processes
- Best suited to study nonlinear sdof systems.

Reference :

J B Roberts and P D Spanos, 1986, Stochastic averaging: an approximate method of solving nonlinear random vibraiton problems, International Journal of Nonlinear Mechanics, 21(2),111-134.

Illustration of deterministic averaging procedure

Consider the free vibration response of a nonlinearly damped sdof system

$$\ddot{u}+\omega_0^2 u=\varepsilon f\left(u,\dot{u}\right);t\geq 0;u\left(0\right)\&\dot{u}\left(0\right) \text{ specified.}$$

$\varepsilon =$ a small non-dimensional number

$\varepsilon=0 \Rightarrow$

$$u\left(t\right)=a\cos\left(\omega_0 t+\beta\right)$$

$$\dot{u}\left(t\right)=-a\omega_0\sin\left(\omega_0 t+\beta\right)$$

$$\ddot{u}+\omega_0^2 u=0$$

$$u(t)=A\cos\omega_0 t+B\sin\omega_0 t$$

$$u(0)=u_0 \Rightarrow A=u_0$$

$$\dot{u}(0)=\dot{u}_0 \Rightarrow B=\frac{\dot{u}_0}{\omega_0}$$

$$u(t)=u_0\cos\omega_0 t+\frac{\dot{u}_0}{\omega_0}\sin\omega_0 t=R\cos(\omega_0 t+\beta)$$

For $\varepsilon \neq 0$, we consider the transformation

$$u\left(t\right)=a\left(t\right)\cos\left[\omega_0 t+\beta\left(t\right)\right]$$

$$\dot{u}\left(t\right)=-a\left(t\right)\omega_0\sin\left[\omega_0 t+\beta\left(t\right)\right]$$

Note: this step involves no approximations

Consider

$$u(t) = a(t)\cos\left[\omega_0 t + \beta(t)\right] = a(t)\cos\phi(t); \phi(t) = \omega_0 t + \beta(t)$$

$$\Rightarrow \dot{u}(t) = \dot{a}(t)\cos\phi(t) - a(t)\left[\omega_0 + \dot{\beta}(t)\right]\sin\phi(t)$$

Since we are taking $\dot{u}(t) = -a(t)\omega_0 \sin\phi(t)$, it follows

$$\dot{a}(t)\cos\phi(t) - a(t)\dot{\beta}(t)\sin\phi(t) = 0 \cdots (A)$$

Consider $\dot{u}(t) = -a(t)\omega_0 \sin\phi(t)$

$$\Rightarrow \ddot{u}(t) = -\dot{a}(t)\omega_0 \sin\phi(t) - a(t)\omega_0\left[\omega_0 + \dot{\beta}(t)\right]\cos\phi(t)$$

$$\Rightarrow -\dot{a}(t)\omega_0 \sin\phi(t) - a(t)\omega_0\left[\omega_0 + \dot{\beta}(t)\right]\cos\phi(t) +$$

$$\omega_0^2 a(t)\cos\phi(t) = \varepsilon f\left[a(t)\cos\phi(t), -a(t)\omega_0 \sin\phi(t)\right]$$

$$\Rightarrow -\dot{a}(t)\omega_0 \sin\phi(t) - a(t)\omega_0 \cos\phi(t) =$$

$$\varepsilon f\left[a(t)\cos\phi(t), -a(t)\omega_0 \sin\phi(t)\right] \cdots (B)$$

$$\dot{a}(t)\cos\phi(t)-a(t)\dot{\beta}(t)\sin\phi(t)=0\cdots(A)$$

$$-\dot{a}(t)\omega_0\sin\phi(t)-a(t)\omega_0\cos\phi(t)=$$

$$\varepsilon f\left[a(t)\cos\phi(t),-a(t)\omega_0\sin\phi(t)\right]\cdots(B)$$

Solving for $\dot{a}(t)$ and $\dot{\beta}(t)$ we get

$$\dot{a}(t)=-\frac{\varepsilon}{\omega_0}\sin\phi(t)\,f\left[a(t)\cos\phi(t),-a(t)\omega_0\sin\phi(t)\right]$$

$$\dot{\beta}(t)-\frac{\varepsilon}{\omega_0 a(t)}\cos\phi(t)\,f\left[a(t)\cos\phi(t),-a(t)\omega_0\sin\phi(t)\right]$$

- Note: no approximations till now.
- If ε is small, $\dot{a}(t)$ and $\dot{\beta}(t)$ are small.

That is $a(t)$ and $\beta(t)$ do not change appreciably over

the time duraiton t to $t+\frac{2\pi}{\omega_0}$.

$$\dot{a}(t) = -\frac{\varepsilon}{\omega_0}\sin\phi(t)\, f\left[a(t)\cos\phi(t), -a(t)\,\omega_0 \sin\phi(t)\right]$$

$$\dot{\beta}(t) - \frac{\varepsilon}{\omega_0 a(t)}\cos\phi(t)\, f\left[a(t)\cos\phi(t), -a(t)\,\omega_0 \sin\phi(t)\right]$$

$$\dot{a}(t) \approx -\frac{\varepsilon}{\omega_0}\frac{1}{2\pi}\int_0^{2\pi} \sin\phi\, f\left[a\cos\phi, -a\omega_0 \sin\phi\right] d\phi$$

$$\dot{\beta}(t) \approx -\frac{\varepsilon}{\omega_0 a}\frac{1}{2\pi}\int_0^{2\pi} \cos\phi\, f\left[a\cos\phi, -a\omega_0 \sin\phi\right] d\phi$$

These are averaged equations.

Example

$$\ddot{u} + \omega_0^2 u = -2\varepsilon\mu\dot{u}$$

$$\dot{a}(t) \approx -\frac{\varepsilon}{\omega_0}\frac{1}{2\pi}\int_0^{2\pi} \sin\phi\left[2\mu a\omega_0 \sin\phi\right]d\phi$$

$$= -\frac{\varepsilon}{\omega_0}\frac{1}{2\pi}\left(2\mu a\omega_0\right)\int_0^{2\pi} \sin^2\phi d\phi = -\varepsilon a\mu$$

$$\dot{\beta}(t) \approx -\frac{\varepsilon}{\omega_0 a}\frac{1}{2\pi}\int_0^{2\pi} \cos\phi\left[2\mu a\omega_0 \sin\phi\right]d\phi = 0$$

$$\Rightarrow u(t) \approx a_0 \exp(-\varepsilon\mu t)\cos\left[\omega_0 t + \beta_0\right]$$

Exercise: compare this with the known exact solution

Example

$$\ddot{u}+\omega_0^2 u=-\varepsilon\dot{u}\left|\dot{u}\right|$$

$$\dot{a}(t)\approx-\frac{\varepsilon}{\omega_0}\frac{1}{2\pi}\int_0^{2\pi}\sin\phi\left[a\omega_0\sin\phi\left|-a\omega_0\sin\phi\right|\right]d\phi$$

$$=-\frac{\varepsilon}{\omega_0}\frac{1}{2\pi}\left(a^2\omega_0^2\right)\int_0^{2\pi}\sin^2\phi\left|\sin\phi\right|d\phi=-\varepsilon\frac{4\omega_0 a^2}{3\pi}$$

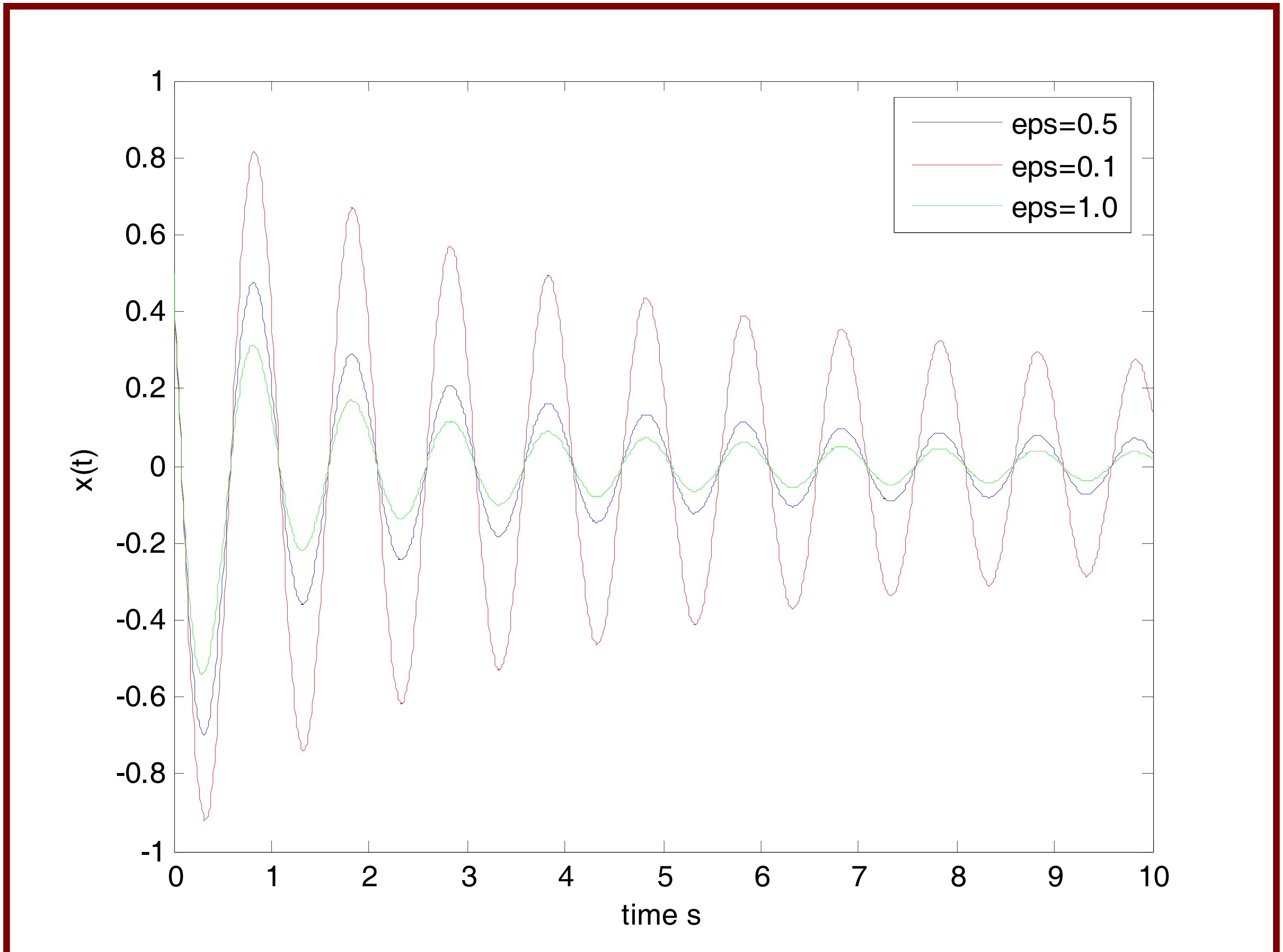
$$\dot{\beta}(t)\approx-\frac{\varepsilon}{\omega_0 a}\frac{1}{2\pi}\int_0^{2\pi}\cos\phi\left[a\omega_0\sin\phi\left|-a\omega_0\sin\phi\right|\right]d\phi=0$$

$$\Rightarrow u(t)\approx\left\{\frac{a_0}{1+\frac{4\varepsilon\omega_0 a_0}{3\pi}t}\right\}\cos\left[\omega_0 t+\beta_0\right]$$

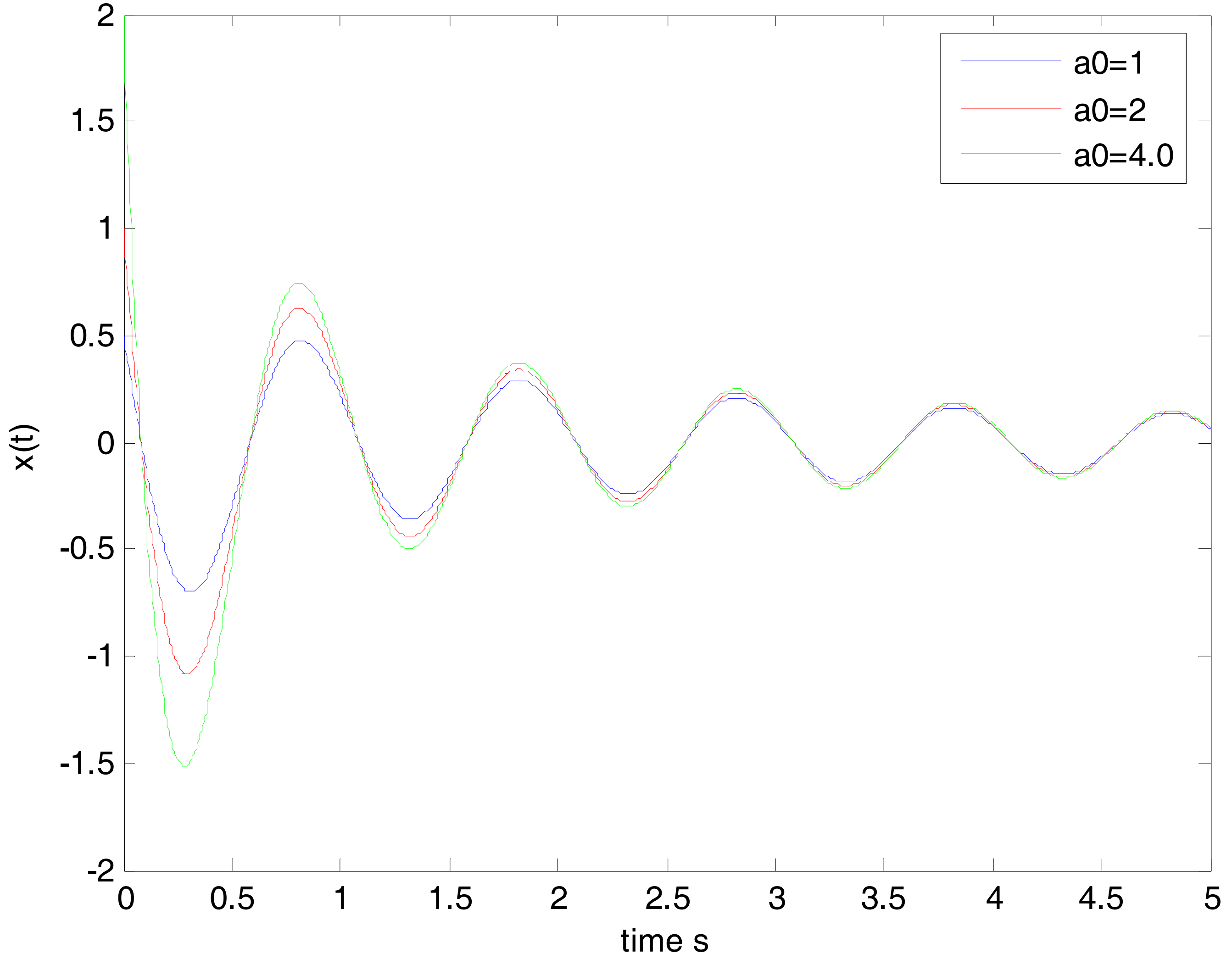
$$\dot{a}=-\epsilon\frac{4\omega_0^2 a^2}{3\pi}$$

$$\frac{da}{a^2}=-\epsilon\frac{dt\,4\omega_0^2}{3\pi}$$

Note: Rate of free vibration decay depends on ics.



eps=0.5
eps=0.1
eps=1.0
x(t)
time s
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0 1 2 3 4 5 6 7 8 9 10



a0=1
a0=2
a0=4.0
x(t)
time s
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Example

$$\ddot{u}+\omega_0^2 u=\varepsilon\left(\dot{u}-\dot{u}^3\right)$$

$$\ddot{u}+\omega_0^2 u-\varepsilon\dot{u}\left(1-\dot{u}^2\right)=0$$

$$\dot{a}=\frac{1}{2}\varepsilon a\left[1-\frac{3}{4}\omega_0^2 a^2\right]\ \&\ \dot{\beta}=0$$

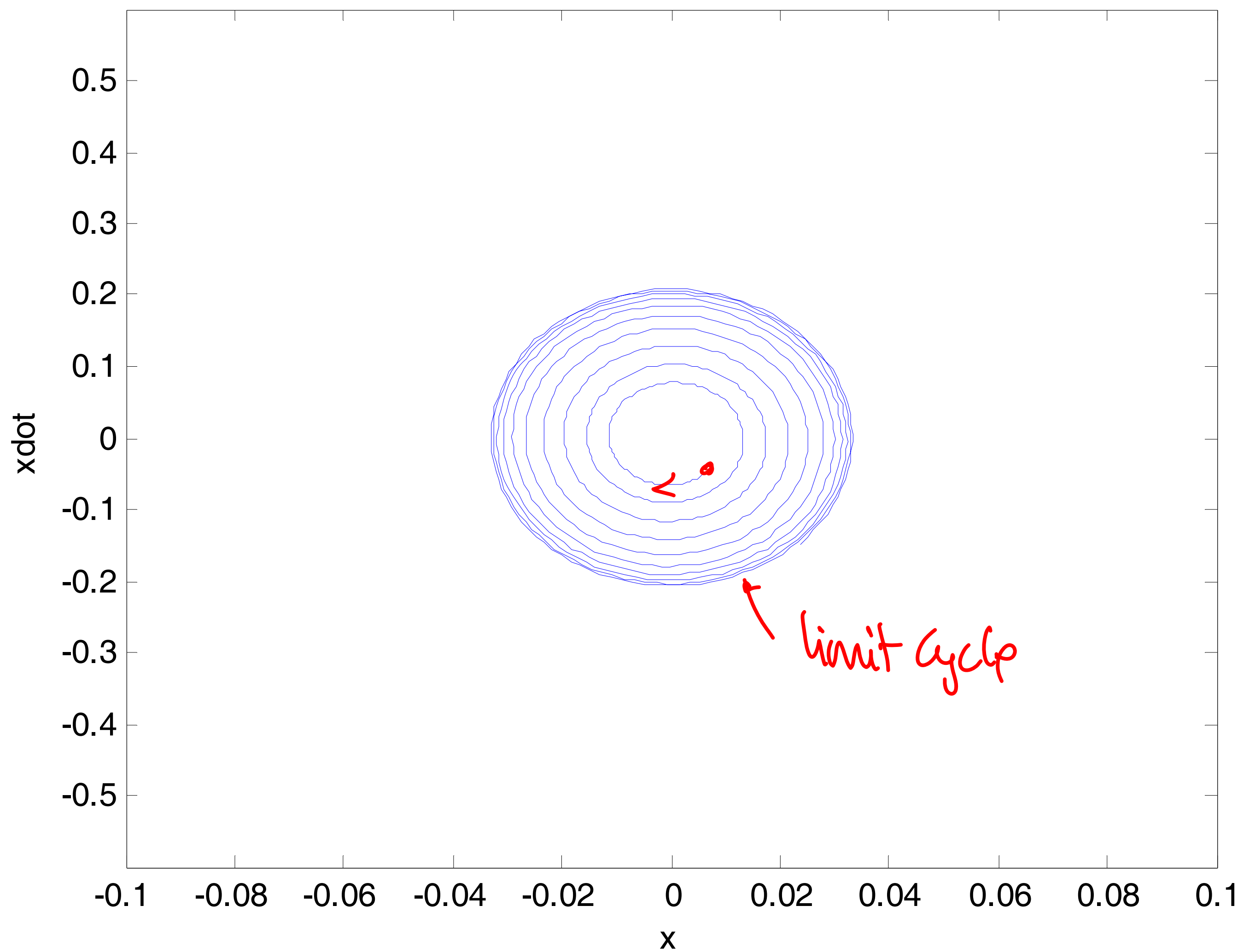
$$a(t)=\left\{\frac{a_0^2}{\frac{3}{4}\omega_0^2 a_0^2+\left[1-\frac{3}{4}\omega_0^2 a^2\right]\exp(-\varepsilon t)}\right\}^{\frac{1}{2}}\ \&\ \beta=\beta_0$$

$$\Rightarrow u(t)\approx\left\{\frac{a_0^2}{\frac{3}{4}\omega_0^2 a_0^2+\left[1-\frac{3}{4}\omega_0^2 a^2\right]\exp(-\varepsilon t)}\right\}^{\frac{1}{2}}\cos\left[\omega_0 t+\beta_0\right]$$

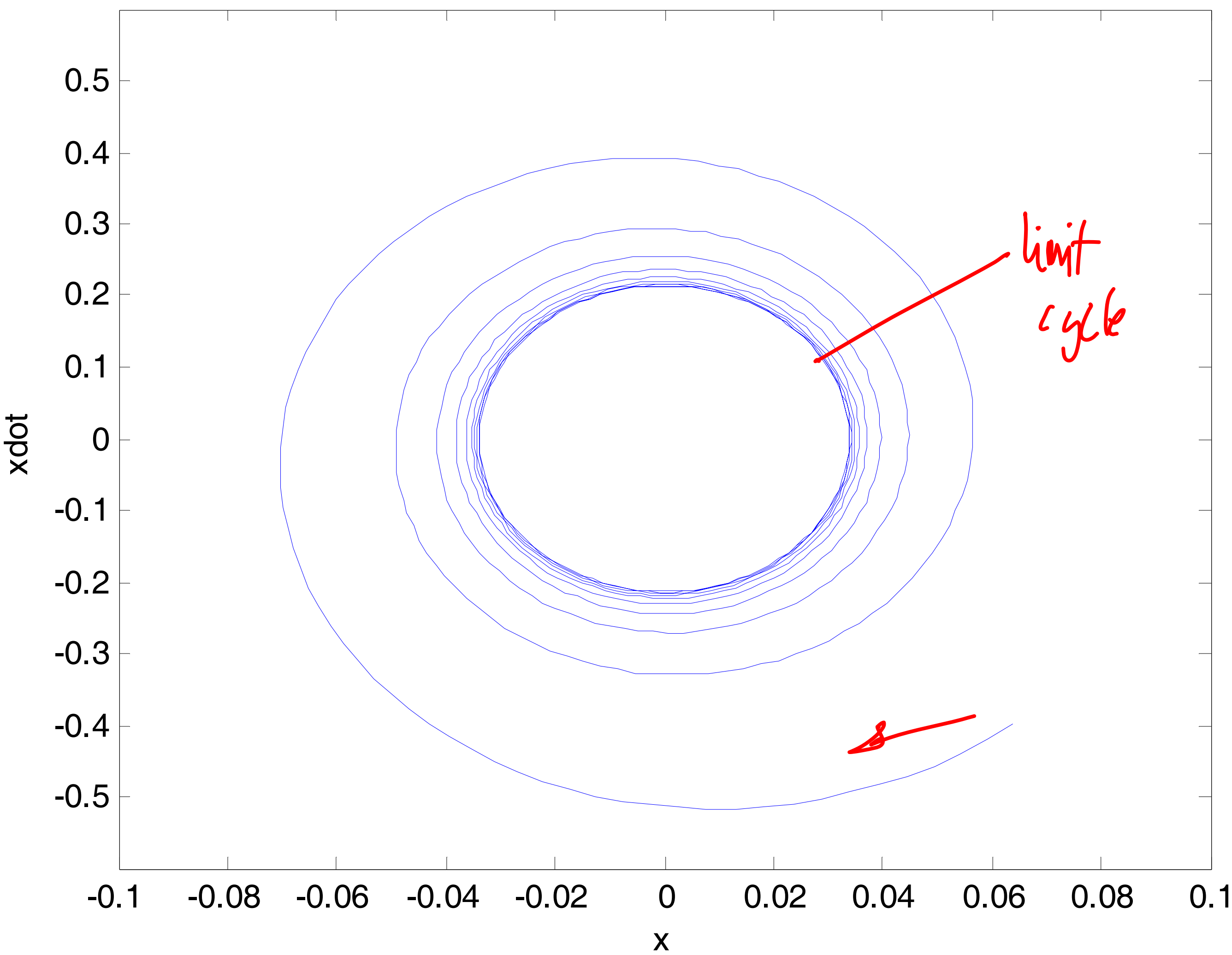
$$u(t) \approx \left\{ \frac{a_0^2}{\frac{3}{4}\omega_0^2 a_0^2 + \left[1 - \frac{3}{4}\omega_0^2 a^2\right] \exp(-\varepsilon t)} \right\}^{\frac{1}{2}} \cos\left[\omega_0 t + \beta_0\right]$$

$$\lim_{t \to \infty} u(t) \to \left\{ \frac{4}{3\omega_0^2} \right\}^{\frac{1}{2}} \cos\left[\omega_0 t + \beta_0\right]$$

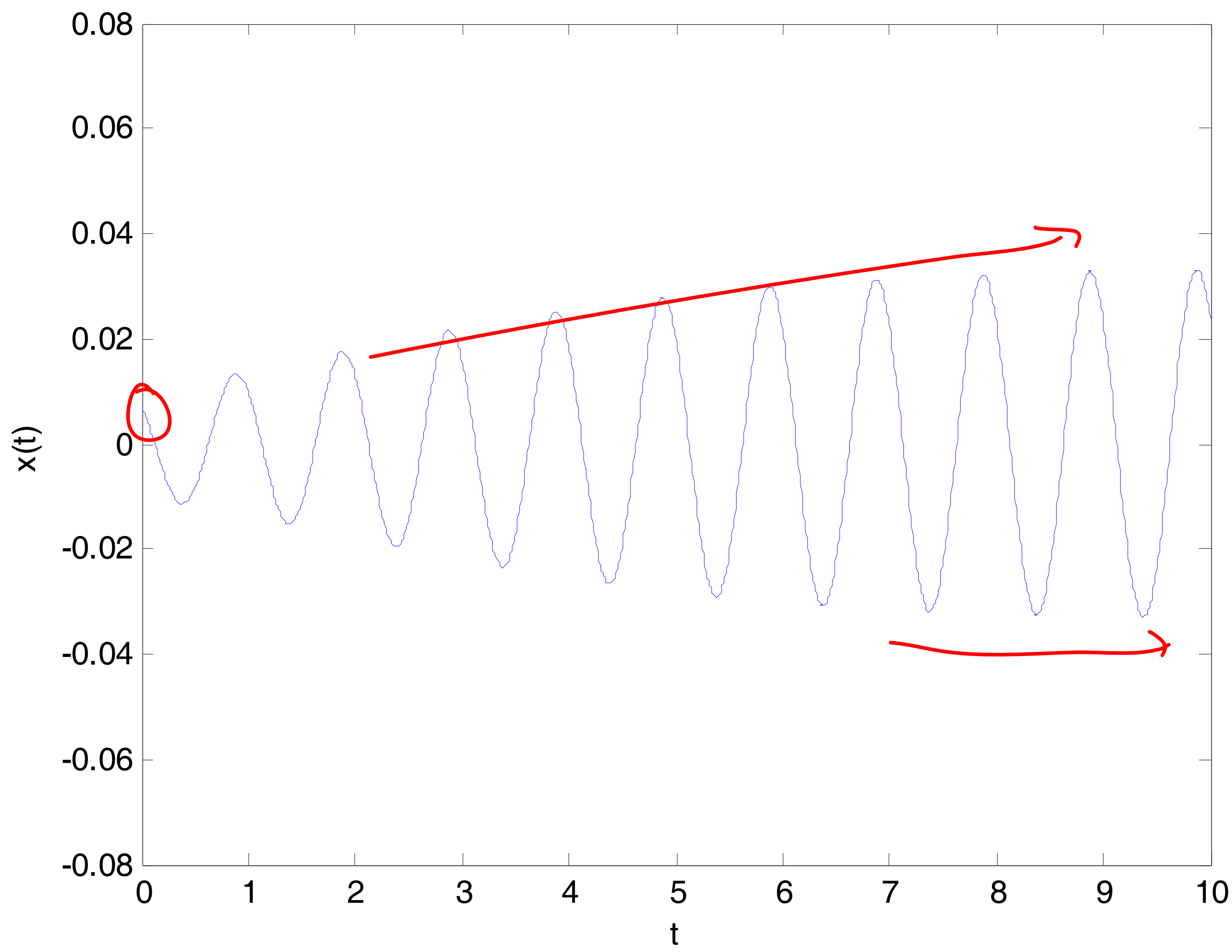
Note: the amplitude of steady state oscillations (in free vibrations) is independent of initial conditions. Contrast this with the response of undamped linear sdof system.



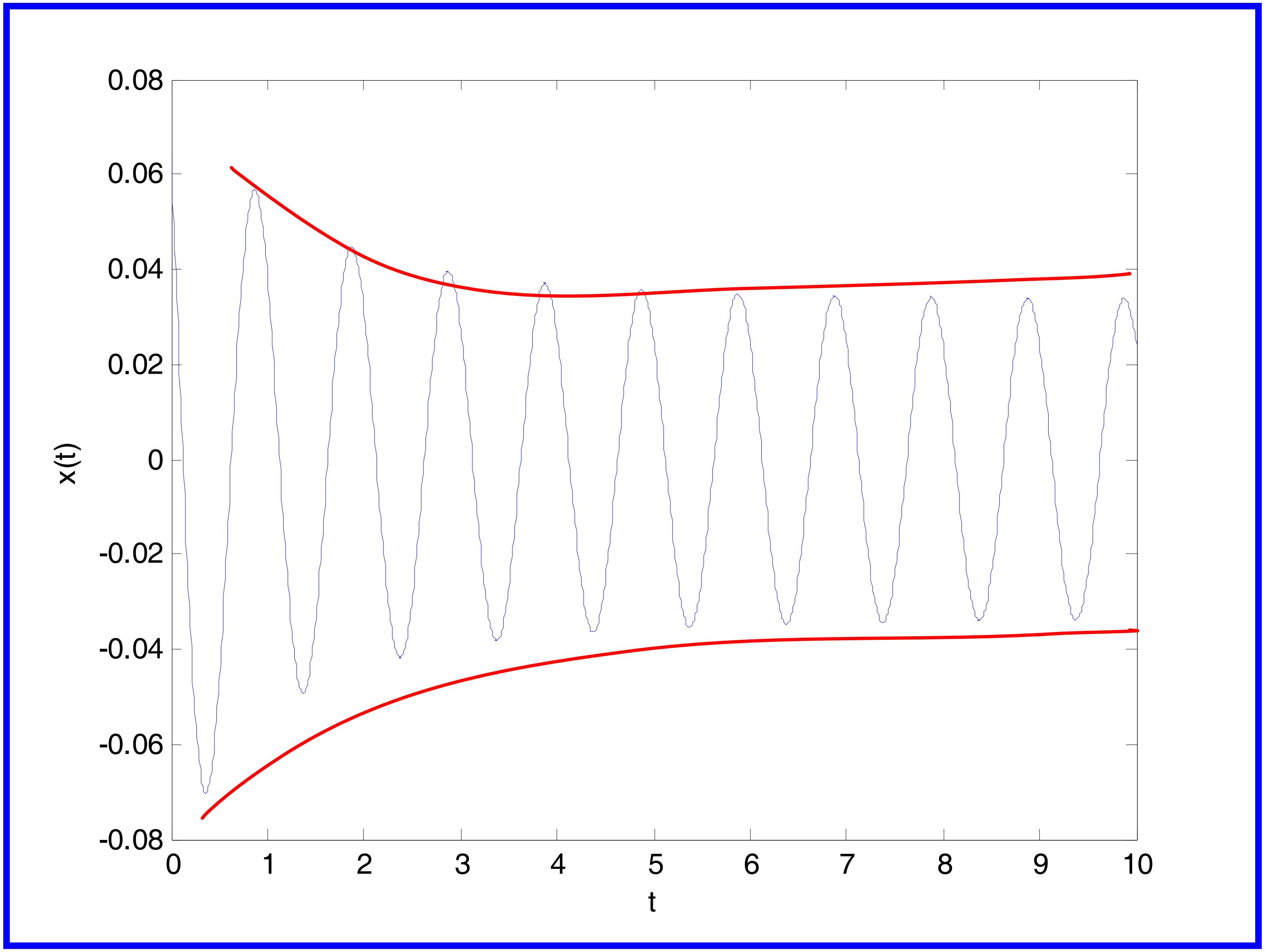
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
xdot
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1
x
limit cycle



limit cycle
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
xdot
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1
x



0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
x(t)
0
1
2
3
4
5
6
7
8
9
10
t



0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
x(t)
0
1
2
3
4
5
6
7
8
9
10
t