INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 3

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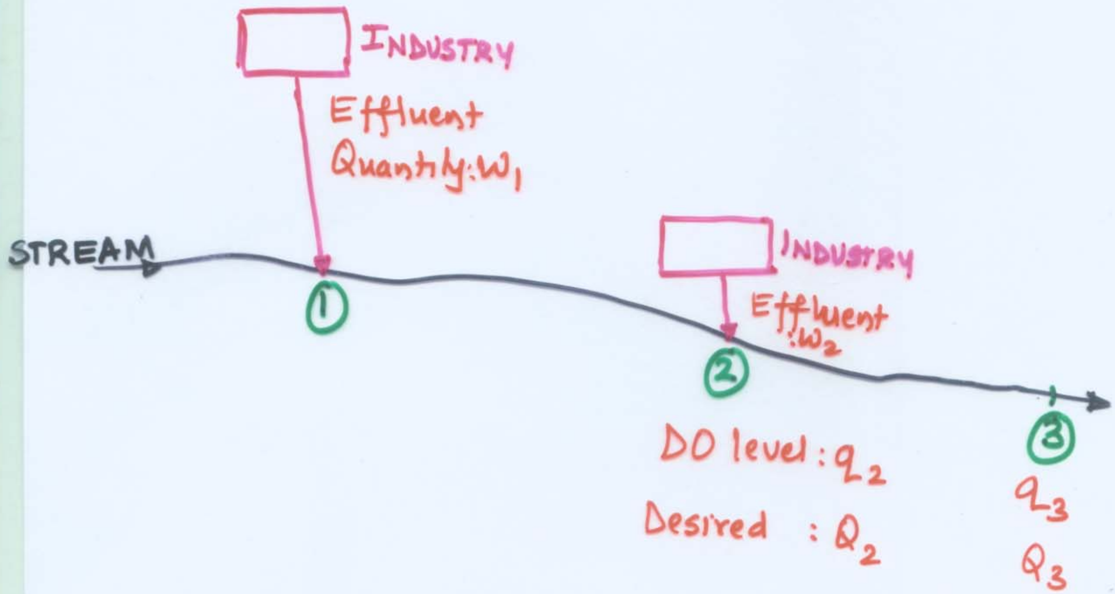
Summary of the previous lecture

- Definition of a system
- Types of systems
  - Simple and complex systems
  - Linear and nonlinear systems
  - Time variant and time invariant systems
  - Continuous, discrete and quantized systems
  - Lumped parameter and distributed parameter systems
  - Deterministic and probabilistic systems
  - Stable and unstable systems

SIMPLE LP FORMULATIONS IN WATER RESOURCES

- WATER QUALITY MANAGEMENT MODELS.



- EACH UNIT OF WASTE LOAD REMOVED AT SITE 1 ENHANCES DO LEVEL AT SITE 2 BY a_{12} & THAT AT SITE 3 BY a_{13}.

- SIMILARLY a_{23} IS THE INCREASE IN DO LEVEL AT SITE 3 FOR EACH UNIT OF WASTE LOAD REMOVED AT SITE 2.

x_1 : FRACTION OF EFFLUENT TREATED AT SITE 1

x_2 : FRACTION OF EFFLUENT TREATED AT SITE 2.

COST OF TREATMENT: $C_1(x_1) + C_2(x_2)$

DECISIONS : x_1 & x_2.

DO LEVEL AT ② AFTER TREATMENT AT ①:

$$= \underbrace{q_2}_{\text{ORIGINAL DO LEVEL}} + \overbrace{a_{12}\, \underbrace{w_1 x_1}_{\text{AMOUNT OF WASTE REMOVED}}}^{\text{IMPROVEMENT IN DO.}}$$

DO LEVEL AT ③ AFTER TREATMENT AT ① & ②:

$$= \underbrace{q_3}_{\text{ORIGINAL DO}} + \underbrace{a_{13} w_1 x_1}_{\text{IMPROVEMENT DUE TO TREATMENT AT ①}} + \underbrace{a_{23} w_2 x_2}_{\text{IMPROVEMENT DUE TO TREATMENT AT ②.}}$$

OPTIMIZATION MODEL:

$$\text{MIN. } C_1(x_1) + C_2(x_2)$$

S.t.

$$q_2 + a_{12} w_1 x_1 \geq Q_2$$

$$q_3 + a_{13} w_1 x_1 + a_{23} w_2 x_2 \geq Q_3$$

$$x_{min} \leq x_1 \leq x_{max}$$

$$x_{min} \leq x_2 \leq x_{max}.$$

Desired @ 2

Desired @ 3

TECHNOLOGICAL LIMITS

Optimization and simulation

- Mathematical expression for optimization problem

  Maximize f(X)

  subject to (s.t.)

  $g_j(X) \leq 0 \qquad j = 1, 2, \ldots\ldots m$

  Where X is vector of decision variables

  $X = [x_1, x_2, x_3, \ldots\ldots x_n]$

  n decision variables, m constraints

- Decision variables are the variables for which decisions are required.
- Complexity of the problem varies depending on nature of function, constraints and the no. of variables and constraints.

Optimization and simulation

- Simulation is a technique used to mimic the behavior of a system.
- Simulation is used to answer “what-if” type of questions.
- Simulation is a powerful technique for analyzing complex systems for performance evaluation
- Decision makers would be interested in examining a number of scenarios rather than just looking at one single solution that is optimal; simulation is useful in such situations.

Optimization and simulation

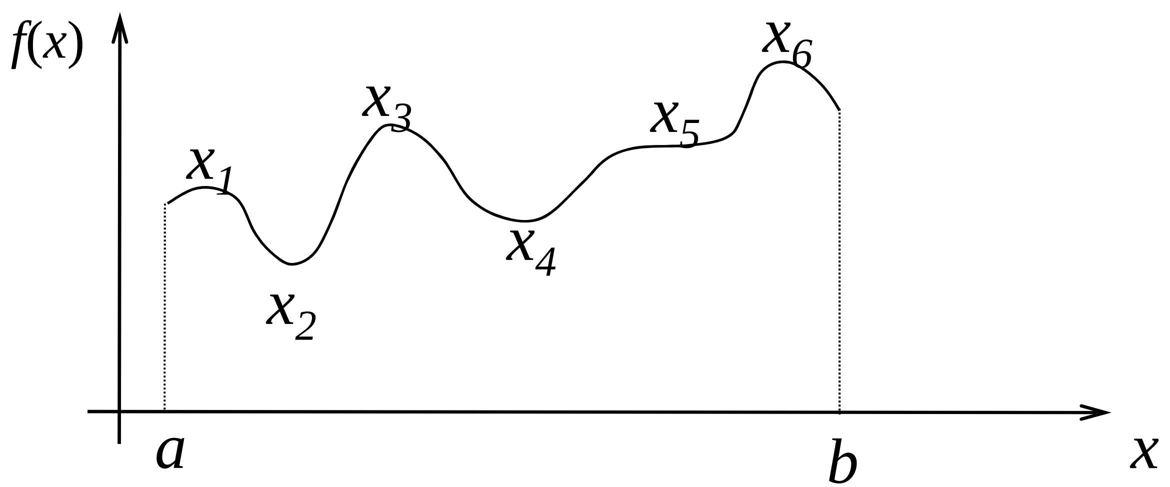
- Possible to obtain near-optimal solutions by repeatedly simulating a system with various sets of inputs.
- Typical examples where simulation is used are
  - Analysis of river basin development alternatives
  - Multi-reservoir operation problems
  - Generating trade-offs of water allocations among various uses
  - Conjunctive use of surface and ground water resources.

Optimization: Methods of Calculus

Optimization: Methods of Calculus

Function of a single variable:

- Let $f(x)$ be a function of a variable x, defined in the range $a < x < b$



- Local maximum: value higher than any other value in neighbourhood; x_1, x_3 and x_6 are local maxima

$$f(x_1 - \Delta x_1) < f(x_1) > f(x_1 + \Delta x_1)$$

Optimization: Methods of Calculus

- Local minimum: value lower than any other value in neighbourhood; x_2 and x_4 are local minima

  $f(x_2 - \Delta x_2) > f(x_2) < f(x_2 + \Delta x_2)$

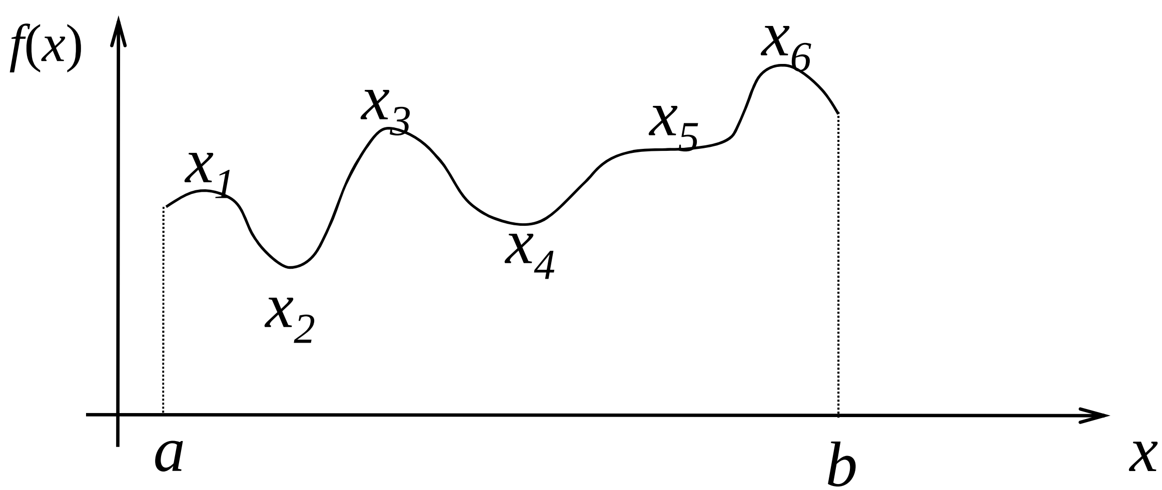
- Saddle point: The slope of the function is zero at saddle point (x_5); value of the function is lower on one side and higher on other (or vice-versa).

  $f(x_5 - \Delta x_5) < f(x_5) < f(x_5 + \Delta x_5)$; Slope of $f(x)$ at $x = x_5$ is zero

Optimization: Methods of Calculus

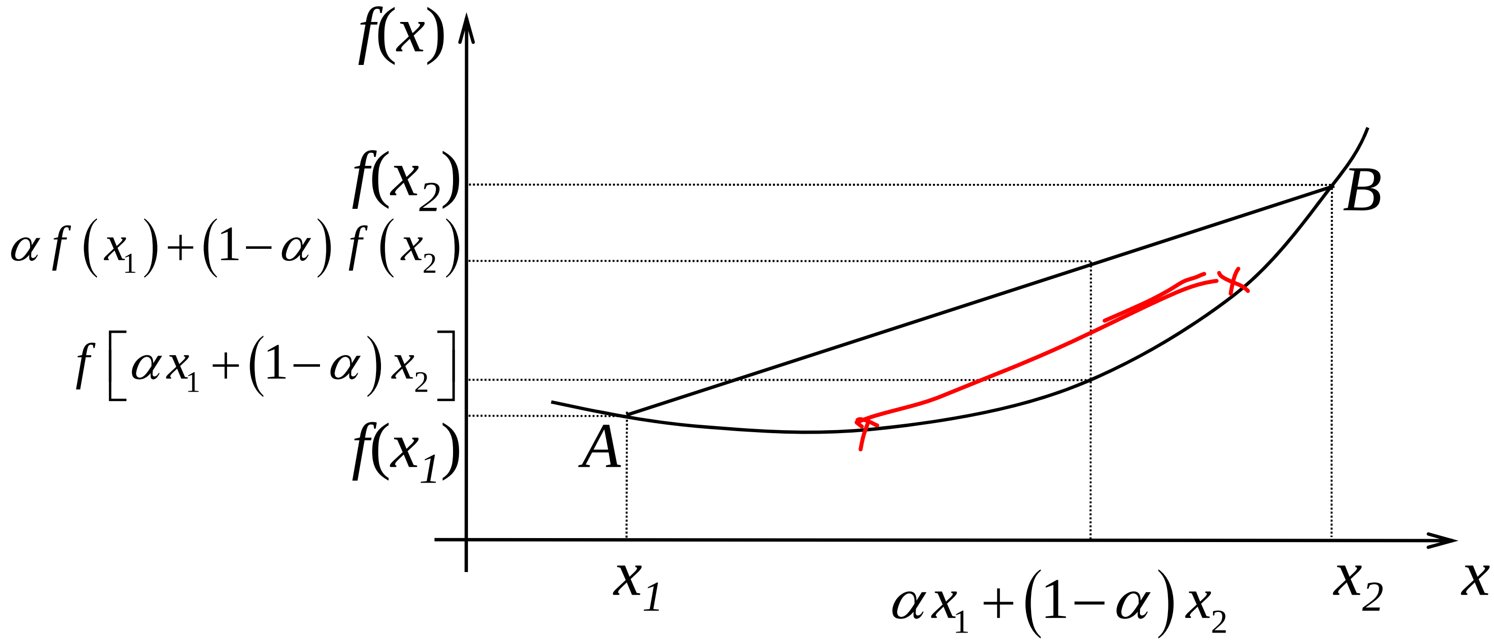
- Global maximum: value of function is higher than any other value in the defined range (point x_6 in the figure)

- Global minimum: value of function is lower than any other value in the defined range (point x_2 in the figure)



Optimization: Methods of Calculus

- Convex functions:



- A straight line (AB) drawn between any two points is above the curve

Optimization: Methods of Calculus

- $f(x)$ is said to be strictly convex if

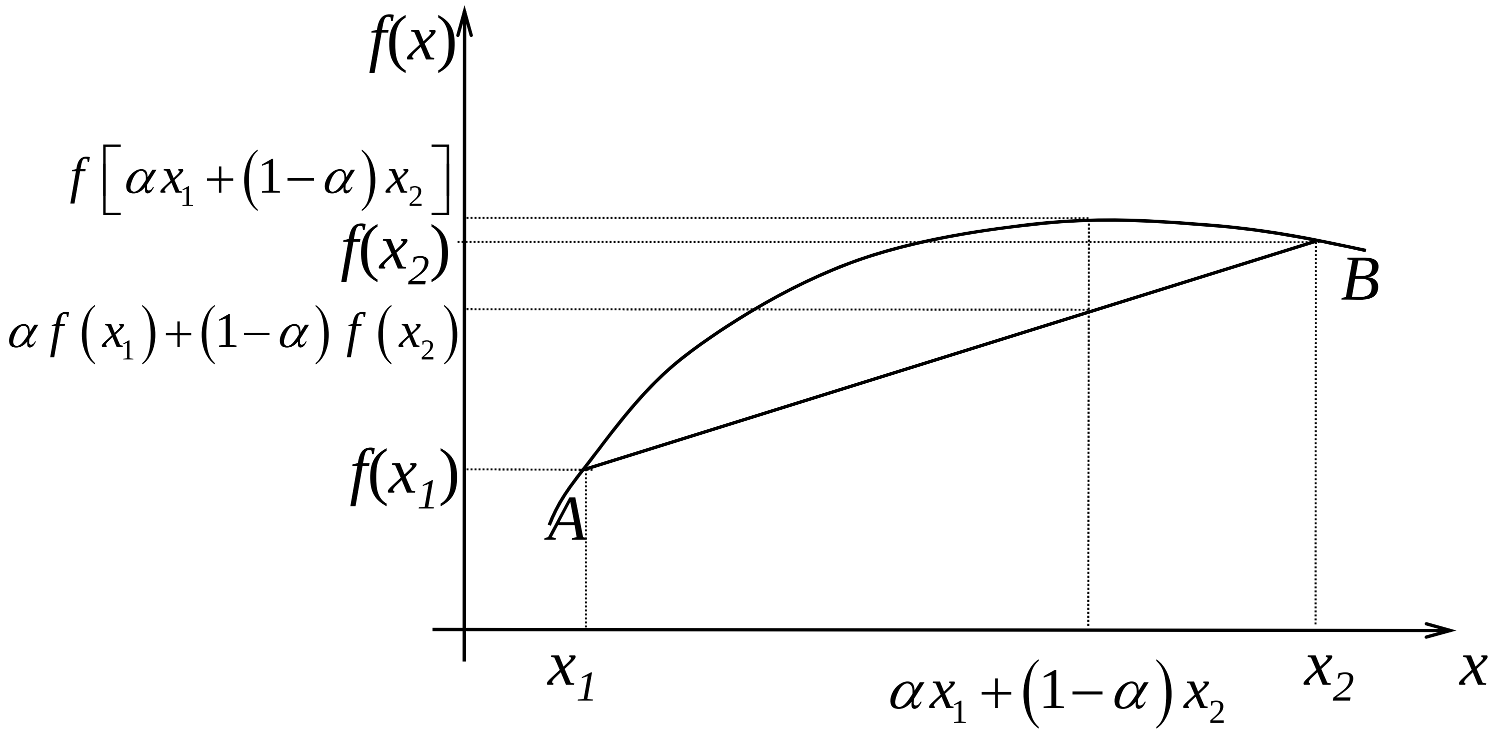
$$f\left[\alpha x_1 + (1-\alpha)x_2\right] < \alpha f(x_1) + (1-\alpha) f(x_2) \qquad 0 \leq \alpha \leq 1$$

- If the inequality sign < is replaced by $\leq$ sign, then $f(x)$ is said to be convex but not strictly convex
- If the inequality sign < is replaced by = sign, $f(x)$ is a straight line and satisfies the condition for convexity mentioned above; A straight line is a convex function
- If a function is strictly convex, slope increases continuously

  For a strictly convex function, $\dfrac{d^2 f}{dx^2} > 0$

Optimization: Methods of Calculus

- Concave function



- A straight line (AB) drawn between any two points is below the curve

Optimization: Methods of Calculus

- $f(x)$ is said to be strictly concave if

$$f\left[\alpha x_1 + (1-\alpha) x_2\right] > \alpha f(x_1) + (1-\alpha) f(x_2) \qquad 0 \le \alpha \le 1$$

- If the inequality sign > is replaced by $\geq$ sign, then $f(x)$ is said to be concave but not strictly concave.
- If the inequality sign < is replaced by = sign, $f(x)$ is a straight line and satisfies the condition for concavity mentioned above; A straight line is a concave function
- If a function is strictly concave, slope increases continuously

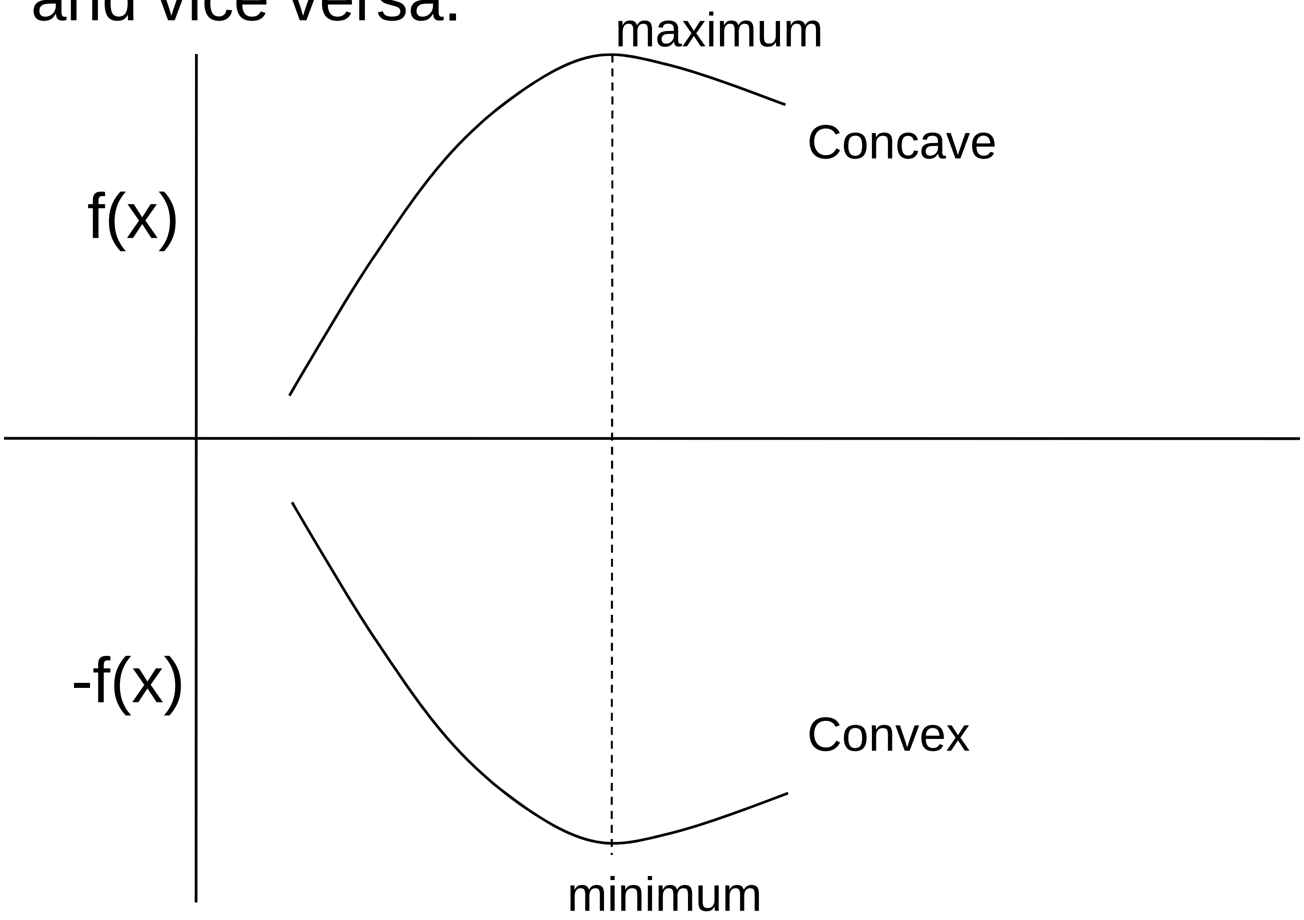
For a strictly concave function, $\dfrac{d^2 f}{dx^2} < 0$

Optimization: Methods of Calculus

- A straight line is both convex and concave and is neither strictly convex nor strictly concave.
- A local minimum of a convex function is also its global minimum.
- A local maximum of a concave function is also its global maximum.
- The sum of strictly convex functions is strictly convex
- The sum of strictly concave functions is strictly concave.

Optimization: Methods of Calculus

- If $f(x)$ is a concave function, $-f(x)$ is a convex function and vice versa.



Optimization: Methods of Calculus

- If $f(x)$ is a convex function and α is a constant,

  $\alpha f(x)$ is a convex function if $\alpha > 0$ and
  $\alpha f(x)$ is a concave function if $\alpha < 0$

Optimization: Methods of Calculus

- At stationary point, the slope of function is zero

$x = x_0$ is a stationary point if $\left.\dfrac{df}{dx}\right|_{x_0} = 0$

Sufficiency condition is examined as follows

- If $\dfrac{d^2 f}{dx^2} > 0$ for all x, $f(x)$ is convex and stationary point is a global minimum